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NOTE ON EXPONENTIAL POLYNOMIALS

LÁSZLÓ SZÉKELYHIDI

It is known that every finite dimensional translation invariant subspace of measurable functions on a σ-compact locally compact Abelian group consists of exponential polynomials. This paper extends this result for continuous functions on arbitrary commutative topological groups. An analogous characterization is proved for trigonometric polynomials using Fourier transformation.

In this paper joining with the investigations of Engert [3] and Laird [5] we prove that every finite dimensional translation invariant subspace of continuous functions on arbitrary commutative topological groups consists of exponential polynomials. Our method is similar to that of [3] but we prove the important lemma of Engert in a simpler way using generalized polynomials. In contrast with [3] and [5] here the main emphasis is on functional equations. In the last part we prove an analogous result for bounded continuous functions using the Fourier transform of almost periodic functions. We note that translation invariant finite dimensional subspaces of the space of finite signed measures on a commutative topological group can be characterized in a similar way by the same technique.

If G is an Abelian group then an additive function on G is a complex valued function a such that $a(x + y) = a(x) + a(y)$ for all x and y in G. A multiplicative function on G is a complex valued function m such that $m(x + y) = m(x)m(y)$ for all x and y in G. If n is a positive integer then we mean by an n-additive function on G a complex valued function on G^n which is additive in each variable. We define generalized polynomials on G as functions satisfying the so called Fréchet equation: $\Delta_y^{n+1}f(x) = 0$. (Here Δ_y denotes the difference operator: $\Delta_y f(x) = f(x + y) - f(x)$ and $\Delta_y^{n+1}f = \Delta_y(\Delta_y^n f)$.) Functions with this property are called generalized polynomials of degree at most n. It is well-known (see e.g., [2], [6], [8]) that every complex valued generalized polynomial of degree at most n can be uniquely expressed in the form $\sum_{k=0}^n A^{(k)}$ where $A^{(k)}$ is the diagonalization of a k-additive, symmetric function A_k, that is $A^{(k)}(x) = A_k(x, x, \cdots, x)$ ($A^{(0)}$ is a constant). For more about generalized polynomials on groups see [6], [8].

If G is a topological Abelian group then by a polynomial on G we mean a function of the form $p(x) = P(a_1(x), \cdots, a_n(x))$ where P is a complex polynomial in n variables and a_i $(i = 1, \cdots, n)$ is a continuous additive function. An exponential polynomial on G is a function of the form $\sum_{i=1}^{n} p_i \cdot m_i$ where p_i is a polynomial and m_i is a continuous multiplicative function. By a trigonometric polynomial on G we mean a linear combination of characters, that is continuous multiplicative functions into the complex unit circle.

A multi-index $p = (p_1, \cdots, p_n)$ is an n-tuple of nonnegative integers and if $(a_1, \cdots, a_n)$ is a complex n-tuple, then a^p is defined to be $a_1^{p_1} \ldots a_n^{p_n}$. (For more details on the notation see [3], [4], [5].)

THEOREM 1. *Let f be a continuous function on the topological group G such that the complex linear space spanned by $\{\Delta_y f\colon y \in G\}$ is a finite dimensional space of polynomials. Then f is a polynomial.*

Proof. Let $a_1, \cdots, a_n$ be a finite set of continuous additive functions such that all polynomials in the subspace V spanned by $\{\Delta_y f\colon y \in G\}$ are built up from these functions and $\{a^p\}$ are linearly independent for $|p| \leqq N$. Then

$$\Delta_y f = \sum_{|p| \leqq N} (P/y) a^p$$

holds for all y in G. We see that f satisfies the Fréchet equation $\Delta_y^{N+2} f = 0$ and hence we have the representation

$$f = \sum_{k=0}^{N+1} A^{(k)} .$$

This yields

$$\Delta_y f(x) = \sum_{k=0}^{N+1} [A^{(k)}(x + y) - A^{(k)}(x)] .$$

On the right hand side we have only one member which is of degree N in x. This is $A_{N+1}(x, x, \cdots, x, y)$. It follows that

$$\sum_{|p|=N} (p/y) a^p = A_{N+1}(x, x, \cdots, x, y)$$

holds for all x and y in G. Since the right hand side is additive in y we have

$$\sum_{|p|=N} [c_p(y + z) - c_p(y) - c_p(z)] a^p = 0 .$$

Here the functions a^p ($|p| = N$) are linearly independent and we conclude that c_p is additive for all $|p| = N$. Hence $A^{(N+1)}$ is a polynomial. Repeating this argument for the function $f - A^{(N+1)}$ we get the statement by induction.

THEOREM 2. *Let V be a translation invariant finite dimen-*

sional vectorspace of continuous functions on a topological Abelian group. Then every function in V is an exponential polynomial.

Proof. Let $g_1, \cdots, g_n$ be a basis for V, then for every function f in V the functional equation

$$f(x + y) = \sum_{i=1}^{n} g_i(x)h_j(y)$$

holds. Let $x_1, \cdots, x_n$ be elements of the group G for which the matrix $(g_i(x_j))$ is regular. We may suppose without loss of generality that this matrix is the identity matrix. We introduce the notations $\widehat{f}(x) = (f(x_1 + x), \cdots, f(x_n + x))$, $M(x) = (g_i(x_j + x))$ $\widetilde{h}(x) = (h_1(x), \cdots, h_n(x))$ for all x in G. Then we have the functional equation

$$\widetilde{f}(x + y) = M(x)\widetilde{h}(y) .$$

which shows that $\widetilde{f} = \widetilde{h}$ and the subspace in $\boldsymbol{C}^n$ generated by the range of $\widetilde{f}$ is invariant under $M(x)$ for all x. ($\boldsymbol{C}$ denotes the set of complex numbers.) We may suppose that this subspace is $\boldsymbol{C}^n$, then we have for all x and y in G

$$M(x + y) = M(x)M(y) .$$

The matrices $M(x)$ commute for all x hence they can be transformed into triangular form simultaneously. Since after a similarity transformation our equation remains valid we may suppose that the matrices are all triangular from above. It is easy to see that the diagonal elements are all multiplicative functions. Let $D(x)$ be the diagonal matrix for which $M(x) - D(x)$ is strictly triangular from above for all x. Then $D(x + y) = D(x)D(y)$ and with the notation $A(x) = D^{-1}(x)M(x)$ we have

$$A(x + y) = A(x)A(y)$$

and all diagonal elements of $A(x)$ are 1. This equation means for the components A_{ij} of A

$$A_{ij}(x + y) = \sum_{k=1}^{n} A_{ik}(x)A_{kj}(y) = \sum_{k=i+1}^{j-1} A_{ik}(x)A_{kj}(y) + A_{ij}(x) + A_{ij}(y) .$$

We prove by induction on $j - i$ that A_{ij} is a polynomial. For $i = j$ this is trivial. Supposing that it is valid for $j - i \leqq l$ we see that

$$\Delta_y A_{i,i+l+1}(x) = A_{i,i+l+1}(y) + \sum_{k=i+1}^{i+l} A_{ik}(x)A_{k,i+l+1}(y)$$

hence the subspace spanned by $\{\Delta_y A_{i,i+l+1}, y \in G\}$ is contained in the subspace spanned by 1, A_{ik} $(k = i + 1, \cdots, i + l)$ (l), that is, consists of polynomials. Thus by Theorem 1 all components of M are exponential polynomials. Finally, by

$$\tilde{f}(x) = M(x)\tilde{f}(0)$$

we conclude that f is an exponential polynomial.

THEOREM 3. *Let V be a translation invariant finite dimensional vectorspace of continuous bounded functions on a topological Abelian group which has sufficiently many characters. Then every function in V is a trigonometric polynomial.*

Proof. Using the notations of the previous theorem we have that for every function f in V the functional equation

$$f(x + y) = \sum_{i=1}^{n} g_i(x)h_i(y)$$

holds. Since the functions g_i, f are bounded and $g_1, \cdots, g_n$ are linearly independent, it follows from [7] that f, g_i, h_i are almost periodic functions on G. If $\hat{G}$ denotes the dual of G we have by Fourier transformation, that

$$\hat{f}(\gamma)\gamma(y) = \sum_{i=1}^{n} \hat{g}_i(\gamma)h_i(y)$$

holds for every y in G and γ in $\hat{G}$. Repeating this argument with respect to y we have that

$$\tilde{f}(\gamma)\hat{\gamma}(\tau) = \sum_{i=1}^{n} \hat{g}_i(\gamma)\hat{h}_i(\tau)$$

holds for all γ and τ in $\hat{G}$. If $\gamma_i, \cdots, \gamma_n$ are elements of $\hat{G}$ such that the matrix $(\hat{g}_i(\gamma_j))$ is regular, then substituting γ_j for γ we have a linear system of equations for the unknowns $\hat{h}_i(\tau)$ $(i = 1, \cdots, n)$ which is homogeneous if $\tau \neq \gamma_j$ $(j = 1, \cdots, n)$. Hence we conclude that $\hat{h}_i(\tau) = 0$ for $\tau \neq \gamma_j$ $(i, j = 1, \cdots, n)$ and by the inversion theorem h_i is a trigonometric polynomial for $i = 1, \cdots, n$. Since f is a linear combination of the functions h_i, we have that f is a trigonometric polynomial.

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KOSSUTH LAJOS UNIVERSITY
4010 DEBRECEN, HUNGARY