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MODULE FINITENESS OF LOW DIMENSIONAL *PI* RINGS

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A partial answer is given to the question: If R is a PI ring finitely generated as an algebra over a commutative ring, and if R has Krull dimension one or zero, when can we conclude that R is finitely generated as a module over its center, or over some commutative subring?

0. Introduction. If S and R are rings, we say R is (*left* or *right*) *module finite* over S if R is a finitely generated (left or right) S-module. If S is contained in the center of R, and R is finitely generated as an S-algebra, then R is said to be *ring finite* over S, cf. [2]. Suppose that R is *PI*, that R is ring finite over its center, and that the classical Krull dimension of R is zero or one. The purpose of this paper is to present some results which give conditions under which we can conclude that R is module finite over some (usually central) commutative subring.

The main results are listed below. Throughout the text, "Noetherian" and "Artinian" means on both sides, unless qualified by "left" or "right". If R is a ring, the Krull dimension $Kd(R)$ of R will always be the classical Krull dimension. All rings are assumed to have an identity, and all subrings referred to are assumed to contain the identity.

THEOREM 2.18. *Let R be a semiprime PI ring whose Krull dimension is no greater than one. Suppose R is ring finite over A. Then if either R or A is Noetherian, R is module finite over its center Z, and both Z and R are Noetherian.*

THEOREM 2.19. *Suppose R is a semiprime PI ring with Krull dimension one. If R is ring finite over a field F, then R is module finite over a central subring of the form $C = F[x]$, where x is transcendental over F.*

THEOREM 2.24. *Let S be a PI ring with the following properties:*

(a) *The Krull dimension of S is one or zero.*

(b) *S is ring finite over a Hilbert ring A.*

(c) *If P is a minimal prime ideal of S, then $\ell(P) \cap r(P) \not\subset P$, where $\ell(P)$ and $r(P)$ are respectively the left and right annihilators in S of P.*

Then if either A or S has the ascending chain condition on ideals, S is module finite over its center Z, and Z and S are Noetherian.

THEOREM 2.25. *Let S be a left Noetherian PI ring which is ring finite over a field F, and which has Krull dimension one. Then S is left module finite over a commutative subring $C = F[x]$, where $x \in S$ is transcendental over F, and x is central modulo every minimal prime of S.*

The result 2.18 is a generalization of the following theorem of A. Braun which is proved in [4].

2.18* Let R be a prime PI ring whose Krull dimension is one. Suppose R is ring finite over A. Then if either R or A is Noetherian, R is module finite over its center Z, and both Z and R are Noetherian.

Braun proved 2.18* by studying the prime spectrums of R and Z vis a vis the primes of certain integral extensions of R and Z. Lance Small later gave a short, elegant proof of 2.18* using central localization and the principal ideal theorem [11]. The proof of 2.18 given here is an extension of Small's proof to the semiprime case.

All the results given above are proved in § 2. In § 1, we furnish some background material which can be skipped initially by the reader. In § 3, we indicate how known examples show that the results of § 2 are very close to being the best possible, and we list a few open questions.

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1. Background.

SMALL'S THEOREM 1.1. *Let R be a semiprime PI ring satisfying the ascending chain condition on annihilator ideals. Then there are only finitely many minimal primes $P_1, P_2, \cdots, P_n$ of R. And R is an order in*

$$Q = Q(R/P_1) \oplus Q(R/P_2) \oplus \cdots \oplus Q(R/P_n)$$

where $Q(R/P_i)$ is the quotient ring of R/P_i. Moreover, Q can be obtained from R by inverting central elements of R. (See [18], [19], *and* [20].)

THEOREM 1.2 (Braun, Small, et al). *If R is a PI ring with Krull dimension zero, and if R is ring finite over a Noetherian* *ring A, then R is module finite over its center Z, and Z and R are Artinian. Moreover, if A is a field then R is finite dimensional over A.* (*See* [4].)

COROLLARY. *If R is a prime PI ring with Krull dimension one, and if R is ring finite over a Noetherian ring A, then R is Noetherian.*

Proof. Suppose $0 \neq I$ is an ideal of R. There is a nonzero $\in I \cap Z$, where Z is the center of R. Since R/Rc has Krull dimension zero, it is Artinian by 1.2. Choose $x_1, x_2, \cdots, x_n$ in R such that the set $\{x_i + Rc\}$ generates I/Rc as a left (R/Rc)-module. Then $I = Rx_1 + Rx_2 + \cdots + Rx_n + Rc$ is finitely generated as a left ideal. Therefore R has the A.C.C. on two-sided ideals; implying R is Noetherian. (See [5].)

SMALL'S GENERALIZED ARTIN-TATE LEMMA 1.3. *Let $A \subset C \subset R$ be rings such that C is commutative, A is in the center of R, and A is Noetherian. Suppose that R is left module finite over C, and ring finite over A. Then C is ring finite over A, and (hence) C is Noetherian and R is left Noetherian.*

Proof. Say $R = A\{x_1, x_2, \cdots, x_n\} = Cy_1 + Cy_2 + \cdots + Cy_m$. For each i such that $1 \leqq i \leqq n$, write

$$\text{(i)} \qquad x_i = c_{i1}y_1 + c_{i2}y_2 + \cdots + c_{im}y_m$$

where each $c_{ij} \in C$. For each pair (i, j) such that $1 \leqq i \leqq m$, $1 \leqq j \leqq n$ write

$$\text{(ii)} \qquad y_ix_j = c_{ij1}y_1 + c_{ij2}y_2 + \cdots + c_{ijm}y_m$$

where each $c_{ijk} \in C$. Let T be the algebra generated over A by all the c_{ij} and all the c_{ijk}. By (i), each x_i is in the left T-module $S = Ty_1 + Ty_2 + \cdots + Ty_m$. So any element of the form x_ix_j is in $Sx_j = Ty_1x_j + Ty_2x_j + \cdots + Ty_mx_j$. By (ii), each y_ix_j is in S; so any x_ix_j is in $TS = S$. By induction, we can show that any monomial formed from elements of $\{x_1, x_2, \cdots, x_n\}$ is in S. We may assume that one of the y_j's is equal to 1. Therefore S is an A-module containing 1, and containing every monomial in the x's—so it must be all of $R = A\{x_1, x_2, \cdots, x_n\}$. So R is a finitely generated left T-module. Now T is Noetherian, by the Hilbert basis theorem; so since $T \subset C \subset R$, C must also be a finitely generated T-module. But now the fact that T is ring finite over A implies C is ring finite over A. Thus C is Noetherian, and R is left Noetherian because it has the A.C.C. on left C-modules. $\square$

THEOREM 1.4 (a variation on 1.3). *Suppose that R is a ring which is left module finite over a commutative subring C, and that R has the ascending chain condition on "extended ideals" IR where I is an ideal of C. Suppose that R is ring finite over $A \subset C$. Then C is ring finite over A, C is Noetherian, and R is left Noetherian.*

Proof. Construct T as in the proof of the previous theorem. As before, R is left module finite over T. By a result [9] of Formanek, T is Noetherian. Therefore C is module finite over T, and Noetherian. It also follows that C is ring finite over A, and that R is left Noetherian. $\square$

The next few results will help us set up "Noetherian induction" arguments in § 2.

THEOREM 1.5. *Say R is ring finite over a Noetherian ring C. If I is an ideal of R and R/I is left module finite over some commutative subring of R/I, then I is finitely generated as a two sided ideal.*

Proof. Suppose $R = C\{a_1, a_2, \cdots, a_P\}$, and that $\bar{R} = R/I$ is left module finite over a commutative ring $\bar{B}$, which we may assume contains $\bar{C} = (C + I)/I$. (We will continue to use a bar to denote images modulo I.) By 1.3, $\bar{B}$ is ring finite over $\bar{C}$. Choose $z_1, z_2, \cdots, z_m$ in R such that $\bar{R} = \sum \bar{B}\bar{z}_j$ and choose $x_1, x_2, \cdots, x_n$ in R such that $\bar{B} = \bar{C}[\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n]$. Let $B \subset R$ be defined by $B = C\{x_1, x_2, \cdots, x_n\}$. Since $R = M + I$ where $M = B + \sum Bz_j$, we can choose a finite set S of elements of I such that

(i) for each i, j such that $1 \leqq i, j \leqq m$, $\exists\, \sigma_{ij} \in S$ such that $z_i z_j - \sigma_{ij} \in M$,

(ii) for each i, j such that $1 \leqq i \leqq m$ and $1 \leqq j \leqq n$, $\exists\, s_{ij} \in S$ such that $z_i x_j - s_{ij} \in M$,

(iii) for each i such that $1 \leqq i \leqq p$, $\exists\, s_i \in S$ such that $a_i - s_i \in M$.

Since the $\bar{x}$ all commute, we can also choose S so that

(iv) for each i, j such that $1 \leqq i, j \leqq n$, $[x_i, x_j] \in S$.

Let $\tilde{I}$ be the two sided ideal generated by S. Let $R/\tilde{I} = \tilde{R}$, and let us use the tilde symbol to denote images in $\tilde{R}$. Using (i)-(iii) it is pretty easy (though tedious) to show that every monomial in the elements $\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_p$ is contained in $\tilde{M} = \tilde{B} + \sum \tilde{B}z_j$, and hence $\tilde{R} = \tilde{M}$; so $\tilde{R}$ is module finite over $\tilde{B}$. One way to proceed to show that all the monomials in the $\tilde{a}$'s are in $\tilde{M}$ is to use (i) to prove that every monomial in the $\tilde{z}$'s is in $\tilde{M}$, then use (ii) to show that every monomial formed from the elements $\tilde{x}_1, \cdots, \tilde{x}_n, \tilde{z}_1, \cdots, \tilde{z}_m$ is in $\tilde{M}$, and then finally to use (iii) to prove each monomial in the $\tilde{a}$'s is in $\tilde{M}$.

By (iv), $\tilde{B}$ is commutative, hence Noetherian, by the Hilbert basis theorem; so $\tilde{R}$ is left Noetherian because it is module finite over $\tilde{B}$. Thus the kernel of the projection $\tilde{R} \to R/I$ is finitely generated. Since I is the kernel of the composition $R \to \tilde{R} \to \bar{R}$, and since the kernel of each map is finitely generated as a two-sided ideal, I is too. □

THEOREM 1.6. *Let R be a ring and C a central subring of R. Let I be an ideal of R maximal among ideals K such that R/K is not a finitely generated C-module. Then I is prime.*

Proof. By passing to R/I, we may assume $I = 0$. We will first prove R is semiprime; then we will prove R is prime. If $a \in R$ and $aRa = 0$ but $a \neq 0$, then $RaR \neq 0$ and since R/RaR is module finite over C, we can write

$$\text{(i)} \qquad R = Cx_1 + Cx_2 + \cdots + Cx_n + RaR$$

where each $x_i \in R$. But then $RaR = (\sum Cx_i + RaR)a(\sum Cx_i + RaR) = \sum Cx_iax_j$; so

$$\text{(ii)} \qquad R = \sum Cx_k + \sum Cx_iax_j\ ,$$

which means R is module finite over C, a contradiction. Therefore R is semiprime. Now if $aRb = 0$ where a and b are nonzero elements of R, then $bRa = 0$ too, else $RbRa$ is a nonzero nilpotent left ideal of R. As before, we can write (i). Similarly we get

$$\text{(iii)} \qquad R = Cy_1 + Cy_2 + \cdots + Cy_m + RbR\ .$$

But then by (i), $RbR = (\sum Cx_i + RaR)b(\sum Cx_j + RaR) = \sum Cx_ibx_j$, which together with (iii) implies R is module finite over C, a contradiction which allows us to conclude that R is prime. □

THEOREM 1.7. *Say R is ring finite over a Noetherian ring C, and suppose D is a central subring of R containing C. Let K be an ideal of R such that R/K is not module finite over D. Then there exists an ideal I of R containing K such that I is maximal with respect to the property that R/I is not module finite over D, and I is a prime ideal.*

Proof. Using Zorn's lemma and 1.5, it is easy to prove that I iexsts. By 1.6, I is prime. □

THEOREM 1.8. *Suppose that R is a left Noetherian ring, that C is a commutative subring of R, and that I is an ideal of R maximal with respect to the property that R/I is not left module finite over C. Then I is prime.*

Proof. By passing to R/I, we may assume $I = 0$. Suppose that A and B are nonzero ideals of R such that $AB = 0$. We can write

$$\text{(i)} \qquad R = \sum Cx_i + A = \sum Cy_j + B$$

$$\text{(ii)} \qquad B = \sum Rb_k .$$

Combining (ii) with the first part of (i) we get

$$\text{(iii)} \qquad B = \sum (\sum Cx_i + A)b_k = \sum Cx_ib_k .$$

Thus by the second part of (i), $R = \sum Cy_j + \sum Cx_ib_k$—contradicting the assumption that R/I is not module finite over C. Therefore I is prime. □

2. **Main results.** In this section, *R will always denote a semiprime PI ring.* The center of a ring A will be denoted $Z(A)$. $Z(R)$ will usually just be denoted by Z. We begin by noting some basic facts about the minimal prime spectrum of semiprime *PI* rings having the A.C.C. (ascending chain condition) on annihilator ideals. Propositions 2.1–2.4 are all easily proved as consequences of 1.1.

PROPOSITION 2.1. *Suppose R has the* A.C.C. *on annihilator ideals. Then the minimal primes of the center Z of R are finite in number and are all of the form $P \cap Z$ where P is a minimal prime of R.*

PROPOSITION 2.2. *Let R be as in* 2.1. *If P_1 and P_2 are minimal primes of R, and $P_1 \cap Z = P_2 \cap Z$, then $P_1 = P_2$.*

PROPOSITION 2.3. *If R is as in* 2.1, *then the mapping $P \mapsto P \cap Z$ is a bijection between the minimal primes of R and the minimal primes of Z.*

PROPOSITION 2.4. *If R is as in* 2.1, *then an element of Z is a zero divisor if and only if it is contained in $\mathfrak{p}_1 \cup \mathfrak{p}_2 \cup \cdots \cup \mathfrak{p}_n$, the union of the minimal primes of Z. Moreover, an ideal $\mathfrak{q}$ of Z which is not contained in any of the $\mathfrak{p}_i$ must contain a regular element.*

We next prove that "many" central localizations of R are semiprime rings. This fact will be used to obtain global information about R from local information.

PROPOSITION 2.5. *Suppose S is a multiplicatively closed set of central elements of R. If all but a finite number of the minimal primes of R survive in RS^{-1}, then RS^{-1} is semiprime, and $Z(RS^{-1}) = ZS^{-1}$.*

Proof. First of all, note that the proposition is obviously true if S consists of regular elements. Now assume that S contains some zero divisors. Then the kernel K of the homomorphism $f: R \rightarrow RS^{-1}$ given by $f(r) = r/1$ is nonzero, since $K = \{k \in R \mid ks = 0, \exists s \in S\}$. Note that $K \subset \cap Q$ where Q ranges over all those minimal primes of R which survive in RS^{-1}. This is so because if $k \in K$ and $s \in S$ such that $ks = 0$, then $s \notin Q$ since Q surviving in RS^{-1} is equivalent to $Q \cap S$ being empty. Thus k is in every such Q. Since $K \neq 0$, $\cap Q \neq 0$; so R has some minimal primes, say $P_1, P_2, \cdots, P_n$, which do not survive in RS^{-1}. Note that $0 = (\cap Q) \cap (\cap P_i)$. Also, since each P_i blows up, we can choose $\sigma_i \in P_i \cap S$ for each i, and thus $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in (\cap P_i) \cap S$.

We claim that $K = \cap Q$. To see this, note that $\sigma\tau = 0$ for every $\tau \in \cap Q$. Thus R/K is semiprime. Since RS^{-1} is obtained from R/K by inverting regular central elements, RS^{-1} is semiprime and $Z(RS^{-1}) = Z(R/K)\bar{S}^{-1}$ where $\bar{S} = (S + K)/K$.

If $x \in Z(RS^{-1})$, then $x = (r + K)(s + K)^{-1} = rs^{-1}$ (a slight abuse of notation) where $(r + K) \in Z(R/K)$ and $s \in S$. Since $(r + K) \in Z(R/K)$, the commutator $[r, t]$ is in K for every $t \in R$. Thus for all $t \in R$, $[r\sigma, t] = \sigma[r, t]$ is in $K \cap (\cap P_i) = 0$. Thus $r\sigma \in Z(R)$; and so $x = (r + K)(s + K)^{-1} = (r\sigma + K)(\sigma s + K)^{-1} = (r\sigma)(\sigma s)^{-1} \in ZS^{-1}$. This shows that $Z(RS^{-1}) \subset ZS^{-1}$. It's trivial now to verify that $Z(RS^{-1}) = ZS^{-1}$. □

One of the problems that makes it difficult to prove the results of this paper is that if R is a ring with center Z and π is an epimorphism $\pi: R \rightarrow S$, then it is possible that $Z(S) \neq \pi(Z)$. However, we are sometimes able to make headway when we know that $\pi(Z)$ contains a nonzero ideal of $Z(S)$. Therefore we want to prove:

PROPOSITION 2.6. *Suppose P is a minimal prime ideal of R, and $P \cap L = 0$ for some nonzero ideal L of R. Then*

$$\bar{I} = ((L + P)/P) \cap Z(R/P)$$

is a nonzero ideal of $Z(R/P)$ which is contained in $(Z(R) + P)/P$.

Proof. Since $P \cap L = 0$ and $L \neq 0$, $L \not\subset P$. Thus $(L + P)/P$ is a nonzero ideal of R/P. Therefore $\bar{I}$ is a nonzero ideal of $Z(R/P)$. (See [8] and [15].)

We claim that $\bar{I}$ is actually contained in $(Z + P)/P$ where $Z = Z(R)$. To see this, suppose $x \in \bar{I}$. Choose $y \in L$ such that $x = (y + P)$. Since $x \in Z(R/P)$, $[y, t] \in P$ for every $t \in R$; since $y \in L$, $[y, t] \in L$ for every $t \in R$. But $P \cap L = 0$; so $y \in Z$. Thus $x \in (Z + P)/P$. □

We now list some standard facts which will be used sooner or later. Some of the proofs are left for the reader.

PROPOSITION 2.7. *Suppose R is ring finite over A, and M is a maximal ideal of R such that $M \cap A$ is a maximal ideal of A. Then R/M is a finite dimensional vector space over $(A + M)/M$.*

Proof. Use Kaplansky's theorem [10], 1.3, and the weak Nullstellensatz [22].

PROPOSITION 2.8. *If R is prime, and $H = \cap M$ where M ranges over all those maximal ideals of R such that the PI degree of R/M equals the PI degree of R* ($\operatorname{pid}(R/M) = \operatorname{pid}(R)$), *then $J(R) = 0$ implies $H = 0$.*

Proof. Let $I = \cap M$ where M ranges over all the maximal ideals of R such that $\operatorname{pid}(R/M) < \operatorname{pid}(R)$. If $J(R) = 0$, then $H \cap I = J(R) = 0$; and so $IH = 0$. Thus H will be zero if $I \neq 0$. But $I = 0$ is impossible because the canonical projections $R \to (R/M)$ induce a ring monomorphism $(R/I) \to \prod (R/M)$ where M ranges over those M such that $\operatorname{pid}(R/M) < \operatorname{pid}(R)$. If I were zero, then this would mean that R could be embedded into a ring with lower *PI* degree. □

PROPOSITION 2.9. *If R is prime and M is a maximal ideal of R such that R/M has the same PI degree as R, then $M \cap Z$ is a maximal ideal of Z.*

Proof. There is a multilinear polynomial f which is central and nonvanishing on R and R/M. (See [14] or [6].) Pick $r_1, r_2, \cdots, r_n$ in R such that $f(\bar{r}_1, \bar{r}_2, \cdots, \bar{r}_n) \neq 0$, where the bar is used to denote images in R/M. Then $Z(R/M)f(\bar{r}_1, \bar{r}_2, \cdots, \bar{r}_n)$ is a nonzero ideal of $Z(R/M)$; so it is equal to $Z(R/M)$ because $Z(R/M)$ is a field. Thus if $z \in Z(R/M)$, say $z = f(\bar{s}\bar{r}_1, \bar{r}_2 \cdots, \bar{r}_n) = \overline{f(sr_1, r_2, \cdots, r_n)}$ where $s \in R$ is chosen so $\bar{s} \in Z(R/M)$, then $f(sr_1, r_2, \cdots, r_n) \in Z = Z(R)$ because f is central on R. So we have shown that $z \in (Z+M)/M$. Therefore $Z(R/M) = (Z+M)/M \cong Z/(M \cap Z)$, proving $M \cap Z$ is maximal. □

PROPOSITION 2.10. *If R is Noetherian and I is an ideal of R then there are only finitely many prime ideals of R minimal over I.*

Now we are ready to prove "lying over" for a certain class of rings R. This will lead to a proof that $Kd(Z) \leqq 1$ if $Kd(R) \leqq 1$, and this will make it possible to prove 2.18. We will begin with three lemmas:

LEMMA 2.11. *If R has the* A.C.C. *on annihilator ideals, Z has a unique maximal ideal m, and Q is a minimal prime ideal of R, then $J(R/Q) = 0$ implies $Q \cap Z = m$.*

Proof. We claim $(Z+Q)/Q$ contains a nonzero ideal $\bar{I}$ of $C = Z(R/Q)$. If $Q = 0$, then $Z = C$ and the claim is trivial. If $Q \neq 0$, then there are other minimal primes $P_1, P_2, \cdots, P_n$ of R (see 2.1). Since no P_i is contained in Q, $P_1P_2\cdots P_n \not\subset Q$; so $L = P_1 \cap P_2 \cap \cdots \cap P_n \not\subset Q$. But $L \cap Q = 0$. So by 2.6, we have proved our claim. By 2.8, $J(R/Q) = 0$ implies $\cap M = 0$, where M ranges over all elements of G—the set of maximal ideals of R/Q which do not "lower the *PI* degree". Moreover, if A is the intersection of all the elements of G not containing $\bar{I}$, and B is the intersection of the elements of G containing $\bar{I}$, then $AB = 0$ and $\bar{I} \subset B \neq 0$; so $A = 0$. Now each element M of G contracts to a maximal ideal of $C = Z(R/Q)$, by 2.9. If M does not contain $\bar{I}$, then $\bar{I}$ has a nonzero image in R/M. That image is an ideal of the image of C, and the image of C is a field. Therefore, since $\bar{I}$ is also an ideal of $(Z+Q)/Q$, the images of $\bar{I}$ and $(Z+Q)/Q$ and C are all the same. Thus M contracts to a maximal ideal of $(Z+Q)/Q$ and $0 = A \cap ((Z+Q)/Q)$ is therefore an intersection of maximal ideals of $(Z+Q)/Q$. Since m is the only maximal ideal of Z, this implies that $(Z+Q)/Q$ is a field, i.e., that $Q \cap Z = m$. This completes the proof. □

LEMMA 2.12. *If R is Noetherian, Z has a unique maximal ideal m, Q is a minimal prime ideal of R such that $Q \cap Z \neq m$, and the Krull dimension of R is no greater than one, then Q is contained in only finitely many maximal ideals.*

Proof. By 2.11, $J(R/Q) \neq 0$. The maximal ideals of R containing Q are in one-one correspondence with the primes minimal over $J(R/Q)$; so we are done, by 2.10. □

LEMMA 2.13. *If the Krull dimension of R is no greater than one, R is Noetherian, and the center of R has only one maximal ideal, then R has only a finite number of prime ideals.*

Proof. If Q is a minimal prime of R, and $Q \cap Z = m$ where m is the maximal ideal of Z, then $Z/(Q \cap Z) \cong (Z + Q)/Q$ is a field. So since $(Z + Q)/Q$ contains a nonzero ideal of $Z(R/Q)$, $Z(R/Q) = (Z + Q)/Q$. (See the first few lines of the proof of 2.11.) Therefore R/Q is simple, by Formanek's theorem [8]; and so Q is maximal. If $Q \cap Z \neq m$, then by 2.12 Q is contained in only finitely many maximal ideals. Since R has only a finite number of minimal primes, we are done. $\square$

Now we come to the lying over theorem we are after. First we will prove the local version, and then the global one.

PROPOSITION 2.14. *Let R be a Noetherian, semiprime PI ring with Krull dimension no greater than one. Then if p is a prime ideal of Z, $p = P \cap Z$ where P is a prime ideal of R.*

Proof. First assume that the center of R has a unique maximal ideal m. We claim that $m = M \cap Z$ where M is a maximal ideal of R. To see this, suppose that $z \in m$; z is not a unit of R because $z^{-1} \in R$ implies $z^{-1} \in Z(R)$. Thus $Rz \neq R$; so $z \in M_i$ for some maximal ideal M_i of R. By 2.13, R has a finite set $M_1, M_2, \cdots, M_n$ of maximal ideals. So

$$m \subset (M_1 \cap Z) \cup (M_2 \cap Z) \cup \cdots \cup (M_n \cap Z) .$$

So $m \subset M_i \cap Z$ for some i.

Now if Z is arbitrary and p is a prime ideal of Z, let $S = Z - p$. By 2.5 and 2.1, RS^{-1} is semiprime and $Z(RS^{-1}) = ZS^{-1}$. Thus by the first part of the proof, $p(ZS^{-1}) = P(RS^{-1}) \cap ZS^{-1}$ where P is a prime ideal of R such that $Z \cap P \subset p$. If $x \in p$, then $(x/1) \in P(RS^{-1})$, i.e., $xst = yt$ where s and t are in $Z - p$, and $y \in P$. So $x(st) \in Z \cap P \subset p$. Since $st \notin p$, $x \in Z \cap P$. Thus $p = Z \cap P$. $\square$

We will now prove a noncommutative version of Krull's principal ideal theorem. The result is due to Jategaonkar and the proof that we give here is one adapted by Goldie from the proof of the commutative version that appears in Kaplansky's book. See [11] and [12].

LEMMA 2.15. *Let u, y be regular central elements of a ring B. Then*

(a) *The modules* $(Bu + By)/Bu$ *and* $(Bu^2 + Buy)/Bu^2$ *are isomorphic.*

(b) *If* $Bu^2 \cap By \subset Buy$, *then the modules* Bu/Bu^2 *and* $(Bu^2+By)/(Bu^2 + Buy)$ *are isomorphic.*

Proof. (a) Multiplication by u is a module isomorphism from $(Bu + By)$ to $(Bu^2 + Buy)$ which maps Bu to Bu^2.

(b) Consider the following diagrams of B-module homomorphisms:

$$(Bu^2 + By) \longrightarrow C = (Bu^2 + By)/Bu^2 \longrightarrow D = (Bu^2 + By)/(Bu^2 + Buy)$$
$$C' = By/(Bu^2 \cap By) \longrightarrow D' = By/Buy\ .$$

Note that D is obtained from C in the same way that D' is obtained from C', i.e., by modding out the image (from $Bu^2 + By$) of Buy. Since C and C' are isomorphic, so are D and D'. But $D' \cong B/Bu \cong Bu/Bu^2$, by the regularity of u and y. □

PROPOSITION 2.16. *If R is Noetherian and ring finite over its center Z, and if Z has a unique maximal ideal m, and if there is a regular $x \in m$ such that m is the only prime of Z containing x, and if R/Rx is Artinian, then the rank of m is no greater than one.*

Proof. Suppose that q is a prime ideal of Z, and that q is not minimal. Then q contains a regular element y (see 2.4). Put $I_k = \{r \in R \mid rx^k \in Ry\}$. I_k is an ideal of R, and $I_n \subset I_m$ whenever n and m are positive integers such that $n \leqq m$. Choose n such that $tx^{2n} \in Ry$ implies $tx^n \in Ry$. Put $u = x^n$. Then $Ru^2 \cap Ry \subset Ruy$, and since x and y are regular, $Zu^2 \cap Zy \subset Zuy$. Since R/Rx is Artinian, so is R/Ru^2. We claim that R/Ru^2 is module finite over $(Z + Ru^2)/Ru^2$. To see this is true, put $\bar{R} = R/Ru^2$ and $\bar{Z} = (Z + Ru^2)/Ru^2$, and choose a prime ideal P of $\bar{R}$ maximal with respect to the property that $\bar{R}/P$ is not module finite over $\bar{Z}$. Put $S = \bar{R}/P$. Now S is an Artinian *PI* ring. Given any regular central element t of S, for some k $St^k = St^{k+1}$. It follows that t is invertible, and so S is simple, by Formanek's theorem. Thus P is maximal. Also $P \cap \bar{Z}$ is $\bar{m}$, the image of m. Therefore the claim follows from 2.7. It follows now from a theorem [7] of Eisenbud that $\bar{Z}$ is Artinian. Consider the exact sequences

$$0 \longrightarrow Zu/Zu^2 \longrightarrow (Zu + Zy)/Zu^2 \longrightarrow (Zu + Zy)/Zu \longrightarrow 0$$
$$0 \longrightarrow (Zu^2 + Zuy)/Zu^2 \longrightarrow (Zu^2 + Zy)/Zu^2$$
$$\longrightarrow (Zu^2 + Zy)/(Zu^2 + Zuy) \longrightarrow 0\ .$$

By 2.15, the module at the end of the first sequence is isomorphic to the module at the beginning of the second sequence, and the module at the beginning of the first sequence is isomorphic to the module at the end of the second sequence. Since $\bar{Z} \cong Z/Zu^2$, it follows that $(Zu + Zy)/Zu^2$ and $(Zu^2 + Zy)/Zu^2$ have equal length as $\bar{Z}$-modules. Since the latter is contained in the former, we conclude $Zu + Zy = Zu^2 + Zy$. Say $u = z_1u^2 + z_2y$. Then $u(1 - z_1u) = z_2y$, implying that $u \in Zy$, since $(1-z_1u)$ is invertible. But then $q=m$. □

We are finally ready to prove a lemma crucial to the proof of 2.18.

LEMMA 2.17. *Suppose R is Noetherian and ring finite over its center Z. Then if the Krull dimension of R is no greater than one, the Krull dimension of Z is no greater than one.*

Proof. First assume that Z has a unique maximal ideal m. R has only a finite number of prime ideals, by 2.13. By 2.14, every prime ideal of Z is a contraction of a prime ideal of R. So Z has only a finite number of prime ideals. Let $M_1, M_2, \cdots, M_s$ be the primes of R which contract to m. If some M is minimal, then m is minimal by 2.3, and we are done. So we can assume that none of the M_i is minimal. Let $P_1, P_2, \cdots, P_t$ be the primes of R which do not contract to m. Since $m \not\subset \cup P_i$, we can choose $x \in m - \cup P_i$. Since all the minimal primes of R are among the P's, $\cup P_i$ contains all the zero divisors of Z (see 2.3 and 2.4). Thus x is regular. Moreover, the prime ideals of $\bar{R} = R/Rx$ are just the images $\bar{M}_i$ of the M's, and for $1 \leqq i \leqq s$, $\bar{M}_i \cap \bar{Z} = \bar{m}$. The primes of $\bar{Z}$ are in one-to-one correspondence with the primes of Z containing x. By 2.14, $\bar{m}$ is therefore the only prime of $\bar{Z}$; and so $Kd(\bar{Z}) = Kd(\bar{R}) = 0$. Also, $\bar{R}$ is module finite over $\bar{Z}$ by 2.7 and by 1.6. Thus by the theorem [9] of Formanek, $\bar{Z}$ is a Noetherian ring. But then $\bar{Z}$ is Artinian, since its Krull dimension is zero. This in turn implies $\bar{R}$ is Artinian because it is module over $\bar{Z}$. Thus rank$(m) \leqq 1$, by 2.16.

Now consider the case where Z is arbitrary. Suppose p is a prime ideal of Z. Put $S = Z - p$. By 2.5 RS^{-1} satisfies the hypotheses above, and its center ZS^{-1} has a unique maximal ideal pZS^{-1}. By the case we considered above, $Kd(ZS^{-1}) \leqq 1$. Therefore $Kd(Z) \leqq 1$. □

We can now prove the first of our main results.

THEOREM 2.18. *Let R be a semiprime PI ring whose Krull* *dimension is no greater than one. Suppose that R is ring finite over A. Then if either R or A is Noetherian, R is module finite over its center Z, and both Z and R and Noetherian.*

Proof. Suppose first that R is Noetherian. If R is not module finite over Z, choose an ideal P maximal with respect to the property that R/P is not module finite over Z. P is prime, by 1.6. Suppose P is minimal. Put $R/P = \bar{R}$, $Z(R/P) = C$, and $(Z + P)/P = \bar{Z}$. For every $0 \neq s \in C$, $\bar{R}/\bar{R}s$ is module finite over $(\bar{Z} + \bar{R}s)/\bar{R}s$, by the maximality property of P. By the theorem [9] of Formanek, each $(\bar{Z} + \bar{R}s)/\bar{R}s$ is Noetherian. Thus $(C + \bar{R}s)/\bar{R}s$ is a finitely generated $(Z + \bar{R}s)/\bar{R}s$ module. This tells us two things. First, since $(C + \bar{R}s)/\bar{R}s \cong C/(C \cap \bar{R}s) = C/Cs$, we see that C/Cs is a Noetherian ring for every $0 \neq s \in C$, which implies C is Noetherian. Second, as is easily proved, C is a finitely generated $\bar{Z}$-module (recall that $\bar{Z}$ contains a nonzero ideal of C, as was shown in the proof of 2.11 in the first few lines). Since C is Noetherian, $\bar{R}$ is module finite over C by Formanek's theorem [8]; and so $\bar{R}$ is module finite over $\bar{Z}$. This is a contradiction so P must not be minimal. We claim that $P \cap Z$ is not minimal. To see this, suppose Q is a minimal prime of R contained in P. Just as we noted above, $\tilde{Z} = (Z + Q)/Q$ contains a nonzero ideal I of $\tilde{C} = Z(R/Q)$. If K is any nonzero ideal of $\tilde{C}$, then $KI \subset K \cap \tilde{Z}$ is a nonzero ideal of $\tilde{Z}$. We conclude that every nonzero ideal of R/Q contracts to a nonzero ideal of $\tilde{Z}$. Thus $(P/Q) \cap \tilde{Z}$ is nonzero, which implies that $P \cap Z \not\subset Q \cap Z$.

Thus by 2.17 and 2.7, R/P is finite dimensional over $(Z + P)/P$. This contradiction shows that R must be module finite over Z. Of course, then Z is Noetherian by [9]. This proves the theorem for the case where R is Noetherian.

Now suppose that A is Noetherian. By what we have done above, the theorem will be proved if we can show that R is Noetherian. Since R is a semiprime *PI* ring, it can be embedded in a ring of $n \times n$ matrices over some commutative ring B containing A (see [1]). Suppose $R = A\{x_1, x_2, \cdots, x_n\}$ where each x_i is a matrix with n^2 entries, $b_{ijk} \in B$. Let $\bar{B}$ be the algebra generated over A by all the b_{ijk}. Then $\bar{B}$ is Noetherian, by the Hilbert basis theorem, and R is contained in the ring of $n \times n$ matrices over $\bar{B}$ (cf. [21]). Therefore R has A.C.C. on annihilators; and so by Small's Theorem 1.1, R has a finite number $P_1, P_2, \cdots, P_m$ of minimal prime ideals. (Note that this argument shows that any semiprime *PI* ring which is ring finite over a Noetherian subring A is a Goldie ring.) Now, R is a subdirect product of the rings R/P_i, which means that the map

$$r \longmapsto (r + P_1, r + P_2, \cdots, r + P_m)$$

from R into $\bigoplus(R/P_i)$ is an imbedding. An easy induction will show that R is Noetherian if and only if each of the R/P_i is Noetherian. But R/P_i is Noetherian by the corollary to 1.2. $\square$

If we assume that A is a field, we can sharpen the conclusion of Theorem 2.18:

THEOREM 2.19. *Suppose R is a semiprime PI ring with Krull dimension one. If R is ring finite over a field F, then R is module finite over a central subring of the form $C = F[x]$, where x is transcendental over F.*

Proof. By 2.18, R is a finitely generated Z-module. Thus, by an application of 1.3, Z is ring finite over F. Also, $Kd(Z) = Kd(R) = 1$ by a theorem [17] of Schelter. By the Noether normalization theorem, Z is therefore module finite over a subring of the form $F[x]$ where $x \in Z$, and x is transcendental over F because R and $F[x]$ must both have Krull dimension one.

We would like to know if the conclusion of Theorem 2.18 will hold if we remove the hypothesis that R is semiprime and replace it with some condition on the prime spectrum of R. So far, we have not quite been able to do this, but we can prove such a result if we make the further assumption that A is a Hilbert ring. We now define the relevant concepts and prove our result.

DEFINITION 2.20. A *G-domain* is an integral domain D whose quotient field $Q(D)$ is equal to $D[1/t]$ for some nonzero $t \in D$.

It is well known that a domain is a G-domain if and only if $Q(D)$ is ring finite over D, if and only if the intersection of the nonzero primes of D is nonzero.

DEFINITION 2.21. A prime ideal P of a commutative ring C is called a *G-ideal* if C/P is a G-domain.

Obviously, a maximal ideal of a commutative ring is a G-ideal. We single out for consideration those commutative rings in which the converse is true.

DEFINITION 2.22. A *Hilbert ring* (also *Jacobson ring*) is a commutative ring in which every G-ideal is maximal.

There are many examples of Hilbert rings. Any commutative algebra which is ring finite over a field is a Hilbert ring, as is any countably generated commutative algebra over an uncountable field. We now prove a result due to Amitsur and Procesi which will be used as a lemma to our next theorem.

LEMMA 2.23. *Suppose that S is a PI ring which is ring finite over A, where A is a Hilbert ring. The maximal ideals of S contract to maximal ideals of A.*

Proof. Let M be a maximal ideal of S. By passing to S/M, we may assume $M = 0$. Using Kaplansky's theorem and a generalized version of the Artin-Tate Lemma 1.4, we find that the center of S is a field which is ring finite over A. By the Weak Nullstellensatz, Z is finite dimensional over the quotient field F of A, and therefore S is finite dimensional over F. Applying our version of the Artin-Tate lemma once more with $C = F$, we find that F is ring finite over A which implies that $A = F$ since A was assumed to be a Hilbert ring. $\square$

THEOREM 2.24. *Let S be a PI ring with the following properties:*

(a) *The Krull dimension of S is one or zero.*

(b) *S is ring finite over a Hilbert ring A.*

(c) *If P is a minimal prime ideal of S, then $\ell(P) \cap \mathscr{r}(P) \not\subset P$, where $\ell(P)$ and $\mathscr{r}(P)$ are respectively the left and right annihilators in S of P.*

Then if either A or S has the ascending chain condition on ideals, S is module finite over its center Z, and Z and S are Noetherian.

Proof. Suppose S is not module finite over Z. Choose a prime ideal P of S maximal with respect to S/P not being module finite over Z. (P exists, by 1.7.) If P is maximal then $P \cap A$ is maximal by 2.23. But then S/P is module finite over A by 2.7. So we may assume that P is minimal.

Put $K = \ell(P) \cap \mathscr{r}(P)$. $T = Z(S/P) \cap ((K + P)/P)$ is a nonzero ideal of $Z(S/P)$. Note that if x, y are elements of K which map into T under the canonical epimorphism $S \to \bar{S} = S/P$, then we have $[xy, r] = x[y, r] + [x, r]y \in KP + PK = 0$, for every $r \in S$. Therefore T^2 is a nonzero ideal of $C = Z(S/P)$ which is contained in $\bar{Z} = (Z + P)/P$ (cf. 2.6).

By 2.18, $\bar{S}$ is module finite over C, and $\bar{S}$ and C are Noetherian. Choose $0 \neq t \in T^2$. Then $\bar{S}t \neq \bar{S}$ and $Ct = \bar{S}t \cap C$ is a nonzero ideal of C contained in $\bar{Z}$. Since $\bar{S}/\bar{S}t$ is module finite over $\bar{Z}/Ct$, $\bar{Z}/Ct$ is

Noetherian [9]. Thus C/Ct is module finite over $\bar{Z}/Ct$. It follows that C is module finite over $\bar{Z}$. Therefore $\bar{S}$ is module finite over $\bar{Z}$. This contradiction completes the proof that S is module finite over Z. If S has the A.C.C on ideals, then Z is Noetherian [9]. If A is Noetherian, then Z is ring finite over A by 1.3, and hence Z is Noetherian. Since Z is Noetherian in either case, S is too, because it is module finite over Z. $\square$

There is one more result that we would like to prove here. It is a generalization of 2.19 which applies to the case where the ring has a nilpotent ideal.

THEOREM 2.25. *Let S be a left Noetherian PI ring which is ring finite over a field F and which has Krull dimension one. Then S is left module finite over a commutative subring $C = F[x]$, where $x \in S$ is transcendental over F and x is central modulo every minimal prime of S.*

Proof. Let $P(S)$ be the lower prime radical of S. By 2.19, we can find $x \in S$ such that x is transcendental over F and $S/P(S)$ is left module finite over $C = F[x]$. By 1.8, an ideal I of S maximal with respect to the property that S/I is not left module finite over C is prime. Since S is module finite over C modulo every prime, this implies S is module finite over C.

3. Examples and open questions. In this section we will prove by giving examples that the main results of § 2 are about as good as can be proved.

EXAMPLE 3.1 (Schelter, Wadsworth). Let F be a field with characteristic zero, and let x be an indeterminate. Put $L_1 = F(x^2)$ and $L_2 = F((x+1)^2)$, the rational functional function fields in x^2 and $(x+1)^2$ respectively. A routine calculation will show that $L_1 \cap L_2 = F$. Let y be another indeterminate, with $[x, y] = 0$, and let (y) be the ideal of $F(x)[y]$ generated by y. Define the ring B to be

$$\begin{pmatrix} L_1 + (y) & (y) \\ (y) & L_2 + (y) \end{pmatrix} = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \middle| \, a_{12}, a_{21} \in (y);\ a_{ii} \in L_i + (y) \quad \text{for} \quad i = 1, 2 \right\} .$$

Since $L_1 + (y)$ and $L_2 + (y)$ are module finite over $L_1[y]$ and $L_2[y]$ respectively, they are Noetherian rings. Since B is module finite over the Noetherian subring

$$\begin{pmatrix} L_1 + (y) & 0 \\ 0 & L_2 + (y) \end{pmatrix}.$$

B is Noetherian. Also, it is obvious that *B is prime, PI,* and $Kd(B) = 1$. But $Z(B)$, which is clearly isomorphic to $(L_1 + (y)) \cap (L_2 + (y)) = F + (y)$, is non-Noetherian. Thus *B is not module finite, over* $Z(B)$. This proves:

Item 3.1.1. We cannot drop from 2.18, 2.19, 2.24, or 2.25 the hypothesis that R is ring finite over Z.

EXAMPLE 3.2 (Small). Let A be the ring

$$\begin{pmatrix} F[x] & F[x] \\ 0 & F \end{pmatrix}$$

where F is any field. *A is PI, ring finite over F, and* $Kd(A) = 1$. The minimal primes of A are

$$P_1 = \begin{pmatrix} F[x] & F[x] \\ 0 & 0 \end{pmatrix} \text{ and } P_2 = \begin{pmatrix} 0 & F[x] \\ 0 & F \end{pmatrix}$$

and we have $r(P_1) = \ell(P_2) = 0$, $\ell(P_1) = P_2$, and $r(P_2) = P_1$. Thus $\ell(P_i) \cap r(P_i) = 0$ for $i = 1, 2$ and therefore A shows that the following is true:

Item 3.2.1. We cannot remove from 2.24 the hypothesis that $\ell(P) \cap r(P) \not\subset P$ for each minimal prime P of S.

Now take an infinite ascending sequence of F-vector subspaces of $F[x]$, $S_1 < S_2 < S_3 < \cdots$. Put

$$I_j = \begin{pmatrix} 0 & S_j \\ 0 & 0 \end{pmatrix}.$$

Then I_j is a right ideal of A. Put

$$B = \begin{pmatrix} F & A \\ 0 & A \end{pmatrix}.$$

B is another example of interest. It is easy to see that *B is a PI ring which is ring finite over F,* that $Z(B) \cong F$, and that $Kd(B) = 1$. Also, for each I_j,

$$\begin{pmatrix} 0 & I_j \\ 0 & 0 \end{pmatrix}$$

is an ideal of B; so B does not have the A.C.C. on ideals. Thus B

cannot be module finite over any commutative subring (1.3). Thus we have:

Item 3.2.2. We cannot remove from 2.18 or 2.19 the hypothesis that R is semiprime.

Item 3.2.3. We cannot remove from 2.25 the hypothesis that S is left Noetherian.

EXAMPLE 3.3 (Cauchon). Let F be a field with characteristic zero. Let $S = F[x, y]$, the commutative polynomial ring. Put

$$B = \left\{ \begin{pmatrix} f + xs & \dfrac{\partial f}{\partial y} + xt \\ xu & f + xv \end{pmatrix} \middle| f, s, t, u, v \in S \right\}.$$

Then $Z(B) \cong F + xS = \{f \in S \mid \partial f/\partial y \in Sx\}$. B is module finite over the ring

$$\bar{S} = \left\{ \begin{pmatrix} f & \dfrac{\partial f}{\partial y} \\ 0 & f \end{pmatrix} \middle| f \in S \right\}$$

which is isomorphic to S, but is not contained in $Z(B)$. Since $\bar{S}$ is Noetherian, *B is Noetherian. B is also ring finite over F,* and *B is prime.* But since $Z(B)$ is not Noetherian, *B cannot be module finite over* $Z(B)$ [9]. Note $Kd(B) = 2$.

Item 3.3.1. Analogues of 2.18, 2.19, 2.24, and 2.25 do not exist for the case where the Krull dimension is greater than one.

EXAMPLE 3.4. Let F be a field with characteristic zero. Put

$$B = F\left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} x & 0 & 0 \\ 1 & x & 0 \\ 0 & 0 & x \end{pmatrix} \right\} = \left\{ \begin{pmatrix} f & 0 & 0 \\ f' & f & 0 \\ h & d & f \end{pmatrix} \middle| f, h, d \in F[x] \right\}.$$

One may verify that

$$Z(B) = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ h & 0 & \alpha \end{pmatrix} \middle| \alpha \in F; h \in F[x] \right\}.$$

B is Noetherian, PI, ring finite over F, and $Kd(B) = 1$, but B is not module finite over $Z(B)$. Thus we have:

Item 3.4.1. We cannot improve 2.25 to yield the conclusion that $x \in Z(S)$.

Note that this example also proves Item 3.2.2. And Example 3.4 is Noetherian, unlike the second example in 3.2.

The foregoing examples show that, aside from the Noetherian hypothesis, none of the hypotheses of 2.18, 2.19, 2.24, or 2.25 can be removed without weakening the conclusions. (It is well known that any ring which is module finite over a commutative subring is *PI*—so the *PI* hypothesis obviously cannot be removed.)

There is an example, which we do not wish to describe here, of a *PI* ring which is Noetherian and ring finite over a field but *not module finite over any commutative subring*. The example is due to G. Bergman, Bergman's example has Krull dimension two, and *I* have recently constructed an example, based on Bergman's, of a *prime PI ring which is Noetherian, ring finite over a field* and *not module finite over any commutative subring*. The new example has Krull dimension three. Both of the examples are explicated in [16].

Here is a list of open questions:

(i) Suppose R is a right Noetherian *PI* ring which is ring finite over a Noetherian ring, and $Kd(R) = 1$. Is R module finite over a commutative subring?

(ii) Is there a prime Noetherian *PI* ring which is not module finite over any commutative ring?

(iii) Can any affine Noetherian *PI* ring be embedded in a ring of matrices over a commutative ring? (cf. [3])

(iv) Is there an example of a prime *PI* ring with Krull dimension one which is ring finite but not module finite over its center? (We feel the answer is yes, but that such examples are difficult to construct.)

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