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ON THE EVALUATION OF PERMANENTS

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Two identities involving permanents are obtained. One of them is used to deduce in a simple way, two well known formulas for the evaluation of permanents, namely the formulas of Binet and Minc and of Ryser.

1. Notation. Let $A = [a_{ij}]$ be an $n \times n$ matrix. The permanent of A is defined by

$$\operatorname{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)} ,$$

where S_n is the symmetric group of degree n.

Let $\Gamma_{r,n}$ denote the set of all n^r sequences $\omega = (\omega_1, \cdots, \omega_r)$ of integers satisfying $1 \leqq \omega_i \leqq n$. Let $Q_{r,n}$ be the set of those sequences in $\Gamma_{r,n}$ which are strictly increasing.

By $G(n)$ we denote the set of all nondecreasing sequences of positive integers $(t_1, \cdots, t_k)$ such that $t_1 + \cdots + t_k = n$.

Given an $n \times n$ matrix A and nonnegative integers $\alpha_1, \cdots, \alpha_n$ $(\beta_1, \cdots, \beta_n)$ satisfying $\alpha_1 + \cdots + \alpha_n = n$ $(\beta_1 + \cdots + \beta_n = n)$, we represent by $A(\alpha_1, \cdots, \alpha_n)$ $(A(\beta_1, \cdots, \beta_n))$ the matrix obtained from A by repeating its first row (column) $\alpha_1(\beta_1)$ times, its second row (column) $\alpha_2(\beta_2)$ times $\cdots$ and its nth row (column) $\alpha_n(\beta_n)$ times. If $\alpha_i = 0(\beta_i = 0)$ the ith row (column) of A is omitted.

Given an $n \times n$ matrix A and nonnegative integers $\alpha_1, \cdots, \alpha_n$, $\beta_1, \cdots, \beta_n$ satisfying $\alpha_1 + \cdots + \alpha_n = n$, $\beta_1 + \cdots + \beta_n = n$, let $A(\alpha_1, \cdots, \alpha_n; \beta_1, \cdots, \beta_n)$ denote the $n \times n$ matrix obtained from A by repeating its first row α_1 times, $\cdots$, its nth row α_n times and also its first column β_1 times $\cdots$ its nth column β_n times. Again $\alpha_i = 0$ or $\beta_i = 0$ means that the ith row or the ith column of A is omitted.

Given the integers α_i, $1 \leqq i \leqq n$, such that $\sum_{i=1}^{n} \alpha_i = n$, let $R_{\alpha_1, \cdots \alpha_n} = (j_1, \cdots, j_n)$ represent the nondecreasing sequence of nonnegative integers $(j_1, \cdots, j_n)$ where i occurs with multiplicity α_i.

2. Two identities involving permanents. Let A be an $n \times n$ matrix and

$$Z = \prod_{i=1}^{n} Z_i$$

with

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$$Z_i = \sum_{j=1}^{n} a_{ij} x_j \,, \qquad i = 1, \cdots, n \,.$$

Since Z is an homogeneous polynomial of degree n in $x_1, \cdots, x_n$ we can write

$$Z = \sum_{\beta_1,\cdots,\beta_n} x_{\beta_1,\cdots,\beta_n} \frac{x_1^{\beta_1}}{\beta_1!} \frac{x_2^{\beta_2}}{\beta_2!} \cdots \frac{x_n^{\beta_n}}{\beta_n!} \,, \tag{2.1}$$

where the summation is over all sequences $(\beta_1, \cdots, \beta_n)$ of nonnegative integers satisfying $\beta_1 + \cdots + \beta_n = n$. The coefficient $x_{\beta_1,\cdots,\beta_n}$ is equal to

$$\frac{\partial^n Z}{\partial x_{j_1} \partial x_{j_2} \cdots \partial x_{j_n}} \,, \qquad (j_1, \cdots, j_n) = R_{\beta_1,\cdots,\beta_n}$$

as can be seen from the following

$$\begin{aligned}
\frac{\partial^n Z}{\partial x_{j_1} \partial x_{j_2} \cdots \partial x_{j_n}} &= \frac{\partial^{\beta_1+\cdots+\beta_n} Z}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}} \\
&= x_{\beta_1,\cdots,\beta_n} \left(\frac{1}{\beta_1!} \frac{\partial^{\beta_1} x_1^{\beta_1}}{\partial x_1^{\beta_1}} \right) \cdots \left(\frac{1}{\beta_n!} \frac{\partial^{\beta_n} x_n^{\beta_n}}{\partial x_n^{\beta_n}} \right) \\
&= x_{\beta_1,\cdots,\beta_n} \frac{\beta_1!}{\beta_1!} \cdots \frac{\beta_n!}{\beta_n!} \\
&= x_{\beta_1,\cdots,\beta_n} \,.
\end{aligned}$$

We now compute $\partial^n Z / \partial x_{j_1} \cdots \partial x_{j_n}$, $(j_1, \cdots, j_n) = R_{\beta_1,\cdots,\beta_n}$. We begin by assuming $(j_1, \cdots, j_n) = R_{1,1,\cdots,1}$. Since Z_i is a linear function of the x_j's we have

$$\frac{\partial^2 Z_i}{\partial x_k \partial x_h} = 0 \tag{2.2}$$

for any k and h and by differentiation

$$\begin{aligned}
\frac{\partial^n Z}{\partial x_{j_1} \partial x_{j_2} \cdots \partial x_{j_n}} &= \frac{\partial^n Z}{\partial x_1 \partial x_2 \cdots \partial x_n} \\
&= \sum_{\sigma} \frac{\partial Z_{\sigma(1)}}{\partial x_1} \frac{\partial Z_{\sigma(2)}}{\partial x_2} \cdots \frac{\partial Z_{\sigma(n)}}{\partial x_n} \,,
\end{aligned}$$

where $\sigma \in S_n$. But

$$\frac{\partial Z_{\sigma(i)}}{\partial x_i} = a_{\sigma(i)i}$$

and therefore

$$\frac{\partial^n Z}{\partial x_1 \cdots \partial x_n} = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{\sigma(i)i} = \operatorname{per}(A) \,. \tag{2.3}$$

Let now $(j_1, \cdots, j_n) = R_{\beta_1,\dots,\beta_n}$. Again by differentiation and recalling (2.2) we have

$$
\begin{aligned}
\frac{\partial^n Z}{\partial x_{j_1}\partial x_{j_2}\cdots\partial x_{j_n}} &= \sum_{\sigma\in S_n}\prod_{i=1}^{n}\frac{\partial Z_{\sigma(i)}}{\partial x_{j_i}} \\
&= \sum_{\sigma\in S_n}\prod_{i=1}^{n} a_{\sigma(i)j_i} \qquad (2.4)\\
&= \operatorname{per}(A(\beta_1, \cdots, \beta_n)) .
\end{aligned}
$$

From (2.1), (2.3) and (2.4) we obtain the following identity

$$
\prod_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}x_j\right) = \sum_{\beta_1,\dots,\beta_n} \operatorname{per}(A(\beta_1, \cdots, \beta_n))\frac{x_1^{\beta_1}}{\beta_1!}\cdots\frac{x_n^{\beta_n}}{\beta_n!}, \tag{2.5}
$$

where the summation is over all the sequences $(\beta_1, \cdots, \beta_n)$ of non-negative integers satisfying $\sum_{i=1}^{n}\beta_i = n$.

Now we shall deduce from (2.5), the following identity expressing the nth power of a bilinear form as a polynomial with permanents as coefficients,

$$
\begin{aligned}
\frac{1}{n!}\left(\sum_{i,j=1}^{n} x_i a_{ij} y_j\right)^n = \sum_{\alpha,\beta} \operatorname{per}(A(\alpha_1, \cdots, \alpha_n; \beta_1, \cdots, \beta_n))\frac{x_1^{\alpha_1}}{\alpha_1!} \qquad (2.6)\\
\times\cdots\frac{x_n^{\alpha_n}}{\alpha_n!}\,\frac{y_1^{\beta_1}}{\beta_1!}\cdots\frac{y_n^{\beta_n}}{\beta_n!},
\end{aligned}
$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\beta = (\beta_1, \cdots, \beta_n)$, with $\alpha_i, \beta_i \geqq 0$ $i = 1, \cdots, n$ and $\sum_{i=1}^{n}\alpha_i = \sum_{i=1}^{n}\beta_i = n$.

Indeed, by the multinomial expansion we have

$$
\frac{1}{n!}\left(\sum_{i,j=1}^{n} x_i a_{ij} y_j\right)^n = \sum_{\alpha}\prod_{i=1}^{n}\left[\frac{x_i^{\alpha_i}}{\alpha_i!}\left(\sum_{j=1}^{n} a_{ij}y_j\right)^{\alpha_i}\right], \tag{2.7}
$$

where $\alpha_i \geqq 0$, $i = 1, \cdots, n$ and $\sum_{i=1}^{n}\alpha_i = n$. Let $(h_1, \cdots, h_n) = R_{\alpha_1,\dots,\alpha_n}$. We have

$$
\begin{aligned}
\prod_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}y_j\right)^{\alpha_i} &= \underbrace{\left(\sum_{j=1}^{n} a_{1j}y_j\right)\cdots\left(\sum_{j=1}^{n} a_{1j}y_j\right)}_{\alpha_1 \text{ times}}\cdots\underbrace{\left(\sum_{j=1}^{n} a_{nj}y_j\right)\cdots\left(\sum_{j=1}^{n} a_{nj}y_j\right)}_{\alpha_n \text{ times}} \\
&= \left(\sum_{j=1}^{n} a_{h_1 j}y_j\right)\cdots\left(\sum_{j=1}^{n} a_{h_n j}y_j\right) \qquad (2.8)\\
&= \prod_{i=1}^{n}\left(\sum_{j=1}^{n} a_{h_i j}y_j\right).
\end{aligned}
$$

Substituting (2.8) into (2.7), applying the identity (2.5) to $\prod_{i=1}^{n}(\sum_{j=1}^{n} a_{h_i j}y_j)$ and noting that the matrix $A = (a_{ij})$ has been replaced by the matrix $A(\alpha_1, \cdots, \alpha_n) = (a_{h_i j})$ we obtain

$$\frac{1}{n!}\left(\sum_{i,j=1}^{n} x_i a_{ij} y_j\right)^n = \sum_{\alpha} \prod_{i=1}^{n} \frac{x_i^{\alpha_i}}{\alpha_i!} \prod_{i=1}^{n}\left(\sum_{j=1}^{n} a_{h_i j} y_j\right)$$

$$= \sum_{\alpha} \prod_{i=1}^{n} \frac{x_i^{\alpha_i}}{\alpha_i!} \sum_{\beta} \operatorname{per}\,(A(\alpha_1, \cdots, \alpha_n; \beta_1, \cdots, \beta_n))\frac{y_1^{\beta_1}}{\beta_1!} \cdots \frac{y_n^{\beta_n}}{\beta_n!}\,.$$

This proves (2.6).

3. About the Binet-Minc formula. As we have seen per $A = \partial^n Z/\partial x_1 \cdots \partial x_n$ which is equivalent to per $A = \operatorname{coef}_{x_1 \cdots x_n} \prod_{i=1}^n \times (\sum_{j=1}^n a_{ij} x_j)$. This result for the case $n = 3$, is due to Muir. Let us evaluate $\partial^n Z/\partial x_1 \cdots \partial x_n$. Let again $Z = \prod_{i=1}^n Z_i$ and $Z_i = \sum_{j=1}^n a_{ij} x_j$, $i = 1, \cdots, n$. Let $T_1, \cdots, T_k$ be a partition of $\{1, \cdots, n\}$ such that

$$T_j = \{w_1^{(j)}, \cdots, w_s^{(j)}\} \in Q_{s,n}\,, \tag{3.1}$$

where $s = t_j$, $j = 1, \cdots, k$ and $t_1 \leqq \cdots \leqq t_k$.

We define

$$\phi^{(w_1^{(j)})} = \sum_{h=1}^{n} \frac{1}{Z_h} \frac{\partial Z_h}{\partial x_{w_1^{(j)}}} \tag{3.2}$$

and

$$\phi^{(T_j)} = \frac{\partial^{t_j - 1} \phi^{w_1^{(j)}}}{\partial x_{w_2^{(j)}} \cdots \partial x_{w_s^{(j)}}} \tag{3.3}$$

with $s = t_j$, $j = 1, \cdots, k$.

It is clear that

$$\phi^{(w_1^{(j)})} = \sum_{h=1}^{n} \frac{a_{hw_1^{(j)}}}{Z_h}\,. \tag{3.4}$$

On the other hand, it is easy to show (by induction) that

$$\phi^{(T_j)} = (-1)^{t_j - 1}(t_j - 1)! \sum_{h=1}^{n} \frac{a_{hw_1^{(j)}} \cdots a_{hw_s^{(j)}}}{Z_h^{t_j}} \tag{3.5}$$

$s = t_j$, $j = 1, \cdots, k$. In fact, suppose

$$\phi^{(T_j - w_s^{(j)})} = \frac{\partial^{s-2} \phi^{(w_1^{(j)})}}{\partial x_{w_2^{(j)}} \cdots \partial x_{w_{s-1}^{(j)}}} = (-1)^{s-2}(s-2)! \sum_{h=1}^{n} \frac{a_{hw_1^{(j)}} \cdots a_{hw_{s-1}^{(j)}}}{Z_h^{s-1}}\,, \tag{3.6}$$

where $s = t_j$, and $T_j - w_s^{(j)}$ denotes the set $\{w_1^{(j)}, \cdots, w_{s-1}^{(j)}\}$. By differentiating (3.6) with respect to $x_{w_s^{(j)}}$ we obtain

$$\frac{\partial^{s-1} \phi^{(w_1^{(j)})}}{\partial x_{w_2^{(j)}} \cdots \partial x_{w_{s-1}^{(j)}} \partial x_{w_s^{(j)}}} = (-1)^{s-1}(s-1)! \sum_{h=1}^{n} \frac{a_{hw_1^{(j)}} \cdots a_{hw_{s-1}^{(j)}}}{Z_h^{s}} \frac{\partial Z_h}{x_{w_s^{(j)}}}$$

$$= (-1)^{s-1}(s-1)! \sum_{h=1}^{n} \frac{a_{hw_1^{(j)}} \cdots a_{hw_{s-1}^{(j)}} a_{hw_s^{(j)}}}{Z_h^{s}}$$

and (3.5) holds.

We finally prove (by induction) that

$$(3.7)\qquad \frac{\partial^n Z}{\partial x_1 \cdots \partial x_n} = Z \sum_{k=1}^{n} \sum_{T_1,\cdots,T_k} \phi^{(T_1)} \cdots \phi^{(T_k)} ,$$

where the summation $\sum_{T_1,\ldots,T_k}$ is over all the partitions $T_1, \cdots, T_k$ of $\{1, \cdots, n\}$.

For $n = 1$ it is trivial to verify (3.7). Suppose that

$$(3.8)\qquad \frac{\partial^{n-1} Z}{\partial x_1 \cdots \partial x_{n-1}} = Z \sum_{k=1}^{n-1} \sum_{T_1,\cdots,T_k} \phi^{(T_1)} \cdots \phi^{(T_k)} ,$$

where $T_1, \cdots, T_k$ is a partition of $\{1, \cdots, n-1\}$. Differentiating (3.8) with respect to x_n we get

$$(3.9)\qquad \frac{\partial^n Z}{\partial x_1 \cdots \partial x_{n-1} \partial x_n} = \sum_{k=1}^{n-1} \sum_{T_1,\cdots,T_k} (\phi^{(T_1,n)} \cdots \phi^{(T_k)} + \cdots + \phi^{(T_1)} \cdots \phi^{(T_k,n)} + \phi^{(T_1)} \cdots \phi^{(T_k)} \phi^{(n)}) Z$$

where $\phi^{(T_h,n)} = (\partial\phi^{(T_h)}/\partial x_n)$, $\phi^{(n)}$ is defined by (3.4) and T_h, n is a shorthand notation for $T_h \cup \{n\}$ $h = 1, \cdots, k$. We observe that in (3.9) the summation over $T_1, \cdots, T_k$ refers to the partition of $\{1, \cdots, n-1\}$. It is obvious that (3.9) is equal to

$$Z \sum_{k=1}^{n} \sum_{T_1,\cdots,T_k} \phi^{(T_1)} \cdots \phi^{(T_k)} ,$$

where $\sum_{T_1 \cdots T_k}$ is now over all the partitions $T_1, \cdots, T_k$ of $\{1, \cdots, n\}$, and (3.7) holds.

THEOREM 3.1. *Let $A = (A_{ij})$ be an $n \times n$ matrix over C. Then*

$$\operatorname{per}(A) = Z \sum_{k=1}^{n} \sum_{T_1,\cdots,T_k} \phi^{(T_1)} \cdots \phi^{(T_k)} ,$$

where $\phi^{(T_j)}$ is defined by (3.5).

REMARK. The theorem is also true for matrices defined over an arbitrary field if we use formal differentiation.

Now we cast (3.7) into another equivalent form. Then we shall see that $\operatorname{per}(A)$ is in fact independent of $x_1, \cdots, x_n$. We need the following additional notation. Let $r_{w_1 * \cdots * w_s}$ denote the sum of the entries in the Hadamard product of columns $w_1, \cdots, w_s$ of A, i.e.,

$$r_{w * \cdots * w_s} = \sum_{j=1}^{n} a_{jw_1} a_{jw_2} \cdots a_{jw_s} .$$

DEFINITION 3.1. For $(t_1, \cdots, t_k) \in G(n)$ we define the R function,

$R(t_1, \cdots, t_k)$, as the symmetrised sum of all distinct products of the $r_{w_1 * \cdots * w_s}$, $s = t_1, \cdots, t_k$, such that in each product the sequences $(w_1, \cdots, w_s) \in Q_{s,n}$, $s = t_1, \cdots, t_k$, partition the set $\{1, \cdots, n\}$.

DEFINITION 3.2. The permanent of an $m \times n$ matrix $A = (a_{ij})$ $m \leqq n$, written Per(A), is defined by

$$\operatorname{Per}(A) = \sum_{\sigma} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{m\sigma(m)} ,$$

where the summation extends over all one-to-one functions from $\{1, \cdots, m\}$ into $\{1, \cdots, n\}$.

The special case $m = n$ is of particular importance. In fact most writers restrict the designation "permanent" to the case of square matrices. However the permanent of a rectangular $m \times n$ matrix A, $m \leqq n$, can be obtained by evaluating the permanent of the $n \times n$ matrix

$$B = \begin{bmatrix} A \\ C \end{bmatrix} ,$$

where C is the $(n - m) \times n$ matrix all of whose entries are 1. Clearly $\operatorname{per}(B) = (n - m)! \operatorname{per}(A)$. In [2, 1] Minc states a formula for the computation of the permanent of an $m \times n$ matrix, $m \leqq n$, in terms of the R functions. This formula is an extension of Binet's formulas for $m \leqq 4$, which are based on the Principle of Inclusion and Exclusion.

If A is an $n \times n$ matrix, then

$$\operatorname{Per}(A) = \sum_{(t_1, \cdots, t_k) \in G(n)} C(t_1, \cdots, t_k) R(t_1, \cdots, t_k)$$

with

$$C(t_1, \cdots, t_k) = (-1)^{n+k} \prod_{h=1}^{k} (t_h - 1)! \ .$$

This is the formula of Binet and Minc of $m = n$. Let us show that (3.7) is precisely this formula.

Let $A' = ((1/Z_i) a_{ij})$ be the matrix obtained from A by multiplying its ith row by $1/Z_i$ $i = 1, \cdots, n$. Thus, we can write (3.5) in the following form

$$\phi^{(T_j)} = (-1)^{t_j - 1} (t_j - 1)! \ r_{w_1^{(j)} * \cdots * w_s^{(j)}} , \tag{3.10}$$

where $r_{w_1^{(j)} * \cdots * w_s^{(j)}}$ denotes the sum of the entries in the Hadamard product of the columns $w_1^{(j)}, \cdots, w_s^{(j)}$ of A, $s = t_j$, $j = 1, \cdots, k$. On the other hand

$$\sum_{k=1}^{n}\sum_{T_1,\cdots,T_k}\phi^{(T_1)}\cdots\phi^{(T_k)}=\sum_{k=1}^{n}\sum_{(t_1,\cdots,t_k)}\sum_{T_1,\cdots,T_k}{}'\phi^{(T_1)}\cdots\phi^{(T_k)}\ ,$$

where $\sum'$ denotes the summation over the partitions $T_1, \cdots, T_k$ whose cardinals $t_i = |T_i|$, $i = 1, \cdots, k$, are kept fixed. Substituting (3.10) into (3.11), applying (3.1) and the definition of R we obtain the formula of Binet and Minc for per (A')

$$\begin{aligned}\sum_{k=1}^{n}\sum_{T_1,\cdots,T_k}\phi^{(T_1)}\cdots\phi^{(T_k)}&=\sum_{k=1}^{n}\sum_{T_1,\cdots,T_k}(-1)^{t_1-1}\cdots(-1)^{t_k-1}(t_1-1)!\\&\quad\times\cdots(t_k-1)!\prod_{j=1}^{k}r_{w_1^{(j)}*\cdots*w_{t_j}^{(j)}}\\&=\sum_{k=1}^{n}\sum_{t_1\cdots t_k}(-1)^{n-k}\prod_{j=1}^{k}(t_j-1)\prod_{j=1}^{k}r_{w_1^{(j)}*\cdots*w_{t_j}^{(j)}}\\&=\sum_{(t_1,\cdots,t_k)\in G(n)}C(t_1,\cdots,t_k)R(t_1,\cdots,t_k)=\operatorname{per}(A')\ .\end{aligned}$$

It is now clear that in fact (3.7) is independent of $x_1, \cdots, x_n$ due to the multilinearity of the permanent, since per $(A) = Z_1 \cdots Z_n$ per $(A') = Z$ per (A').

4. Ryser's formula. In this section we shall prove Ryser's formula [3] from the identity (2.5), without using the Principle of Inclusion and Exclusion.

THEOREM 4.1. *Let $A = (a_{ij})$ be a matrix of size n by n defined over a field F. Let $w = (w_1, \cdots, w_k) \in Q_{k,n}$ and $\bar{w} = (\bar{w}_1, \cdots, \bar{w}_{n-k}) \in Q_{n-k,n}$ be sequences complementary to each other. We shall denote by A_w the matrix obtained from A by replacing columns $\bar{w}_1, \cdots, \bar{w}_{n-k}$ by zeros. Let $S(A_w)$ represent the product of the row sums of A_w. Then*

$$\operatorname{per}(A)=\sum_{k=1}^{n}\sum_{w\in Q_{k,n}}(-1)^{n-k}S(A_w)\ . \tag{4.1}$$

Proof. Let $A' = (a_{ij}x_j)$ be the $n \times n$ matrix obtained from A by multiplying its jth column by x_j, $j = 1, \cdots, n$. From (2.5) it follows that

$$\begin{aligned}S(A'_w)&=\prod_{j=1}^{n}\left(\sum_{t=1}^{k}a_{jw_t}x_{w_t}\right)\\&=\sum_{\beta}\operatorname{per}(A_w(\beta_1,\cdots,\beta_n))\frac{x_1^{\beta_1}}{\beta_1!}\cdots\frac{x_n^{\beta_n}}{\beta_n!}\ .\end{aligned}\tag{4.2}$$

Note that if one of the $\beta_{\bar{w}_t}$, $t = 1, \cdots, n - k$, is different from zero, the matrix $A_w(\beta_1, \cdots, \beta_n)$ will have columns of zeros and per $(A_w(\beta_1, \cdots, \beta_n)) = 0$. If the condition $\beta_{\bar{w}_1} = \cdots = \beta_{\bar{w}_{n-k}} = 0$ holds, then per $(A_w(\beta_1, \cdots, \beta_n))$ will be independent of w.

From (4.2) we have

$$\begin{aligned}
&\sum_{k=1}\sum_{w\in Q_{k,n}}(-1)^k S(A'_w)\\
(4.3)\qquad &= \sum_{\beta}\frac{x_1^{\beta_1}}{\beta_1!}\cdots\frac{x_n^{\beta_n}}{\beta_n!}\sum_{k=1}^{n}\sum_{w\in Q_{k,n}}(-1)^k \operatorname{per}(A_w(\beta_1,\cdots,\beta_n))\,,
\end{aligned}$$

where $\sum_{i=1}^{n}\beta_i = n$. In order to evaluate (4.3) we first prove that

$$\sum_{k=1}^{n}\sum_{w\in Q_{k,n}}(-1)^k \operatorname{per}(A_w(\beta_1,\cdots,\beta_n)) = 0$$

provided that $\beta_i \neq 1$ for some i. For this purpose let $(\beta_{w_1},\cdots,\beta_{w_h})$, $w' = (w_1,\cdots,w_h)\in Q_{h,n}$, are the only nonzero elements in $(\beta_1,\cdots,\beta_n)$. It is clear that $\beta_{\bar{w}_1} = \cdots = \beta_{\bar{w}_{n-h}} = 0$ and $\operatorname{per}(A_w(\beta_1,\cdots,\beta_n)) = 0$ unless $\bar{w}\subset\bar{w}' \Leftrightarrow w\supset w'$. Since there are $\binom{n-h}{k-h}$ different sequences w, $w\in Q_{k,n}$, such that $w\supset w'$ for a fixed w', we have

$$\begin{aligned}
\sum_{k=1}^{n}\sum_{w}(-1)^k \operatorname{per}(A_w(\beta_1,\cdots,\beta_n)) &= \sum_{k=1}^{n}\sum_{\substack{w\\ w\supset w'}}(-1)^k \operatorname{per}(A(\beta_1,\cdots,\beta_n))\\
(4.4)\qquad &= (-1)^h\sum_{k=h}^{n}(-1)^{k-h}\binom{n-h}{k-h}\operatorname{per}(A(\beta_1,\cdots,\beta_n))\\
&= (-1)^h(1-1)^{n-h}\operatorname{per}(A(\beta_1,\cdots,\beta_n))\\
&= 0
\end{aligned}$$

if $n\neq h$. Consider now the case $n = h$, that is $\beta_1 = \cdots = \beta_n = 1$. Clearly $\operatorname{per}(A_w(1,\cdots,1)) = 0$ unless $w = \{1,\cdots,n\}$. From (4.3) and (4.4) it finally follows

$$(4.5)\qquad \sum_{k=1}^{n}\sum_{w\in Q_{k,n}}(-1)^k S(A'_w) = (-1)^n \operatorname{per}(A)\,x_1\cdots x_n\,.$$

Since $\operatorname{per}(A) = (1/x_1\cdots x_n)\operatorname{per}(A')$, (4.5) gives Ryser's formula for the evaluation of $\operatorname{per}(A)$.

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