STRUCTURE SPACES FOR SANDWICH SEMIGROUPS

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For a given semigroup, a structure space is associated
in a natural way with each left ideal of that semigroup.
In this paper, we determine this space when the semigroup
is a sandwich semigroup satisfying appropriate conditions
and the left ideal is the kernal of the semigroup.

Most of the attention in [5] and in subsequent papers [6], [7]
was focused on the case where the left ideal is the kernel of the
semigroup in which case the term $\mathscr{K}$-structure space was used.
We recall briefly the definition. Let S be any semigroup with
kernel K. The requirement that S have a kernel (i.e., smallest
two-sided ideal) is, of course, a restriction since the intersection of
all two-sided ideals of a semigroup may well be empty. A *bond* on
S is any subset A of $S \times K$ such that for each finite subset
$\{(s_i, k_i)\}_{i=1}^n$ of A, the set of equations $\{s_i x = k_i\}_{i=1}^n$ has a common
solution x in K. Maximal bonds are called ultrabonds and the
collection of all ultrabonds on S is denoted by $\mathscr{U}(S)$. For each
$(s, k) \in S \times K$, define

$$H(s, k) = \{A \in \mathscr{U}(S) \colon (s, k) \in A\}$$

and topologize $\mathscr{U}(S)$ by taking all sets of the form $H(s, k)$ as a
subbasis for the closed subsets. $\mathscr{U}(S)$, with this topology, is the
$\mathscr{K}$-*structure space* of S. It was shown in [5] and [6] that for
many topological spaces X, the $\mathscr{K}$-structure space of $S(X)$, the
semigroup of all continuous selfmaps of X turns out to be βX, the
Stone-Čech compactification of X.

In [5] a number of results were proved about the $\mathscr{K}$-structure
space of an abstract semigroup with certain conditions being placed
on that semigroup. One of the conditions was that the semigroup
contain a left identity. This didn't seem to be much of a restric-
tion at the time since we were primarily interested in applying
these results to $S(X)$ which has a two-sided identity. But once
we became more interested in sandwich semigroups, the requirement
that the semigroup contain a left identity became a serious obstacle.
Let us recall the definition of a sandwich semigroup $S(X, Y, \alpha)$. X
and Y are topological spaces and α is a continuous function from
Y into X. $S(X, Y, \alpha)$ is the semigroup of all continuous functions
from X into Y where the product fg of two elements f and g in
$S(X, Y, \alpha)$ is defined by $fg = f \circ \alpha \circ g$. The semigroup $S(X, Y, \alpha)$ is
referred to as a *sandwich semigroup* with sandwich *function* α.

The problem is that $S(X, Y, \alpha)$ contains a left identity if and only if α maps Y homeomorphically onto a retract of X and this is far too restrictive for our purposes here. However, Anne Hudson (private communication) and H. O. Kim [2] observed independently that the requirement that the semigroup contain a left identity is not really necessary, at least, not in order to get a number of the results we obtained in [5]. So this meant that it would not be unreasonable to try to use these results (or more accurately, slightly amended versions of these results) to determine just what the structure space of $S(X, Y, \alpha)$ is. This problem is the main thrust of this paper and we show that for certain spaces X and Y, if α is any closed function from Y into X satisfying some rather mild additional conditions, then the $\mathscr{K}$-structure space $S(X, Y, \alpha)$ is the Stone-Čech compactification of the range of α. The results in the present paper generalize results in both [5] and [6] since by taking $X = Y$ and α to be the identity map we get results in the latter as special cases.

In § 2, we state (without proof) revisions of results in [5] that we need in order to prove the main theorem here. Section 3 contains a brief discussion of sandwich semigroups and some of their elementary properties and it is there that the main theorem of the paper is proved. In § 4, we discuss some applications of the main result and some examples.

2. Revisions of some previous results. Although we are primarily interested in $\mathscr{K}$-structure spaces, whose definition we have recalled in the introduction, there are generally many structure spaces for a semigruop, one for each left ideal. We now give the more general definition so we can state the revision of the results in [5] that we need in this paper. The proofs of these results when T has no left identity are the same as in [5] or at worst are obvious modifications thereof

DEFINITION 2.1. Let S be any semigroup and Z any left ideal of S. A nonempty subset A of $S \times Z$ is called a bond if for each finite subset $\{(s_i, z_i)\}_{i=1}^n$ of A the system of equations $\{s_i x = z_i\}_{i=1}^n$ has a common solution x in Z. A maximal bond is called an ultrabond. The set of all ultrabonds is denoted by $\mathscr{U}(S, Z)$.

For any $v \in Z$, the set $A_v = \{(s, sv): s \in S\}$ is an ultrabond. In [5] it was assumed that S has a left identity to prove this. A_v is an ultrabond regardless of whether or not S has a left identity.

DEFINITION 2.2. An ultrabond of the form A_v is called a princi-

pal ultrabond and the set of all principal ultrabonds is denoted by $\mathscr{R}(S, Z)$.

We topologize $\mathscr{U}(S, Z)$ by taking the sets of the form

$$H(s, z) = \{A \in \mathscr{U}(S, Z): (s, z) \in A\}$$

as a subbasis for the closed subsets.

DEFINITION 2.3. $\mathscr{U}(S, Z)$ with the topology just described is called the structure space of the pair (S, Z). The subspace consisting of all the principal ultrabonds will be denoted by $\mathscr{R}(S, Z)$.

Each element $a \in S$ iduces in a natural way a mapping λ_a from $\mathscr{R}(S, Z)$ into $\mathscr{R}(S, Z)$. Simply, define $\lambda_a(A_v) = A_{av}$.

THEOREM 2.4. *Let S be a semigroup and let Z be a left ideal of S. Then for each element a in S, the function λ_a is a continuous mapping from $\mathscr{R}(S, Z)$ into $\mathscr{R}(S, Z)$.*

DEFINITION 2.5. The pair (S, Z) is said to be admissible if for any ultrabond $A \in \mathscr{C}H(s_1, z_1)$ ($\mathscr{C}$ denotes complement), there exist (s_2, z_2) and (s_3, z_3) in $S \times Z$ such that

$$A \in \mathscr{C}H(s_2, z_2) \subsetneqq H(s_3, z_3) \subsetneqq \mathscr{C}H(s_1 z_1) \ .$$

The following will be crucial for the proof of our main result.

THEOREM 2.6. *Suppose the pair (S, Z) is admissible. Then $\mathscr{U}(S, Z)$ is a Hausdorff compactification of $\mathscr{R}(S, Z)$. Moreover, for each $a \in S$, the function λ_a in $S(\mathscr{R}(S, Z))$ has a unique extension to a function $\hat{\lambda}_a$ in $S(\mathscr{U}(S, Z))$.*

3. **The $\mathscr{K}$-structure space of a sandwich semigroup.** Sandwich semigroups were essentially introduced in [3]. For additional results on sandwich semigroups as well as references to other related papers, one may consult [4], [10], [11] and [13] through [15]. Actually in some of the latter papers, the domains of the functions comprising the sandwich semigroup did not necessarily need to be the entire space. In this paper, of course, they do so that what we call a sandwich semigroup here is a special case of what is studied in some of the latter papers. This section contains the main theorem of the paper and the major portion of this section is devoted to a proof of that theorem. However, before we begin that task, we verify a few of the other remarks made about sandwich semigroups in the introduction. The first is

THEOREM 3.1. $S(X, Y, \alpha)$ *has a left identity if and only if* α *maps* Y *homeomorphically onto a retract of* X.

Proof. Let l be a left identity of $S(X, Y, \alpha)$. The function $\alpha \circ l$ is a continuous selfmap of X. Moreover, we see that

$$(\alpha \circ l) \circ (\alpha \circ l) = \alpha \circ (l \circ \alpha \circ l) = \alpha \circ (ll) = \alpha \circ l$$

so that $\alpha \circ l$ is an idempotent (with respect to composition) continuous selfmap of X. Thus, Ran $(\alpha \circ l)$ (i.e., the range of $\alpha \circ l$)) is a retract of X. Evidently, Ran $(\alpha \circ l) \subseteq$ Ran α. To verify the reverse inclusion, take any $x \in$ Ran α. Then $x = \alpha(y)$ for some $y \in Y$. Define the function $\langle y \rangle$ by $\langle y \rangle(x) = y$ for all $x \in X$ and get

$$(\alpha \circ l)(x) = \alpha \circ l \circ \alpha(y) = \alpha \circ l \circ \alpha \circ \langle y \rangle(x) = \alpha \circ (l \langle y \rangle)(x)$$
$$= \alpha \circ \langle y \rangle(x) = \alpha(y) = x \ .$$

We draw two conclusions from this. The first is that Ran $(\alpha \circ l) =$ Ran α and the second is that $\alpha \circ l$ is the identity on Ran α.

Now choose any $x \in X$ and $y \in Y$ and we have

$$(l \circ \alpha)(y) = l \circ \alpha \circ \langle y \rangle(x) = (l \langle y \rangle)(x) = \langle y \rangle(x) = y \ .$$

That is, $l \circ \alpha$ is identity on Y. Thus, Ran α is, indeed, a retract of X and α is a homeomorphism from Y onto Ran α.

Conversely suppose α is a homeomorphism from Y onto Ran α which is a retract of X. Then Ran $\alpha =$ Ran v where v is some idempotent continuous selfmap of X. Define $l = \alpha^{-1} \circ v$. Then $l \in S(X, Y, \alpha)$ and using the fact that v is the identity on Ran α, one readily verifies that l is a left identity for $S(X, Y, \alpha)$.

When we say that $S(X, Y, \alpha)$ separates points, we mean as usual that if a and b are distinct points of X then $f(a) \neq f(b)$ for some $f \in S(X, Y, \alpha)$.

THEOREM 3.2. *Suppose* $S(X, Y, \alpha)$ *separates points. Then* $S(X, Y, \alpha)$ *has a right identity if and only if* α *maps some retract of* Y *homeomorphically onto* X.

Proof. Let r be a right identity of $S(X, Y, \alpha)$. Then $r \circ \alpha$ is a continuous selfmap of Y and moreover

$$(r \circ \alpha) \circ (r \circ \alpha) = (r \circ \alpha \circ r) \circ \alpha = (rr) \circ \alpha = r \circ \alpha \ .$$

That is, $r \circ \alpha$ is idempotent. We wish to show next that Ran $(r \circ \alpha) =$ Ran r. We immediately have Ran $(r \circ \alpha) \subseteq$ Ran r. On the other hand, take any $y \in$ Ran r. Then $y = r(x)$ for some $x \in X$ and we have

$$(r \circ \alpha)(y) = (r \circ \alpha \circ r)(x) = (rr)(x) = r(x) = y \ .$$

This not only tells us that $\operatorname{Ran}(r \circ \alpha) = \operatorname{Ran} r$, in which case $\operatorname{Ran} r$ is a retract of Y, but it also tells us that $r \circ \alpha$ is the identity on $\operatorname{Ran} r$. To complete the proof we need only show that $\alpha \circ r$ is the identity on X. So let any $x \in X$ and $f \in S(X, Y, \alpha)$ be given. Then

$$f(\alpha \circ r(x)) = (f \circ \alpha \circ r)(x) = (fr)(x) = f(x) \ .$$

Since $S(X, Y, \alpha)$ separates points, it follows that $\alpha \circ r(x) = x$. Thus, $\alpha \circ r$ is the identity on X which means α maps $\operatorname{Ran} r$ homeomorphically onto X and the proof is complete.

REMARK. One cannot simply delete the requirement that $S(X, Y, \alpha)$ separate points and hope to prove the theorem. Examples are abundant which illustrate this. Take X to be any connected space, Y any totally disconnected space and α any continuous function from Y into X. Then $S(X, Y, \alpha)$ is a left zero semigroup which means that everything is a right identity.

COROLLARY 3.3. *Suppose* $S(X, Y, \alpha)$ *separates points. Then* $S(X, Y, \alpha)$ *has a two-sided identity if and only if* α *is a homeomorphism from* Y *onto* X.

Proof. This follows immediately from the previous two theorems.

COROLLARY 3.4. *Suppose* $S(X, Y, \alpha)$ *separates points and has an identity. Then* $S(X, Y, \alpha)$, $S(X)$ *and* $S(Y)$ *are all mutually isomorphic.*

Proof. According to the previous corollary, α is a homeomorphism from Y onto X. The mapping which carries f into $\alpha \circ f$ is an isomorphism from $S(X, Y, \alpha)$ onto $S(X)$ and the mapping which carries f into $f \circ \alpha$ is an isomorphism from $S(X, Y, \alpha)$ onto $S(Y)$.

REMARK. The previous results can all be proven in considerably more generality. For example they all have analogues within the framework of $\varDelta$-structures [9] and the proofs given here carry over intact. In order to present the more general results, we would have to devote considerable space to a discussion of definitions, terminology, etc. and we feel that to be not appropriate here.

Before we can state the main theorem, we need to recall some facts about E-compact spaces. We use the terminology and notation of [6]. Let E be a Hausdorff space. A space X is E-completely

regular if it is homeomorphic to a subspace of some cartesian product of copies of E and it is E-compact if it is homeomorphic to a closed subspace of such a product. An E-compact space Y which contains X as a dense subspace is called an E-compactification of X. An important fact about E-compactifications is that if E is compact in the usual sense, then each E-completely regular space X has a largest E-compactification $\beta_E X$ in the sense that any other E-compactification of X is a continuous image of $\beta_E X$ under a map which keeps the points of X fixed. This is an immediate consequence of Theorem 4.14 of [12, p. 177]. Before stating our main theorem, we recall another concept introduced in [6].

DEFINITION 3.5. Let E be Hausdorff. A space X is E-separated if it is Hausdorff and for each pair H and K of disjoint closed subsets of X, there exists a continuous function f from X into E and two distinct points p and q of E such that $f(x) = p$ for $x \in H$ and $f(x) = q$ for $x \in K$.

It is clear that every E-separated space is normal. It is also easy to see that every closed subspace of an E-separated space is E-separated. We are now in a position to state our main result which concerns the $\mathscr{K}$-structure space of a sandwich semigroup.

MAIN THEOREM 3.6. *Suppose E is compact, X is E-separated and Y contains a copy of E on which α is injective. Suppose also that α is a closed map of Y into X and every continuous function on $\alpha(Y)$ to E can be continuously extended to a function of X to E. Then the $\mathscr{K}$-structure space of $S(X, Y, \alpha)$ is $\beta_E(\alpha(Y))$, the largest E-compactification of $\alpha(y)$.*

Note that $\alpha(Y)$ is E-completely regular, being a closed subspace of an E-separated space X and hence it is meaningful to talk of $\beta_E(\alpha(Y))$.

We will prove the theorem after a series of lemmas. Let S denote the semigroup $S(X, Y, \alpha)$ and K its kernel. $K = \{\langle y \rangle : y \in y\}$ where $\langle y \rangle$ denotes the constant map which maps every element of X into y. Furthermore, it is easy to show that

(1) For any $f \in S$ and $\langle y \rangle$, $\langle z \rangle$ in K, $f \cdot \langle y \rangle = \langle z \rangle$ if and only if $f(\alpha(y)) = z$.

(2) A subset A of $S \times K$ is a bond if and only if $\{f^{-1}(z) : (f, \langle z \rangle) \in A\} \cup \{\text{Ran } \alpha\}$ has the finite intersection property.

(3) A is an ultrabond if and only if $(f, \langle z \rangle) \notin A$ implies $f^{-1}(z) \cap g_1^{-1}(y_1) \cap \cdots \cap g_n^{-1}(y_n) \cap \text{Ran } \alpha = \varnothing$ for a finite subfamily $\{(g_i, \langle y_i \rangle)\}_{i=1}^n$ of A.

We also need the following lemma which is easy to prove.

LEMMA 3.7. *Suppose X is E-separated. For any pair H and K of disjoint closed subsets of X, there exists an $f \in S$ and two distinct points a and b of Y such that $f(x) = a$ for $x \in H$ and $f(x) = b$ for $x \in K$.*

Throughout the rest of this section, X, Y, α are supposed to satisfy the hypothesis of the main theorem.

LEMMA 3.8. *The pair (S, K) is admissible.*

Proof. Suppose A is an ultrabond and $A \in \mathscr{C}H(f, \langle z \rangle)$. Then there exist $\{(g_i, \langle y_i \rangle)\}_{i=1}^{K} \in A$ such that

$$(4) \qquad f^{-1}(z) \cap W = \varnothing, \quad \text{where} \quad W = \bigcap_{i=1}^{N} g_i^{-1}(y_i) \cap \operatorname{Ran} \alpha \ .$$

Since $W \subseteq \mathscr{C}f^{-1}(z)$ and X is normal, there exists an open set G and a closed set K such that $W \subseteq G \subseteq K \subseteq \mathscr{C}f^{-1}(z)$. By Lemma 3.7 there exists two functions $h, k \in S$ and points a, b, p, q in Y such that $a \neq b$ and $p \neq q$ and

$$h(x) = a \quad \text{for} \quad x \in X - G$$
$$h(x) = b \quad \text{for} \quad x \in W$$
$$k(x) = p \quad \text{for} \quad x \in K$$

and

$$k(x) = q \quad \text{for} \quad x \in f^{-1}(z) \ .$$

Then $W \subseteq \mathscr{C}h^{-1}(a) \subseteq k^{-1}(p) \subseteq \mathscr{C}f^{-1}(z)$. To complete the proof we show that $A \in \mathscr{C}H(h, \langle a \rangle) \subseteq H(k, \langle p \rangle) \subseteq \mathscr{C}H(f, \langle z \rangle)$.

Since $W \subseteq \mathscr{C}h^{-1}(a)$ it follows that $A \in \mathscr{C}H(h, \langle a \rangle)$. Suppose now $B \in \mathscr{C}H(h, \langle a \rangle)$. Then there exist $\{(h_i, \langle a_i \rangle)\}_{i=1}^{M}$ in B such that

$$(5) \qquad h^{-1}(a) \cap \bigcap_{i=1}^{M} h_i^{-1}(a_i) \cap \operatorname{Ran} \alpha = \varnothing \ .$$

If $B \notin H(k, \langle p \rangle)$ then there exist $\{(k_i, \langle b_i \rangle)\}_{i=1}^{K}$ in B such that

$$(6) \qquad k^{-1}(p) \cap \bigcap_{i=1}^{K} k_i^{-1}(b_i) \cap \operatorname{Ran} \alpha = \varnothing \ .$$

Since $\mathscr{C}h^{-1}(a) \subseteq k^{-1}(p)$, (5) and (6) together imply that $\bigcap_{i=1}^{M} h_i^{-1}(a_i) \cap \bigcap_{i=1}^{K} k_i^{-1}(b_i) \cap \operatorname{Ran} \alpha = \varnothing$.

This contradicts that B is a bond and hence $B \in H(k, \langle p \rangle)$, proving that $\mathscr{C}H(h, \langle a \rangle) \subseteq H(k, \langle p \rangle)$.

To show that $H(k, \langle p \rangle) \subseteq \mathscr{C}H(f, \langle z \rangle)$, suppose $B \in H(k, \langle p \rangle)$ and

$B \in H(f, \langle z \rangle)$. Then $(f, \langle z \rangle)$ and $(k, \langle p \rangle)$ both belong to B but $k^{-1}(p) \cap f^{-1}(z) = \varnothing$, which is a contradiction. This proves that $H(k, \langle p \rangle) \subseteq \mathscr{C}H(f, \langle z \rangle)$, and the lemma is proved.

The following lemma is easy to prove.

LEMMA 3.9. *For* $y, z \in Y$, $A_{\langle y \rangle} = A_{\langle z \rangle}$ *if and only if* $\alpha(y) = \alpha(z)$.

Let $\mathscr{U}$ denote $\mathscr{U}(S)$ and $\mathscr{R}$ denote the set $\{A_{\langle y \rangle} : y \in y\}$. By Lemma 3.8 and Theorem 2.6 $\mathscr{U}$ is a Hausdorff compactification of $\mathscr{R}$. Let e be the mapping of Y into $\mathscr{R}$ defined by $e(y) = A_{\langle y \rangle}$.

LEMMA 3.10. *e is continuous and closed.*

Proof. Since sets of the form $H(f, \langle z \rangle) \cap \mathscr{R}$ constitute a sub-base of closed subsets of $\mathscr{R}$ and $e^{-1}[H(f, \langle z \rangle) \cap \mathscr{R}] = (f \circ \alpha)^{-1}(z)$, it follows that e is continuous. Suppose now that K is a closed subset of Y and $A_{\langle y \rangle} \notin e[K]$. Then $\alpha(y) \notin \alpha[K]$ and $\alpha[K]$ is closed. By Lemma 3.7, there exist $f \in S$ and $a, b \in Y$, $a \neq b$ such that $f(\alpha(y)) = a$ and $f(x) = b$ for every $x \in \alpha[K]$.

Then $\mathscr{C}H(f, \langle b \rangle)$ is a neighborhood of $A_{\langle y \rangle}$ which does not intersect $e[K]$ showing that $e[K]$ is closed.

Lemma 3.10 shows that $\mathscr{R}$ has the quotient topology induced by the map e on Y. Since α is by hypothesis a continuous closed map of Y it follows that $\alpha[Y]$ also has the quotient topology. Now there is a natural map g of $\alpha[Y]$ onto $\mathscr{R}$ defined by $g(\alpha(y)) = A_{\langle y \rangle}$.

Lemma 3.9 shows that this is well defined and injective and the preceding discussion implies that g is a homeomorphism. Thus we have proved

COROLLARY 3.11. $\alpha[Y]$ *is homeomorphic to* $\mathscr{R}$.

Proof of the main theorem. We begin by proving that the structure space $\mathscr{U}$ is E-compact. We have already observed that $\mathscr{U}$ is a compact Hausdorff space. Thus, if an embedding into a cartesian product of copies of E exists, $\mathscr{U}$ must necessarily be embedded as a closed subset. Consequently, we need only prove the existence of an embedding. According to Theorem 2.1 of [12, p. 165], it will be sufficient to show that for each closed subset W of $\mathscr{U}$ and each $A \in \mathscr{U}$ with $A \notin W$, there exists a continuous function f from $\mathscr{U}$ into E and a point q in E such that $f(B) = q$ for $B \in W$ and $f(A) \neq q$. The proof which follows is a modification of the corresponding proof in [6].

Now W is closed and $A \notin W$. Therefore there exists a finite

subfamily $\{(g_i, \langle y_i \rangle)\}_{i=1}^{N}$ of $S \times K$ such that

$$(7) \qquad\qquad A \notin W^*, \quad W \subseteq W^*$$

where $W^* = \bigcup_{i=1}^{N} H(g_i \langle y_i \rangle)$. Since $A \notin W^*$, there exist by (3), finite subfamilies $\{(h_{ij}, \langle v_{ij} \rangle)\}_{i=1}^{N_i}$ of A such that $g_i^{-1}(y_i) \cap V_i = \varnothing$ where

$$V_i = \bigcap_{i=1}^{N_i} h_{ij}^{-1}(v_{ij}) \cap \operatorname{Ran} \alpha \text{ for } 1 \le i \le N .$$

Now let $V^* = \bigcup_{i=1}^{N} V_i$ and let $H = \bigcup_{i=1}^{N} g_i^{-1}(y_i)$. Then V^* and H are two disjoint closed subsets of X and hence there exist a continuous function s mapping X into E^* and two distinct points p and q of E^* such that

$$(8) \qquad \begin{array}{l} s(x) = p \text{ for } x \in V^* \text{ and } s(x) = q \text{ for } x \in H, \text{ where } E^* \text{ is } h(E), \\ h \text{ being a homeomorphism of } E \text{ into } Y \text{ which results from} \\ \text{the hypothesis.} \end{array}$$

Now the mapping $\lambda_s : \mathscr{R} \to \mathscr{R}$ defined by $\lambda_s(A_{\langle y \rangle}) = A_{\langle s(\alpha(y)) \rangle}$ is continuous by Theorem 2.4 and has a unique extension to a continuous selfmap $\hat{\lambda}_s$ of $\mathscr{U}$ by Lemma 3.8 and Theorem 2.6. Now we assert that

$$(9) \qquad\qquad \hat{\lambda}_s(\mathscr{U}) \subseteq e[E^*]$$

and

$$(10) \qquad \hat{\lambda}_s(A) = e(p) \text{ and } \hat{\lambda}_s(B) = e(q) \text{ for } B \in W^* .$$

Now $s(X) \subseteq E^*$. Therefore $\hat{\lambda}_s(A_{\langle y \rangle}) = A_{\langle s(\alpha(y)) \rangle} \in e[E^*]$ for every $y \in Y$. Since $\mathscr{R}$ is dense in $\mathscr{U}$ and E^* is compact, we have

$$\hat{\lambda}_s(\mathscr{U}) = \hat{\lambda}_s(\operatorname{cl} \mathscr{R}) \subseteq \operatorname{cl}(\hat{\lambda}_s(\mathscr{R})) \subseteq \operatorname{cl} e[E^*] = e[E^*] ,$$

where cl denotes the closure. This verifies (9). Now we shall show that

$$(11) \qquad\qquad A \in \operatorname{cl}\{A_{\langle y \rangle} : \alpha(y) \in V^*\} .$$

Let $\mathscr{C}[\bigcup_{i=1}^{M} H(k_i, \langle r_i \rangle)]$ be a basic neighborhood of A. Then by (3), there exist finite subfamilies $\{(t_{ij}, \langle a_{ij} \rangle)\}_{j=1}^{M_i}$ of A such that $k_i^{-1}(r_i) \cap U_i = \varnothing$ where $U_i = \bigcap_{j=1}^{M_i} t_{ij}^{-1}(a_{ij}) \cap \operatorname{Ran} \alpha$.

Now there exists y in Y such that $\alpha(y) \in V^* \cap U_1 \cap \cdots \cap U_M$. But, then $\alpha(y) \notin k_i^{-1}(r_i)$, $i = 1, 2, \cdots, M$ so that $(k_i, \langle r_i \rangle) \notin A_{\langle y \rangle}$. Therefore, $A_{\langle y \rangle} \in \mathscr{C}[\bigcup_{i=1}^{M} H(k_i, \langle r_i \rangle)]$, and this proves (11). Now for $\alpha(y) \in V^*$, $s(\alpha(y)) = p$ and we have $\hat{\lambda}_s(A_{\langle y \rangle}) = A_{\langle s(\alpha(y)) \rangle} = e(p)$.

This, together with (11) implies that $\hat{\lambda}_s(A) = e(p)$, which is the first half of (10). In the same way as we verified (11), we can show that if $B \in W^*$ then $B \in \operatorname{cl}\{A_{\langle y \rangle} : \alpha(y) \in H\}$ and since $s(x) = q$

for $x \in H$, it follows that $\widehat{\lambda}_s(A_{\langle y \rangle}) = e(q)$ for every y such that $\alpha(y) \in$ H. Therefore $\widehat{\lambda}_s (B) = e(q)$ for each $B \in W^*$ and this completes the proof of the statement (10). Now, e is injective on E^* and $\widehat{\lambda}_s(\mathscr{U}) \subseteqq$ $e[E^*]$. Therefore, $(e \circ h)^{-1} \circ \widehat{\lambda}_s = f$ is a continuous function from $\mathscr{U}$ to E and we have a point $a(=h^{-1}(q))$ in E such that $f(B) = a$ for every $B \in W$ and $f(A) \neq a$, which was to be shown in order to conclude that $\mathscr{U}$ is E-compact.

Now we are in a position to show that $\mathscr{U}$ is $\beta_E(\alpha(y))$. Actually, we show that $\mathscr{U}$ is $\beta_E(\mathscr{R})$ but since the canonical map g (described after Lemma 3.10) maps $\alpha(E)$ homeomorphically onto $\mathscr{R}$ we identify the two spaces. In order to conclude that $\mathscr{U}$ is $\beta_E(\mathscr{R})$ it is sufficient, according to Theorem 4.14 of [12, p. 177] to show that $\mathscr{U}$ is E-compact and that every continuous function from $\mathscr{R}$ to E can be continuously extended to a function which maps $\mathscr{U}$ into E. We have yet to verify the latter and this is what we shall do now. Let f be a continuous function from $\mathscr{R}$ to E. Then $f \circ g$ is a continuous function on $\alpha(Y)$ to E. Let $(f \circ g)^*$ denote an extension of $f \circ g$ to a continuous function on X to E. Now $h \circ (f \circ g) = k$ belongs to S and

$$\lambda_k(A_{\langle y \rangle}) = A_{\langle k(\alpha(y)) \rangle} = A_{\langle h \circ f \circ g(\alpha(y)) \rangle} = A_{\langle h \circ f(A_{\langle y \rangle}) \rangle} = e \circ h \circ f(A_{\langle y \rangle}) \ .$$

Therefore, $\lambda_k = e \circ h \circ f$. Now λ_k has a unique continuous extension $\widehat{\lambda}_k$ to a selfmap of $\mathscr{U}$. We then have $\widehat{\lambda}_k(\mathscr{U}) = \widehat{\lambda}_k(\mathrm{cl}\,\mathscr{R}) \subseteqq$ $\mathrm{cl}\,(\widehat{\lambda}_k(\mathscr{R})) = \mathrm{cl}\,(e \circ h \circ f(\mathscr{R})) \subseteqq \mathrm{cl}\,(e \circ h(E)) = e \circ h(E)$.

The last equality is obtained because E is compact and $e \circ h$ is a homeomorphism on E. The function $(e \circ h)^{-1} \circ \widehat{\lambda}_k$ is the desired extension of f because $(e \circ h)^{-1} \circ \widehat{\lambda}_k(A_{\langle y \rangle}) = (e \circ h)^{-1} \circ \lambda_k(A_{\langle y \rangle}) = (e \circ h)^{-1} \circ e \circ h \circ f(A_{\langle y \rangle}) = f(A_{\langle y \rangle})$.

This completes the proof of the main theorem. If, in the main theorem, we take E to be the closed unit interval, we immediately get

COROLLARY 3.12. *Suppose X is normal and Hausdorff, α a continuous closed map of Y into X and Y contains an arc on which α is injective. Then the $\mathscr{K}$-structure space of $S(X, Y, \alpha)$ is the Stone-Čech compactification of $\alpha(Y)$.*

COROLLARY 3.13. *Let X be a normal 0-dimensional Hausdorff space and let α be a continuous closed map of Y into X. Then the $\mathscr{K}$-structure space of $S(X, Y, \alpha)$ is the Stone-Čech compactification of $\alpha(Y)$.*

Proof. Here again we apply the main theorem and in this case we take E to be the two point discrete space $\mathscr{D}$. First we assume

that α is not constant, then Y contains a copy of $\mathscr{D}$ on which α is injective. One can show that X is $\mathscr{D}$-separated exactly as it is done in the proof of Corollary 2 of [6]. Now we show that every continuous function f on $\alpha(Y)$ to $\mathscr{D}$ can be continuously extended to X. Let p and q be the distinct points of $\mathscr{D}$. Then $f^{-1}(p)$ and $f^{-1}(q)$ are disjoint closed subsets of $\alpha(Y)$ and consequently that of X as $\alpha(Y)$ is closed in X. Since X is $\mathscr{D}$-separated there exists a continuous function $\hat{f}$ on X to $\mathscr{D}$ such that $\hat{f}(x) = p$ for $x \in f^{-1}(p)$ and $\hat{f}(x) = q$ for $x \in f^{-1}(q)$. This $\hat{f}$ is the desired extension. Now our main theorem implies that the $\mathscr{K}$-structure space of $S(X, Y, \alpha)$ is $\beta_{\mathscr{D}}(\alpha(y))$, the largest $\mathscr{D}$-compactification of $\alpha(y)$. Now one proves exactly as in the proof of Corollary 2 of [6] that $\beta(\alpha(y)$ is a $\mathscr{D}$-compactification of $\alpha(y)$ which in turn implies that $\beta_{\mathscr{D}}(\alpha(Y)) = \beta(\alpha(Y))$ and the $\mathscr{K}$-structure space of $S(X, Y, \alpha)$ is $\beta(\alpha(Y))$.

Suppose now that α is a constant map of Y into X, say $\alpha(y) = x_0$ for every $y \in Y$. Let A be an ultrabond for the pair (S, K) where K is the kernel of $S(X, Y, \alpha)$. Suppose $(f, \langle y \rangle) \in A$. Then by definition there exists $y' \in Y$ such that $f \cdot \langle y' \rangle = \langle y \rangle$. But $f \cdot \langle y' \rangle = \langle f(\alpha(y')) \rangle = \langle f(x_0) \rangle$. Thus $A \subseteq \{(f, \langle f(x_0) \rangle) : f \in S(X, Y, \alpha)\}$ which itself is an ultrabond. Therefore $A = \{(f, \langle f(x_0) \rangle) : f \in S\}$ and $\mathscr{U}(S)$ consists of just one point and consequently is the Stone-Čech compactification of $\alpha(Y)$.

4. **Examples, applications and concluding remarks.** If one takes $X = Y$ and α to be the identity map, then $S(X, Y, \alpha)$ is just $S(X)$ the semigroup, under composition, of all continuous selfmaps of X. It follows immediately from the results of the previous section that for a number of spaces X, the $\mathscr{K}$-structure space of $S(X)$ is βX, the Stone-Čech compactification of X. This generalizes results in both [5] and [6].

For further examples, take X to be any normal space, choose any closed subspace Y which contains an arc and denote by $S(X, Y)$ the subsemigroup of $S(X)$ consisting of all functions which map X into Y. Then $S(X, Y)$ is really a sandwich semigroup (the sandwich map is the identity map on Y) which satisfies the hypothesis of the main theorem and it follows that its $\mathscr{K}$-structure space is βY. For a specific example of this type, take X to be the space of real numbers R and take Y to be union of the unit interval with the subspace N of natural numbers. Then the $\mathscr{K}$-structure space of $S(R, Y)$ is the free union of βN with an arc.

For any closed continuous function α from R into R which is injective on some nondegenerate subinterval, there are precisely three possibilities for the $\mathscr{K}$-structure space of $S(R, R, \alpha)$. The space $\mathscr{U}$ is either an arc, βR or βR^+ where R^+ is the space of

nonnegative real numbers. One easily verifies that any continuous selfmap of any Euclidean n-space which carries unbounded sets into unbounded sets, is also a closed map. Thus all polynomial functions P from R into R are closed and any polynomial function Q from C into C is also closed where C denotes the field of complex numbers. Of course, Q must actually map C *onto* C since C is algebraically closed. There are only two possibilities for the $\mathscr{K}$-structure space $\mathscr{U}$ of $S(R, R, P)$. The space $\mathscr{U}$ is βR if and only if P is of odd degree and it is βR^+ if and only if P is of even degree. The $\mathscr{K}$-structure space of $S(C, C, Q)$ is always βR^2 regardless of the degree of Q. All these assertions follow immediately from the theorem.

Now let S and T be any two semigroups with kernels. If S and T are isomorphic then certainly the two $\mathscr{K}$-structure spaces $\mathscr{U}(S)$ and $\mathscr{U}(T)$ are homeomorphic. The converse, of course, is far from being true. For example if G is any group, then $\mathscr{U}(G)$ has the cofinite topology [8, p. 266]. The cardinality of $\mathscr{U}(G)$ agrees with that of G so if G and H are any two groups of the same order (finite or infinite) then $\mathscr{U}(G)$ and $\mathscr{U}(H)$ will be homeomorphic. For another type of example, $S(R)$ and $S(\beta R)$ are not isomorphic [8, p. 256] but $\mathscr{U}(S(R)) = \mathscr{U}(S(\beta R)) = \beta R$, the Stone-Čech compactification of the reals.

Nevertheless, with certain restrictions on X and Y, the existence of a homeomorphism between $\mathscr{U}(S(X))$ and $\mathscr{U}(S(Y))$ will imply the existence of an isomorphism between $S(X)$ and $S(Y)$. This is included in the following

THEOREM 4.1. *Suppose that both X and Y are first countable normal spaces and suppose that each of them either contains an arc or is 0-dimensional. Then the following statements are equivalent*

  4.1.1 *The two $\mathscr{K}$-structure spaces $\mathscr{U}(S(X))$ and $\mathscr{U}(S(Y))$ are homeomorphic.*

  4.1.2 *The two semigroups $S(X)$ and $S(Y)$ are isomorphic.*

  4.1.3 *The spaces X and Y are homeomorphic.*

Proof. Suppose 4.1.1 holds. Then by the results of the previous section there is a homeomorphism h from βX onto βY. No point in $\beta X - X$ is a G_δ in βX while each point in X is a G_δ in βX, in fact it has a countable base in βX [1, p. 132]. Of course, the same thing is true of Y and βY so that the homeomorphism h must carry X onto Y and we see that 4.1.1 implies 4.1.3. It is immediate that 4.1.3 implies 4.1.2 and that 4.1.2 implies 4.1.1.

We conclude the paper with one more result of this type. Recall that a subspace Y is tamely embedded in a space X if every

homeomorphism from Y to some copy of Y in X can be extended to a homeomorphism mapping X onto X.

THEOREM 4.2. *Let X be normal space and let Y and Z be two first countable closed subspaces which are tamely embedded in X. Suppose further that X is either O-dimensional or that both Y and Z contain arcs. Let $S(X, Y)$ and $S(X, Z)$ denote the semigroups, under composition of all continuous functions mapping X into Y and X into Z respectively. Then the following statements are equivalent.*

4.2.1 *The two $\mathscr{K}$-structure spaces $\mathscr{U}(S(X, Y))$ and $\mathscr{U}(S(X, Z))$ are homeomorphic.*

4.2.2 *The two semigroups $S(X, Y)$ and $S(X, Z)$ are isomorphic.*

4.2.3 *The two subspaces Y and Z are homeomorphic.*

Proof. Suppose 4.2.1 holds. By results in the previous section βY is homeomorphic to βZ and just as in the proof of the previous theorem, this implies that Y and Z are homeomorphic. Thus 4.2.1 implies 4.2.3. If Y and Z are homeomorphic, then there exists a homeomorphism h from X onto X carrying Y onto Z since the latter are tamely embedded in X. The mapping which takes f into $h \circ f \circ h^{-1}$ is an isomorphism from $S(X, Y)$ onto $S(X, Z)$ so that 4.2.3 implies 4.2.2. It is evident that 4.2.2 implies 4.2.1 and the proof is complete.

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