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A SPECTRAL MAPPING THEOREM FOR LOCALLY COMPACT GROUPS OF OPERATORS

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If U is a suitably continuous representation of a locally compact abelian group G by means of isometries on a Banach space X, $\mu \to U(\mu)$ its extension to a representation of the convolution algebra $M(G)$ and $\mathrm{sp}(U)$ the spectrum of U, then the spectrum of $U(\mu)$ is not always equal to $\hat{\mu}(\mathrm{sp}(U))^-$, but it is so if the continuous part of μ is absolutely continuous.

1. Introduction. To be more explicit, given a representation U of G as above one forms a representation of $M(G)$, the Banach algebra of bounded regular measures on G, given by

$$U\colon \mu \in M(G) \longrightarrow U(\mu) = \int U(g)d\mu(g) \in B(X) .$$

In particular, if $G = \boldsymbol{R}$, $U(\mu)$ can be interpreted in a more classical way as a function of the infinitesimal generator $D = i(d/dg)U(g)|_{g=0}$ and denoted by $\hat{\mu}(D)$, where $\hat{\mu}$ is the Fourier transform of μ. Notice that in this case $\sigma(D) = \mathrm{sp}(U)$ [5, 9], where σ is the usual spectrum of the linear operator D and $\mathrm{sp}(U)$ is the spectrum of the representation U (see [2]).

Thus it is natural to study how far this functional calculus can be extended and a spectral mapping theorem holds. The setting of our study will be the algebra of local multipliers of $L^1(G)$.

If μ is a Dirac measure, A. Connes [3] proved that

$$\sigma(U(\mu)) = \hat{\mu}(\mathrm{sp}(U))^- .$$

Even if such a result does not always extend (we shall exhibit counterexamples) we prove it for the class of measures whose continuous part belongs to $L^1(G)$.

2. Statement of the main result. Let G be a locally compact abelian group; by a representation U of G on a Banach space X we mean a pointwise $\sigma(X, X^*)$-continuous homomorphism of G into the group of $\sigma(X, X^*)$-continuous isometries of X, where X^* is the dual of X or X is the dual of X^*. The case of bounded representations reduces to this one.

Let $M(G)$ be the Banach algebra of all bounded regular measures on G. Given any algebra $L^1(G) \subset M \subset M(G)$, we can form the representation of M induced by U:

$$U(\mu) = \int U(g)d\mu(g)\ , \quad \mu \in M\ . \tag{2.1}$$

The spectrum of U is a closed subset of the dual $\hat{G}$ of G defined by means of Fourier transforms (see [2] for this and related matter)

$$\mathrm{sp}(U) = \{p \in \hat{G} / U(\mu) = 0 \Longrightarrow \hat{\mu}(p) = 0\ , \quad \mu \in M\} \tag{2.2}$$

and it does not depend on M.

The following lemma will be later generalized.

LEMMA 1. *For every $\mu \in M(G)$ we have $\sigma(U(\mu)) \supset \hat{\mu}(\mathrm{sp}(U))^-$, where σ denotes the usual spectrum in $B(X)$.*

Proof. We have to show that if $p \in \mathrm{sp}(U)$, then $\hat{\mu}(p) \in \sigma(U(\mu))$. Indeed if $p \in \mathrm{sp}(U)$ there exists a net $\{x_i\} \subset X$, $\|x_i\| = 1$, such that $\|U(g)x_i - (p, g)x_i\| \to 0$ uniformly for g varying in a compact subset $K \subset G$. Let $\varepsilon > 0$ and $K \subset G$ such that $|\mu|(G \backslash K) < \varepsilon/4$ and choose $x \in \{x_i\}$ such that

$$\left\| \int_K U(g)d\mu(g)x - \int_K (p, g)d\mu(g)x \right\| < \varepsilon/2$$

then

$$\begin{aligned} \| U(\mu)x_i - \hat{\mu}(p)x_i \| \leqq & \left\| \int_K U(g)d\mu(g)x_i - \int_K (p, g)\, d\mu(g)\, x_i \right\| \\ & + \left\| \int_{G\backslash K} U(g)d\mu(g)x_i - \int_{G\backslash K} (p, g)d\mu(g)x_i \right\| < \varepsilon \end{aligned}$$

that entails the lemma. □

REMARK 1. The reverse inclusion in the above lemma is not true for every $\mu \in M(G)$: in fact, if G is not discrete, $X = L^1(G)$ and $U(g)$ is the translation by g, then, due to asymmetry of $M(G)$, there exists $\mu_0 \in M(G)$ such that $\sigma(U(\mu_0)) \neq \hat{\mu}_0(\hat{G})^-$ (see [11]). By the same reasoning we can give a counterexample for automorphism groups of factors. Indeed let $\alpha = U'$ be the transposed action on $L^\infty(G)$; we have $\hat{\mu}_0(\hat{G}) \subsetneqq \sigma(U(\mu_0)) = \sigma(U'(\mu_0)) = \sigma(\alpha(\mu_0))$. If $\tilde{\alpha}$ is an extension of α to $B(L^2(G))$, then $\hat{\mu}_0(\hat{G}) \subsetneqq \sigma(\alpha(\mu_0)) \subset \sigma(\tilde{\alpha}(\mu_0))$.

THEOREM 1. *For every $\mu \in M(G)$ whose continuous part belongs to $L^1(G)$ we have $\sigma(U(\mu)) = \hat{\mu}(\mathrm{sp}(U))^-$.*

The proof of this theorem requires some lemmas.

3. Identification of spectra. Let M be a subset of $M(G)$; by

$A(M)$ we denote the closure in $B(X)$ of $\{U(\mu): \mu \in M\}$. We recall the following identification of $\mathrm{sp}(U)$. [3, Prop. 2.3.7].

PROPOSITION 2. *The map* $p \in \mathrm{sp}(U) \to j(p) \in \sigma(A(L^1(G)))$ *defined by*

$$j(p)(U(f)) = \hat{f}(p) \quad f \in L^1(G) \tag{3.1}$$

establishes an homeomorphism of $\mathrm{sp}(U)$ *onto the spectrum of* $A(L^1(G))$.

If M is a Banach algebra and $L^1(G) \subset M \subset M(G)$ we split $\sigma(A(M))$ into two disjoint sets $\sigma(A(M)) = H(M) \cup \Omega(M)$ where $H(M) = \{\chi \in \sigma(A(M))/\chi \upharpoonright A(L^1(G)) = 0\}$ and $\Omega(M)$ is the complementary subset.

LEMMA 3. (i) *The map* $\chi \in \Omega(M) \to \chi \upharpoonright A(L^1(G))$ *is an homeomorphism of* $\Omega(M)$ *onto the spectrum of* $A(L^1(G))$.

(ii) *Let* $\pi: A(M) \to A(M)/A(L^1(G))$ *be the quotient map. Then* $\varphi \in \sigma(A(M)/A(L^1(G))) \to \varphi \cdot \pi$ *is an homeomorphism of* $\sigma(A(M)/A(L^1(G)))$ *onto* $H(M)$.

Proof. (i) Let $\chi_0 \in \sigma(A(L^1(G)))$. By Proposition 2 there exists $p \in \mathrm{sp}(U)$ such that $j(p) = \chi_0$. It is enough to show that χ_0 uniquely extends to $\chi \in \sigma(A(M))$ determined by $\chi(U(\mu)) = \hat{\mu}(p)$. In fact let $f \in L^1(G)$, $\hat{f}(p) \neq 0$. Then $\chi(U(\mu)U(f)) = \chi(U(\mu^* f))$, $\mu \in M$, thus $\chi(U(\mu))\hat{f}(p) = \hat{\mu}(p)\hat{f}(p)$ and $\chi(U(\mu)) = \hat{\mu}(p)$ for any extension χ of χ_0.

(ii) This fact is known to be valid in more general situations [10, §15]. □

Let G_d be the group obtained equipping G with the discrete topology and U_d the representation of G_d naturally derived by U. It follows that $\mathrm{sp}(U_d) = \mathrm{sp}(U)^-$ [2], where the closure will be always taken in $\hat{G}_d$ the Bohr compactification of $\hat{G}$. Proposition 2, with $G = G_d$, gives rise to a natural identification of $\mathrm{sp}(U_d)$ with $\sigma(A(M_d(G)))$

$$\begin{aligned} &p \in \mathrm{sp}(U_d) \longrightarrow j_d(p) \in \sigma(A(M_d)) \\ &j_d(p)(U(\mu)) = \hat{\mu}(p) \end{aligned} \tag{3.2}$$

where $M_d(G) = M(G_d) = L^1(G_d)$ is the Banach algebra of discrete measure on G and $\hat{\mu}$ is the Fourier transform of μ as an element of $L^1(G_d)$.

The Banach algebra of measures of interest to us will be

$$\mathscr{M} = \{\mu \in M(G)/\mu_c \in L^1(G)\}$$

where μ_c is the continuous part of μ. Let $\mathscr{A} = A(\mathscr{M})$, $\mathscr{H} = H(\mathscr{M})$ and $\Omega = \Omega(\mathscr{M})$ which is homeomorphic to $\mathrm{sp}(U)$. We define

$$\mathrm{sp}_d(U) = \{p \in \mathrm{sp}(U_d)/\exists \chi \in \mathscr{H} \text{ s.t. } j_d(p) = \chi \restriction A(M_d)\} . \tag{3.3}$$

LEMMA 4. *If G is nondiscrete $\mathrm{sp}_d(U)$ is naturally homeomorphic to $\mathscr{H}$ by the following map:*

$$p \in \mathrm{sp}_d(U) \longrightarrow \chi \restriction A(M_d) = j_d(p) . \tag{3.4}$$

Proof. If $\chi \in \mathscr{H}$ and $\chi \restriction A(M_d) \neq 0$, then by (3.2) there exists $p \in \mathrm{sp}(U_d)$ such that $\chi(U(\mu)) = \hat{\mu}(p)$ for every $\mu \in M(G_d)$. Obviously $j_d(p) = \chi \restriction A(M_d)$, therefore $p \in \mathrm{sp}_d(U)$ by definition, and the map in (3.4) is continuous. On the other hand for any $p \in \mathrm{sp}_d(U)$, $j_d(p)$ extends to $\chi \in \mathscr{H}$ by $\chi(U(\mu)) = \hat{\mu}(p)$ establishing a continuous inverse of the above map. □

4. Topological lemmas. Let G and G_d be as above. We shall identify $M_d(G)$ and $L^1(G_d)$. No confusion will arise since, if $\mu \in L^1(G_d)$, then $\hat{\mu} \restriction \hat{G}$ is the Fourier transform of μ as an element of $M_d(G)$.

LEMMA 5. *For each compact subset K in $\mathrm{sp}(U)$ we have*

$$\mathrm{sp}_d(U) \subset \overline{\overline{\mathrm{sp}(U)}\backslash K} .$$

Proof. Let us assume that there is a $p \in \mathrm{sp}_d(U)$ such that p does not belong to $\overline{\overline{\mathrm{sp}(U)}\backslash K}$. This will lead to a contradiction. Indeed if the thesis is not fulfilled there is an open set V in $\hat{G}_d$ such that V contains $p \in \mathrm{sp}_d(U)$ and $V \cap \overline{(\mathrm{sp}(U)\backslash K)} = \varnothing$. This means that $V \cap \overline{\mathrm{sp}(U)} \subset K$. Let μ be a measure in $L^1(G_d) = M(G_d)$ such that $\mathrm{supp}\,(\hat{\mu}) \subset V$, $\hat{\mu}(p) = 1$. Therefore $\mathrm{supp}\,(\hat{\mu}) \cap \mathrm{sp}(U) \subset K$. As K is compact there exists $f \in L^1(G)$ such that $U(\mu) = U(f)$. If $\chi \in H$ is the character corresponding to $p \in \mathrm{sp}_d(U)$ as in (2.4), then $\chi\,(U(f)) = 0$ and $0 = \chi(U(f)) = \chi(U(\mu)) = 1$. □

The following lemma can be proved by elementary consideration.

LEMMA 6. *Let K be a compact set, F a closed set with $K \subset F \subset \hat{G}$. Then $\overline{\overline{F}\backslash K} \subset \overline{F\backslash K}$, where, as always, the closure are taken in $\hat{G}_d$. In particular for any compact set $K \subset \mathrm{sp}(U)$, we have $\mathrm{sp}_d(U) \subset \overline{\mathrm{sp}(U)\backslash K}$.*

5. Proof of Theorem 1. Let $\mu \in M(G)$ be such that $\mu = \mu_c + \mu_d$ with $\mu_c \in L^1(G)$ and $\mu_d \in M_d(G)$. We have to show that $\hat{\mu}(\mathrm{sp}(U))^- \supset \sigma(U(\mu))$. Since $\sigma(U(\mu)) \subset \sigma_A(U(\mu))$ (where σ_A is the spectrum relative to $\mathscr{A}$), it is sufficient to prove that

$$\hat{\mu}(\mathrm{sp}(U))^- \supset \sigma_A(U(\mu)) .$$

It is enough to show that if $0 \notin \hat{\mu}(\mathrm{sp}(U))^-$ then $U(\mu)$ is invertible in $\mathscr{A}$. That is $\chi(U(\mu)) \neq 0$ for every $\chi \in \sigma(\mathscr{A})$. Assume $0 \notin \hat{\mu}\ (\mathrm{sp}(U))^-$ and let $\varepsilon_0 > 0$ be such that

$$(5.1) \qquad |\hat{\mu}(p)| \geqq \varepsilon_0 \quad p \in \mathrm{sp}(U) .$$

If $\chi \in \sigma(\mathscr{A})$ there are two possibilities, $\chi \in \Omega$ or $\chi \in \mathscr{H}$ (see 3).

(a) If $\chi \in \Omega$ then there exists $p \in \mathrm{sp}(U)$ such that $\chi(U(\mu)) = \hat{\mu}(p)$ for every $\mu \in \mathscr{M}$, therefore

$$|\chi(U(\mu))| = |\hat{\mu}(p)| \geqq \varepsilon_0 > 0 .$$

(b) If $\chi \in \mathscr{H}$ let $p_0 \in \mathrm{sp}_d(U)$ be such that (cf Lemma 4)

$$(5.2) \qquad \chi(U(\mu)) = \hat{\mu}_d(p_0)$$

where μ_d is considered as an element of $L^1(G_d)$. Let

$$K = \{p \in \mathrm{sp}(U) / |\hat{\mu}_c(p)| \geqq \varepsilon_0/2\} ,$$

then, since $\hat{\mu}_c$ vanishes at infinity, K is compact. Since $|\hat{\mu}_c(p) + \hat{\mu}_d(p)| \geqq \varepsilon_0$ for $p \in \mathrm{sp}(U)$, we have $|\hat{\mu}_d(p)| \geqq \varepsilon_0/2$ for $p \in \mathrm{sp}(U) \backslash K$. Since $\hat{\mu}_d$ is continuous on $\hat{G}_d$ we have $|\hat{\mu}_d(p)| \geqq \varepsilon_0/2$ for every p in $\overline{\mathrm{sp}(U) \backslash K} \supset \mathrm{sp}_d(U)$, and therefore, by (4.2), $|\chi(U(\mu))| = |\hat{\mu}_d(p_0)| \geqq \varepsilon_0/2 > 0$ because $p_0 \in \mathrm{sp}_d(U)$. □

6. Functional calculus for local multipliers. We consider now an involutive algebra $\mathfrak{M} = \mathfrak{M}(G, U)$ of local multipliers for $L^1(G)$, namely $\hat{F} \in \mathfrak{M}$ iff $\hat{F}$ is a complex function defined on a neighborhood of $\mathrm{sp}(U)$ and locally belongs to $L^1(G)$ at every point $p \in \mathrm{sp}(U)$.

Let $D_0(U)$ be the union of the spectral subspaces $X(E, U)$ of U corresponding to compact subsets E of G (cf [2, 12])

$$(6.1) \qquad D_0(U) = \bigcup_E X(E, U) , \quad E \text{ compact subset of } G .$$

Owing to the regularity of $L^1(G)$, we can define, for every $\hat{F} \in \mathfrak{M}$, the linear operator $U(F)\colon D_0(U) \subset X \to X$ by

$$(6.2) \qquad U(F)x = U(f)x , \quad x \in X(E, U) , \quad E \text{ compact}$$

where f is an arbitrary element of $L^1(G)$ such that $\hat{f}$ is equal to $\hat{F}$ on a neighborhood of E. In such a way $\{U(F), \hat{F} \in \mathfrak{M}\}$ becomes an involutive algebra of operators of X on the common dense invariant domain $D_0(U)$ (with involution given by $U(F) \to U(\bar{F})$). Every $U(F)$ is closable because $D(U(F)')$, the domain of the transposed of $U(F)$, is dense in X^*, as shown in the following lemma. Note that U', the transposed representation of U, is $\sigma(X^*, X)$-continuous; if $\mu \in$

$M(G)$, $U'(\mu)$ is a bounded linear operator of X^* and $(U(\mu))' = U'(\mu)$. Define $D_0(U') \subset X^*$ as in (6.1).

LEMMA 7. *$D_0(U')$ is contained in $D(U(F)')$ for every $\hat{F} \in \mathfrak{M}$.*

Proof. Fix a compact $E \subset \hat{G}$ and $\varphi \in X^*(E, U')$. If $x \in D_0(U)$ there exists K compact $K \subset \hat{G}$ such that $x \in X(K, U)$. Let $f \in L^1(G)$ such that $\hat{f}(p) = \hat{F}(p)$ if p belongs to a neighborhood of $E \cup K$; we have

$$(U(F)x, \varphi) = (U(f)x, \varphi) = (x, U(f)'\varphi) = (x, U'(f)\varphi) = (x, U'(F)\varphi)$$

that shows $U(F)' \supset U'(F)$. □

We recall that if T is a linear operator on a Banach space X, the extended spectrum $\sum(T)$ is defined as the set of the singularities of the resolvent of T in $\boldsymbol{C} \cup \{\infty\}$.

LEMMA 8. *For every $\hat{F} \in \mathfrak{M}$ we have $\hat{F}(\mathrm{sp}(U))^- \subset \sum(U(F))$.*

Proof. As $\sum(U(F))$ is closed, it is enough to prove that $\sum(U(F)) \supset \hat{F}(\mathrm{sp}(U))$. To show this, we consider the representation $U^E : g \in G \to U(g) \restriction X(E, U)$ obtained by reducing U to the spectral subspace relative to $E \subset \hat{G}$. Let $E \subset \hat{G}$ be a compact set and $f \in L^1(G)$ such that $\hat{f} = \hat{F}$ on a neighborhood of E, so that $U(F) \restriction X(E, U) = U(f) \restriction X(E, U) = U^E(f)$. Owing to the regularity of $L^1(G)$ we have $\mathrm{sp}(U^E) \subset \mathrm{sp}(U) \cap E$, hence

$$\sum(U(F)) \supset \sum(U(F)) \restriction X(E, U) = \sum(U^E(f)) = \hat{f}(\mathrm{sp}(U^E)) = \hat{F}(\mathrm{sp}(U^E)$$

where the second equality is justified by Theorem 1. Since

$$\mathrm{sp}(U) = \bigcup_E \mathrm{sp}(U^E)\ , \quad E \text{ compact subset of } G\ ,$$

the lemma is proved. □

The reverse inclusion in the above lemma cannot be proved for every bounded $\hat{F}$.

PROPOSITION 9. *Let $\hat{F}$ be a bounded continuous function in $\mathfrak{M}$ which is not Fourier transform of a measure of $M(G)$. If U is the representation of G on $L^1(G)$ by translations given by $(U(g)f)(h) = f(g^{-1}h)$, $f \in L^1(G)$ then $\sum(U(F)) \supsetneqq \hat{F}(\mathrm{sp}(U).)$*

Proof. We shall derive from our hypotheses that $\sum(U(F))$ cannot be compact. Assuming the contrary there exists a regular closed Jordan curve Γ containing $\sum(U(F))$ in the interior $\mathring{\Gamma}$. Let $P = (-1/2\pi i)\oint_\Gamma (U(F) - \lambda)^{-1} d\lambda$. P is a projection of $B(L^1(G))$ that commutes with $U(g)$, $g \in G$ and decomposes $U(F)$ according to $U(F) = U(F)P + U(F)(I - P)$. We have $\sum(U(F) \upharpoonright PX) = \sum(U(F)) \cap \mathring{\Gamma} = \sum(U(F))$, $\sum(U(F) \upharpoonright (I-P)X) = \sum(U(F)) \cap (\boldsymbol{C}\backslash\mathring{\Gamma}) = \varnothing$. As $U(F)P$ is bounded and commutes with $U(g)$, $g \in G$, $U(F)P$ is a multiplier of $L^1(G)$ [11]. Therefore $U(F)(I - P) = U(F) - U(F)P$ is a local multiplier. As

$$\sum(U(F)(I - P)) \subset \{0\},$$

by Lemma 8 we have that $U(F)(I - P)$ is a multiplier by a function vanishing on $\mathrm{sp}(U) = \hat{G}$, thus $U(F)(I - P) = 0$. □

REMARK 2. The case of unbounded local multiplier F often reduces to that of a bounded one, for example $(\hat{F} - \lambda)^{-1}$, if λ does not belong to the closure of the range of F. Note that if $G = \boldsymbol{R}$ and D is the generator of U, the spectral mapping theorem for $\hat{F}(D) = U(F)$ assumes the usual form $\sum(\hat{F}(D)) = \hat{F}(\sum(D))$.

Some functions may be of particular interest. If $\hat{F}(t) = e^t + e^{-t}$, the closure of $\hat{F}(D)$ is the inverse of the symmetric resolvent of D [1]. If $\hat{F}(t) = e^t$, then $\hat{F}(D)$ is the analytic generator of U [1]; in this case the spectral mapping theorem does not hold [4, 13], indeed either $\sum(\hat{F}(D)) = \hat{F}(\sum(D))$ or $\sum(\hat{F}(D)) = \boldsymbol{C}$. The second alternative being true for every nontrivial one parameter *-automorphism group of a commutative C^*-algebra.

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