

BIORTHOGONAL POLYNOMIALS SUGGESTED BY THE JACOBI POLYNOMIALS

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In this paper we introduce and study a pair of biorthogonal polynomials that are suggested by the classical Jacobi polynomials. Let $\alpha > -1$, $\beta > -1$ and $J_n(\alpha, \beta, k; x)$ and $K_n(\alpha, \beta, k; x)$, $n=0, 1, 2, \dots$ be respectively the polynomials of degree n in x^k and x , where x is real, k is a positive integer such that these two polynomial sets satisfy biorthogonality conditions with respect to the weight function $(1-x)^\alpha(1+x)^\beta$, namely

$$(1) \quad \int_{-1}^1 (1-x)^\alpha(1+x)^\beta J_n(\alpha, \beta, k; x) x^i dx \text{ is } \begin{cases} 0 & \text{for } i=0, 1, \dots, n-1; \\ \text{not } 0 & \text{for } i=n; \end{cases}$$

and

$$(2) \quad \int_{-1}^1 (1-x)^\alpha(1+x)^\beta K_n(\alpha, \beta, k; x) (1-x)^{k_i} dx \text{ is } \begin{cases} 0 & \text{for } i=0, 1, \dots, n-1; \\ \text{not } 0 & \text{for } i=n. \end{cases}$$

It follows from (1) and (2) that

$$(3) \quad \int_{-1}^1 (1-x)^\alpha(1+x)^\beta J_n(\alpha, \beta, k; x) K_m(\alpha, \beta, k; x) dx \text{ is } \begin{cases} 0 & \text{for } m, n=0, 1, \dots; m \neq n; \\ \text{not } 0 & \text{for } m=n; \end{cases}$$

and conversely.

For $k=1$ both these sets are reduced to the Jacobi polynomial sets. We obtain generating functions, recurrence relations for both these sets and explicitly show that they satisfy biorthogonality conditions.

1. Introduction. The notion of orthogonal polynomials has been extended by Konhauser [7] to two sets of polynomials that satisfy the following biorthogonality relation which happens to be an extension of the usual orthogonality condition:

$$(4) \quad \int_a^b p(x) R_m(x) S_n(x) dx \text{ is } \begin{cases} 0 & \text{for } m, n = 0, 1, \dots; m \neq n; \\ \text{not } 0 & \text{for } m = n. \end{cases}$$

Here $p(x)$ is an admissible weight function over an interval (a, b) and $R_m(x)$ and $S_n(x)$ are polynomials of degree m and n respectively

in the basic polynomials $r(x)$ and $s(x)$ both of which are polynomials in x .

Above condition (4) is known to be equivalent to the following two conditions.

$$\int_a^b p(x)[r(x)]^i S_n(x) dx \text{ is } \begin{cases} 0 & \text{for } i = 0, 1, \dots, n-1; \\ \text{not } 0 & \text{for } i = n; \end{cases}$$

and

$$\int_a^b p(x)[s(x)]^i R_m(x) dx \text{ is } \begin{cases} 0 & \text{for } i = 0, 1, \dots, m-1; \\ \text{not } 0 & \text{for } i = m. \end{cases}$$

Using his basic results of the general theory of biorthogonal polynomials, Konhauser [8] introduced the following pair of biorthogonal polynomials $Z_n^\alpha(x; k)$ and $Y_n^\alpha(x; k)$ that are suggested by the Laguerre polynomials:

$$(5) \quad Z_n^\alpha(x; k) = \frac{\Gamma(kn + \alpha + 1)}{n!} \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{x^{kj}}{\Gamma(kj + \alpha + 1)};$$

see [8, p. 304]

$$(6) \quad Y_n^\alpha(x; k) = \frac{1}{n!} \sum_{r=0}^n \frac{x^r}{r!} \sum_{s=0}^r (-1)^s \binom{r}{s} \left(\frac{s + \alpha + 1}{k} \right)_n;$$

see [2, p. 427].

These two sets are biorthogonal with respect to the weight function $x^\alpha e^{-x}$, ($\alpha > -1$) over the interval $(0, \infty)$ and have been extensively studied.

There is a classical result due to Feldheim [5] which connects the classical Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$ with the Laguerre polynomials $L_n^\alpha(x)$ in the following manner.

$$(7) \quad \Gamma(1 + \alpha + \beta + n) P_n^{(\alpha, \beta)}(x) = \int_0^\infty t^{\alpha + \beta + n} e^{-t} L_n^\alpha\left(\frac{1-x}{2}t\right) dt.$$

This result has made it possible for us to introduce, the first set from the pair of biorthogonal polynomials $J_n(\alpha, \beta, k; x)$ and $K_n(\alpha, \beta, k; x)$ that are suggested by the Jacobi polynomials.

Let us define the first set $J_n(\alpha, \beta, k; x)$ by

$$(8) \quad \begin{aligned} & \Gamma(1 + \alpha + \beta + n) J_n(\alpha, \beta, k; x) \\ &= \int_0^\infty t^{\alpha + \beta + n} e^{-t} Z_n^\alpha\left(\frac{1-x}{2}t; k\right) dt, \quad \alpha + \beta > -1, \quad n = 0, 1, 2, \dots \end{aligned}$$

Using (5) one obtains by routine calculations

$$(9) \quad J_n(\alpha, \beta, k; x) = \frac{(1+\alpha)_{kn}}{n!} \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{(1+\alpha+\beta+n)_{kj}}{(1+\alpha)_{kj}} \left(\frac{1-x}{2}\right)^{kj}.$$

In fact $J_n(\alpha, \beta, k; x)$ has the following hypergeometric form

$$(10) \quad J_n(\alpha, \beta, k; x) = \frac{(1+\alpha)_{kn}}{n!} {}_{k+1}F_k \left[\begin{matrix} -n, \Delta(k, 1+\alpha+\beta+n); \\ \Delta(k, 1+\alpha); \end{matrix} \left(\frac{1-x}{2} \right)^k \right];$$

where $\Delta(m, \delta)$ stands for the sequence of m parameters

$$\frac{\delta}{m}, \frac{\delta+1}{m}, \dots, \frac{\delta+m-1}{m}, \quad m \geq 1.$$

The polynomials $\{J_n(\alpha, \beta, k; x)\}$ were first introduced by Chai [4] and Carlitz [3] published the proof of their biorthogonality to x^i (i.e., of type (1)) with respect to $x^\alpha(1-x)^\beta$ on $(0, 1)$. Chai's proposal was on $(0, 1)$ instead of our $(-1, 1)$. This also reminds one of the transition of the classical Jacobi polynomials first denoted by $F_n(\alpha, \beta; x)$ and orthogonal with respect to the weight function $x^\alpha(1-x)^\beta$ on $(0, 1)$ to that of Szegő's standardized Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$ which are orthogonal with respect to the weight function $(1-x)^\alpha(1+x)^\beta$ over the interval $(-1, 1)$.

We introduce the second set $K_n(\alpha, \beta, k; x)$ in the form of the following explicit series representation

$$(11) \quad K_n(\alpha, \beta, k; x) = \sum_{r=0}^n \sum_{s=0}^r (-1)^{r+s} \binom{r}{s} \frac{(1+\beta)_n}{n! r! (1+\beta)_{n-r}} \left(\frac{s+\alpha+1}{k} \right)_n \left(\frac{x-1}{2} \right)^r \left(\frac{x+1}{2} \right)^{n-r}.$$

For $k=1$, both $K_n(\alpha, \beta, k; x)$ and $J_n(\alpha, \beta, k; x)$ get reduced to the Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$.

It is easy to observe that

$$(12) \quad \begin{cases} \lim_{\beta \rightarrow \infty} J_n\left(\alpha, \beta, k; 1 - \frac{2x}{\beta}\right) = Z_n^\alpha(x; k) \\ \lim_{\beta \rightarrow \infty} K_n\left(\alpha, \beta, k; 1 - \frac{2x}{\beta}\right) = Y_n^\alpha(x; k). \end{cases}$$

For $k=1$, each of (12) gives well known connection relation between the Jacobi polynomials and the Laguerre polynomials; see [12].

2. Biorthogonality. Employing the explicit formulas (9) and (11) we shall show that the pair of polynomials $K_n(\alpha, \beta, k; x)$ and

$J_n(\alpha, \beta, k; x)$ satisfies the biorthogonality condition (3).

In fact, we have

$$\begin{aligned}
 I_{n,m} &= \int_{-1}^1 (1-x)^\alpha (1+x)^\beta J_n(\alpha, \beta, k; x) K_m(\alpha, \beta, k; x) dx \\
 &= \frac{\Gamma(1+\alpha+kn)\Gamma(1+\beta+m)}{2^m n! m! \Gamma(1+\alpha+\beta+n)} \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{\Gamma(1+\alpha+\beta+n+kj)}{2^{kj} \Gamma(1+\alpha+kj)} \\
 &\quad \times \sum_{r=0}^m \sum_{s=0}^r (-1)^s \binom{r}{s} \left(\frac{s+\alpha+1}{k} \right) \frac{1}{m r! \Gamma(1+\beta+m-r)} \\
 &\quad \times \int_{-1}^1 (1-x)^{\alpha+kj+r} (1+x)^{\beta+m-r} dx \\
 &= 2^{1+\alpha+\beta} \frac{\Gamma(1+\alpha+kn)\Gamma(1+\beta+m)}{n! m! \Gamma(1+\alpha+\beta+n)} \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{\Gamma(1+\alpha+\beta+n+kj)}{\Gamma(2+\alpha+\beta+m+kj)} \\
 &\quad \times \sum_{r=0}^m \binom{\alpha+kj+r}{r} \sum_{s=0}^r (-1)^s \binom{r}{s} \left(\frac{s+\alpha+1}{k} \right)_m.
 \end{aligned}$$

Recall the following result of Carlitz [2, p. 429]:

$$\left(\frac{x+\alpha+1}{k} \right)_n = \sum_{r=0}^n \binom{-x+r-1}{r} \sum_{s=0}^r (-1)^s \binom{r}{s} \left(\frac{s+\alpha+1}{k} \right)_n.$$

Using this, we have

$$\begin{aligned}
 I_{n,m} &= 2^{1+\alpha+\beta} \frac{\Gamma(1+\alpha+kn)\Gamma(1+\beta+m)}{n! m! \Gamma(1+\alpha+\beta+n)} \\
 &\quad \times \sum_{j=0}^n (-1)^j \binom{n}{j} (-j)_m \frac{(1+\alpha+\beta)_{n+kj}}{(1+\alpha+\beta)_{m+kj+1}} \\
 &= 2^{1+\alpha+\beta} \frac{\Gamma(1+\alpha+kn)\Gamma(1+\beta+m)}{n! \Gamma(1+\alpha+\beta+n)} (-1)^m \binom{n}{m} \\
 &\quad \times \sum_{j=m}^n (-1)^j \binom{n-m}{j-m} \frac{(1+\alpha+\beta)_{n+kj}}{(1+\alpha+\beta)_{m+kj+1}} \\
 &= 2^{1+\alpha+\beta} \frac{\Gamma(1+\alpha+kn)\Gamma(1+\beta+m)}{n! \Gamma(1+\alpha+\beta+n)} \binom{n}{m} \\
 &\quad \times \sum_{j=0}^{n-m} (-1)^j \binom{n-m}{j} \frac{(1+\alpha+\beta)_{n+km+kj}}{(1+\alpha+\beta)_{m+km+kj+1}} \\
 &= 2^{1+\alpha+\beta} \frac{\Gamma(1+\alpha+kn)\Gamma(1+\beta+m)}{n! \Gamma(1+\alpha+\beta+n)} \binom{n}{m} \\
 &\quad \times \sum_{j=0}^{n-m} (-1)^j \binom{n-m}{j} D^{n-m-1} x^{\alpha+\beta+n+km+kj} \Big|_{x=1} \\
 &= 2^{1+\alpha+\beta} \frac{\Gamma(1+\alpha+kn)\Gamma(1+\beta+m)}{n! \Gamma(1+\alpha+\beta+n)} \binom{n}{m} \\
 &\quad \times D^{n-m-1} x^{\alpha+\beta+n+km} (1-x^k)^{n-m} \Big|_{x=1}
 \end{aligned}$$

which is 0 for $n \neq m$ and nonzero for $n = m$. In particular,

$$I_{n,n} = 2^{1+\alpha+\beta} \frac{\Gamma(1+\alpha+kn)\Gamma(1+\beta+n)}{n! \Gamma(1+\alpha+\beta+n)(1+\alpha+\beta+n+kn)}.$$

3. Generating functions and recurrence relations. From the explicit representation (9) we can obtain by usual series techniques the following generating functions

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(1+\alpha+\beta)_n J_n(\alpha, \beta, k; x) t^n}{(1+\alpha)_{kn}} \\ &= (1-t)^{-1-\alpha-\beta} \\ (13) \quad & {}_{k+1}F_k \left[\begin{matrix} \Delta(k+1, 1+\alpha+\beta); \\ \Delta(k, 1+\alpha); \end{matrix} \left(\frac{(k+1)(1-x)}{2k} \right)^k \left(\frac{-(k+1)t}{(1-t)^{k+1}} \right) \right], \end{aligned}$$

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{J_n(\alpha, \beta-n, k; x) t^n}{(1+\alpha)_{kn}} \\ (14) \quad &= e^t {}_kF_k \left[\begin{matrix} \Delta(k, 1+\alpha+\beta); \\ \Delta(k, 1+\alpha); \end{matrix} -t \left(\frac{1-x}{2} \right)^k \right]. \end{aligned}$$

For $k=1$, (14) reduces to a generating function for Jacobi polynomials which is due to Feldheim [5].

Differentiating (9) with respect to x , we can obtain the following recurrence relation.

$$\begin{aligned} (15) \quad & DJ_n(\alpha, \beta, k; x) \\ &= k 2^{-k} (1-x)^{k-1} (1+\alpha+\beta+n)_k J_{n-1}(\alpha+k, \beta+1, k; x). \end{aligned}$$

By using the result (6) of Konhauser [8], the result on page 638 of Karande and Thakare [6] involving $Z_n^\alpha(x; k)$ we can obtain in view of the definition (8) the following recurrence relations

$$\begin{aligned} (16) \quad & (x-1)DJ_n(\alpha, \beta, k; x) \\ &= nk J_n(\alpha, \beta, k; x) - k(kn-k+\alpha+1)_k J_{n-1}(\alpha, \beta+1, k; x), \end{aligned}$$

$$\begin{aligned} (17) \quad & (x-1)DJ_n(\alpha, \beta, k; x) \\ &= (kn+\alpha)J_n(\alpha-1, \beta+1, k; x) - \alpha J_n(\alpha, \beta, k; x). \end{aligned}$$

Alternatively the results (15) and (16) can be obtained by direct computation by adopting the technique given by Konhauser [8] for evaluating the recurrence relations for $Z_n^\alpha(x; k)$.

Since $J_n(\alpha, \beta, k; x)$ are essentially ${}_{k+1}F_k$ -type generalized hypergeometric polynomials, we can obtain the following differential equation

$$[(x-1)D((x-1)D + \alpha - k + 1)_k - 2^{-k}(1-x)^k((x-1)D - nk) \\ \times ((x-1)D + \alpha + \beta + n + 1)_k]J_n(\alpha, \beta, k; x) = 0.$$

From (11), we have after routine computation

$$\sum_{n=0}^{\infty} K_n(\alpha, \beta - n, k; x)(2w)^n/(x+1)^n \\ = ((x+1)/2)^{-\beta}(1-w)^{-(1+\alpha+\beta)/k}[(x-1)/2 + (1-w)^{1/k}]^{\beta}$$

This readily yields the formula

$$(18) \quad K_n(\alpha, \beta - n, k; x) = \frac{1}{n!} \left(\frac{x+1}{2} \right)^{n-\beta} \left\{ \frac{\partial^n}{\partial w^n} (1-w)^{-(1+\alpha+\beta)/k} \right. \\ \left. \left[\frac{x-1}{2} + (1-w)^{1/k} \right]^{\beta} \right\} \Big|_{w=0}.$$

In (18) replace α by $\alpha + \nu n$, β by $\beta + n + \delta n$ and apply to the resulting expression the Lagrange's expansion formula [9, p. 146] to obtain the generating function in the form

$$\sum_{n=0}^{\infty} K_n(\alpha + \nu n, \beta + \delta n, k; x)u^n = k \left(\frac{x+1}{2} \right)^{-\beta} (1-w)^{-(1+\alpha+\beta)/k} \\ \left[\frac{x-1}{2} + (1-w)^{1/k} \right]^{\beta} \left[\frac{k(1-w) - \nu w}{1-w} - \frac{(1+\delta)w(x-1)/2}{(1-w) \left[\frac{x-1}{2} + (1-w)^{1/k} \right]} \right]^{-1}$$

where

$$u = w \left(\frac{x+1}{2} \right)^{\delta} (1-w)^{(1+\delta+\nu)/k} \left[\frac{x-1}{2} + (1-w)^{1/k} \right]^{-1-\delta}$$

With the substitution $w = 1 - (1+t)^{-k}$, the above generating relation can be put in the form

$$\sum_{n=0}^{\infty} K_n(\alpha + \nu n, \beta + \delta n, k; x)u^n = k(t+1)^{\alpha} \left(1 + \frac{x-1}{x+1}t \right)^{\beta} \\ \times \left\{ \frac{k - \nu[(t+1)^k - 1]}{t+1} - \frac{(1+\delta)(x-1)[(t+1)^k - 1]}{x+1 + (x-1)t} \right\}^{-1}$$

where $u = 2(x+1)^{\delta}[(t+1)^k - 1](t+1)^{-(\nu+k)}(x+1 + xt - t)^{-(1+\delta)}$. The special cases with $\nu = 0$, $\delta = -1$ and $\nu = -k$, $\delta = -1$ yield quite compact forms and they reduce for $k=1$ to the generating functions for Jacobi polynomials due to Feldheim [5]; see also Carlitz [1].

4. **Concluding remarks.** (i) Spencer and Fano [11] utilized biorthogonality of polynomials in x and polynomials in x^2 with respect to the weight function $x^\alpha e^{-x}$ (α , nonnegative integer) over the interval $(0, \infty)$, in carrying out calculations involving the penetration of gamma rays through matter. We are also optimistic that the particular cases of pair of biorthogonal polynomials considered here would, certainly, be of use in physical problems.

(ii) It is needless to say that these polynomials will yield for $\alpha = \beta = 0$, a pair of biorthogonal polynomials suggested by Legendre polynomials, for $\alpha = \beta = \pm 1/2$ a pair of biorthogonal polynomials related to Chebychev polynomials of the second kind and first kind respectively and when $\alpha = \beta$ we would have a pair of biorthogonal polynomials suggested by ultraspherical polynomials.

(iii) Prabhakar and Kashyap [10] have discussed the polynomials (which are just constant multiples of $J_n(\alpha, 0, k; x)$),

$$U_n^\alpha(x; k) = \frac{1}{(1/k)_n} \sum_{j=0}^n \frac{(-n)_j}{j!} \left(\frac{1 + \alpha + j}{k} \right)_n \left(\frac{1 - x}{2} \right)^j,$$

$$V_n^\alpha(x; k) = \frac{1}{n!} \sum_{j=0}^n \frac{(-n)_j}{j!} (1 + \alpha + kj)_n \left(\frac{1 - x}{2} \right)^{kj}$$

and showed that they form a pair of biorthogonal polynomials over $(-1, 1)$ with respect to the weight function $((1 - x)/2)^\alpha$. It is to be noted that these two sets are related to each other by the following

$$V_n^\alpha(x; k) = \frac{(k)_n}{n!} U_n^{(\alpha+1-k)/k} \left(1 - 2 \left(\frac{1-x}{2} \right)^k; 1/k \right);$$

and as such are not independent of each other.

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