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DUALITY AND COHOMOLOGY FOR ONE RELATOR GROUPS

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1. Introduction. Let G be a group having a one relator presentation and some fundamental integral class $[G] \in H_2(G)$. The object of this paper is to study the cap product homomorphism $[G] \cap \cdot$: $H^i(G; A) \to H_{2-i}(G; \bar{A})$ where A is a left G module and $\bar{A}$ is the right G module identified with A as an abelian group and whose scalar multiplication is given by $ag = g^{-1}a$ for $a \in A$, $g \in G$. If this homomorphism is an isomorphism we say that G satisfies *Poincaré duality* with respect to A.

For example consider the fundamental group G of an orientable surface M. In this case the homomorphism $[G] \cap \cdot$ is an isomorphism for all G modules A. Such a group is said to satisfy *Poincaré duality.* Recently Müller [11, 12] has shown that a one relator group satisfying Poincaré duality over A for all G modules A is isomorphic to the fundamental group of some orientable surface; thus answering a question of Johnson and Wall in [9]. Actually Müller's result is stronger than this since it applies to a larger class of groups than one relator groups. However, we will restrict our attention to one relator groups and study duality for fixed coefficients A.

In § 2 various preliminary work relating Fox derivatives and Magnus expansions is given and in § 3 there are some results for $\boldsymbol{Z}$ coefficients. In particular Theorem 3.4 proves that any group satisfying Poincaré duality over the integers has a presentation of the form $\{x_1, \cdots, x_{2g} \mid [x_1, x_2] \cdots [x_{2g-1}, x_{2g}]W = 1\}$ where W lies in the third term of the lower central series of the free group on $x_1, \cdots, x_{2g}$. Note that if $W = 1$ then the presentation reduces to that of a surface group. This result has been proved independently by Ratcliffe, [15].

In § 4 an explicit formula for the homomorphism $[G] \cap \cdot$ on the chain level is given in terms of a Hessian matrix $\partial_i(\overline{\partial_j V})$ of Fox derivatives, where V is the relator.

Using the theory developed in this paper and results from [16] it is routine to verify the claims made in the following examples.

EXAMPLE. The group $G = \{x_1, x_2 \mid [x_1, x_2][x_2, [x_2, x_1]] = 1\}$ satisfies Poincaré duality over $\boldsymbol{Z}$. Now let A be the Laurent polynomial ring $\boldsymbol{Z}[\boldsymbol{Z}]$ on the generator t with the G module structure induced from the homomorphism $\phi: G \to \boldsymbol{Z}[t]$ defined by $\phi(x_1) = 1$, $\phi(x_2) = t$. If G were to satisfy Poincaré duality over A then it would be true that the ideal in A generated by the Fox derivatives $\phi(\partial V/\partial x_1)$, $\phi(\partial V/\partial x_2)$, where $V = [x_1, x_2][x_2, [x_2, x_1]]$, is the augmentation ideal $(1 - t)$. But a simple calculation gives $\phi(\partial V/\partial x_2) = 0$, $\phi(\partial V/\partial x_1) = 1 - t + (1 - t)^2$, and hence G does not satisfy duality with respect to A.

EXAMPLE. Consider the group $G = \{x_1, \cdots, x_4 | V = 1\}$, where $V = [x_1, x_2][x_3, x_4][x_1, [x_2, x_3]]$. Let A be the integral Laurent polynomial ring in variables $t_1, \cdots, t_4$ made into a G module by the homomorphism $\phi\colon Z[G] \to A$, $\phi(x_i) = t_i$. Then the ideal generated by the Fox derivatives $\phi(\partial_i V)$ is the augmentation ideal $(1 - t_1, \cdots, 1 - t_4)$ and hence $[G] \cap \cdot\colon H^2(G; A) \to H_0(G; \bar{A})$ is an isomorphism. A short calculation gives $H^0(G; A) = 0$, $H_2(G; \bar{A}) = 0$, and yet G does not satisfy Poincaré duality over A since if it did the matrix $[\phi\partial_i(\overline{\partial_j V})]$ would be invertible over A. But the ideal generated by the first row is $(t_2 - 1, 1 - 2t_3)$ and therefore this matrix is not invertible.

2. The free differential calculus and Magnus expansions. In this section we collect various results on Fox derivatives. Standard references are [4, 5, 6, 7, 8]. Throughout F will denote the free group on $x_1, \cdots, x_n$ and $\varepsilon\colon \boldsymbol{Z}[F] \to \boldsymbol{Z}$ will denote the augmentation homomorphism. The usual anti-automorphism $\boldsymbol{Z}[F] \to \boldsymbol{Z}[F]$ will be written $f \to \bar{f}$.

For $1 \leqq i \leqq n$ we let ∂_i be the Fox derivative $\partial/\partial x_i$ and for any finite sequence $I = (i_1, \cdots, i_r)$, where $1 \leqq i_k \leqq n$, we let ∂_I denote the higher order derivative $\partial_{i_1} \cdots \partial_{i_r}$. If $r = 0$ put $\partial_I = \text{id}$ and set ε_I equal to the composite $\varepsilon\partial_I$ for any I.

If multiplication of sequences is by juxtaposition then induction on the length of a sequence will prove:

LEMMA 2.1. *For any sequence K and $f, g \in \boldsymbol{Z}[F]$ we have $\varepsilon_K(fg) = \sum_{IJ=K} \varepsilon_I(f)\varepsilon_J(g)$, where the summation is over all ordered pairs (I, J), including (K, ϕ) and (ϕ, K), such that $IJ = K$.*

Thus it follows that $\varepsilon_i\colon F \to \boldsymbol{Z}$ gives the exponent sum of x_i in a word and $\varepsilon_{ij}[g, h] = \varepsilon_i(g)\varepsilon_j(h) - \varepsilon_i(h)\varepsilon_j(g)$ for $g, h \in F$. Now let $\mathfrak{a}$ be the free associative power series ring on the noncommuting variables $a_1, \cdots, a_n$ and with coefficients in $\boldsymbol{Z}$. For any sequence $I = (i_1, \cdots, i_r)$ let a_I denote the monomial $a_{i_1} \cdots a_{i_r}$, where $a_\phi = 1$ by convention. The Magnus expansion is defined to be the homomorphism $M\colon F \to \mathfrak{a}$, $x_i \to 1 + a_i$. Induction on word length easily proves:

LEMMA 2.2. *For any K and $f \in F$ we have $\varepsilon_K(f) = M_K(f)$.*

The following describes chain rules for Fox derivatives. Thus suppose F is free on $x_1, \cdots, x_n$ and G is free on $y_1, \cdots, y_p$. If $\phi\colon G \to F$ is a group homomorphism then

LEMMA 2.3. (a) $\varepsilon_i(\phi(g)) = \sum_{k=1}^{p} \varepsilon_i(\phi(y_k))\varepsilon_k(g)$,
(b) *for* $g \in [G, G]$ *we have* $\varepsilon_{ij}(\phi(g)) = \sum_{k,l=1}^{p} \varepsilon_i(\phi(y_k))\varepsilon_j(\phi(y_l))\varepsilon_{kl}(g)$.

As an example suppose G is free on $y_1, \cdots, y_{2g}$ and $W = [y_1, y_2] \cdots [y_{2g-1}, y_{2g}]$. Then

$$\varepsilon_{kl}(W) = \begin{cases} +1 & \text{if } (k, l) = (2i-1, 2i) \text{ for some } i, \quad 1 \leqq i \leqq g \\ -1 & \text{if } (k, l) = (2i, 2i-1) \text{ for some } i, \quad 1 \leqq i \leqq g \\ 0 & \text{otherwise}. \end{cases}$$

Thus the $2g$ by $2g$ matrix composed of the second order partials $\varepsilon_{kl}(W)$ is the skew symmetric matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

It is not a coincidence that this matrix is also the cup product matrix for the orientable surface of genus g.

3. Poincaré duality with untwisted Z-coefficients. Throughout this section $K = \{x_1, \cdots, x_n \mid V = 1\}$ will denote a one relator presentation of the group G where the relator V belongs to $[F, F]$ and is assumed not to be a proper power.

If $1 \to R \to F \to G \to 1$ is the exact sequence of this presentation then the Hopf formula gives $H_2(K) \cong R/[R, F] \cong Z$ with generator $[K] = V \cdot [R, F]$. The other homology groups can be described as follows: $H_1(K)$ is free abelian on the cosets $\bar{x}_1, \cdots, \bar{x}_n \bmod [F, F]$, $H^1(K)$ is free abelian on the dual classes $x_1^*, \cdots, x_n^*$ and $H^2(K) \cong Z$ by evaluation $u \to \langle u, [K]\rangle$.

Define the *cup product matrix* $A = (a_{ij})$ over the integers by the formula

$$a_{ij} = \langle x_i^* \cup x_j^*, [K]\rangle = \langle x_i^*, [K] \cap x_j^*\rangle .$$

Now $[K] \cap \cdot$ is automatically an isomorphism for $i = 0, 2$ and so K satisfies Poincaré duality over Z if and only if $[K] \cap \cdot\colon H^1(K) \to H_1(K)$ is an isomorphism. This implies the following well known result.

THEOREM 3.1. *Using the notation above K satisfies Poincaré duality over Z if and only if $A \in GL_n(Z)$.*

See for example [15].

Suppose now that $n = 2g$ and $V = [x_1, x_2] \cdots [x_{2g-1}, x_{2g}]$ so that K is a surface. Then it is easily checked that the cup product matrix (a_{ij}) is equal to the matrix (ε_{ij}) defined in the previous section. This is also a consequence of the following general result.

THEOREM 3.2. *Suppose* $K = \{x_1, \cdots, x_n \mid V = 1\}$ *is such that* $V \in [F, F]$ *is not a proper power. Then the cup product matrix* $a_{ij} = \langle x_i^* \cup x_j^*, [K] \rangle = \varepsilon_{ij}(V)$.

Proof. See Porter [14] or Fenn, Sjerve [3].

COROLLARY. *K satisfies Poincaré duality over* $\boldsymbol{Z}$ *if and only if the* $n \times n$ *matrix* $\varepsilon_{ij}(V)$ *is invertible over* $\boldsymbol{Z}$.

There are several effective procedures for computing $\varepsilon_{ij}(V)$. For example we can use the Magnus expansion or if $V = [U_1, V_1] \cdots [U_g, V_g]$ then

$$\varepsilon_{ij}(V) = \sum_{k=1}^{g} \{\varepsilon_i(U_k)\varepsilon_j(V_k) - \varepsilon_i(V_k)\varepsilon_j(U_k)\} .$$

It follows that if we write V in the form
$V = \prod_{1 \leq i < j \leq n} [x_i, x_j]^{a_{ij}} V'$, where $V' \in [F, [F, F]] \cdots$ *

then
$$\varepsilon_{ij}(V) = \begin{cases} a_{ij} & \text{if } i < j \\ 0 & \text{if } i = j \\ -a_{ji} & \text{if } i > j . \end{cases}$$

This together with 3.2 gives the following result due to Labute and Shapiro-Sonn, [10] and [17].

THEOREM 3.3. *Suppose* $K = \{x_1, \cdots, x_n \mid V = 1\}$ *where* V *is written in the form given by *. Then the cup product matrix for* K *is given by the skew symmetric matrix*

$$A = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ -a_{12} & 0 & \cdots & a_{2n} \\ \vdots & & & \\ -a_{1n} & -a_{2n} & \cdots & 0 \end{bmatrix} .$$

If K satisfies Poincaré duality over $\boldsymbol{Z}$ then the following theorem, which has been proved independently by Ratcliffe [15], shows that the relator V can be made almost like that of a surface.

THEOREM 3.4. *Suppose* K *satisfies Poincaré duality over* $\boldsymbol{Z}$.

Then K has the homotopy type of

$$L = \{x_1, \cdots, x_{2g} \mid [x_1, x_2] \cdots [x_{2g-1}, x_{2g}]V'\}$$

where $V' \in [F, [F, F]]$.

Proof. If $N \in \mathrm{Aut}(F)$ is an automorphism then the complex $\{x_1, \cdots, x_n \mid V = 1\}$ has the homotopy type of $\{x_1, \cdots, x_n \mid N(V) = 1\}$. Let A, B be the respective cup product matrices. Then there exists $U \in GL_n(Z)$ such that $B = UAU^T$. Conversely if B is congruent to A then there is an $N \in \mathrm{Aut}(F)$ such that B is the cup product matrix of $\{x_1, \cdots, x_n \mid N(V) = 1\}$ as can be seen using routine calculations with Nielsen transformations.

Now if K satisfies Poincaré duality then A is a nonsingular skew symmetric matrix and so by well known results in linear algebra is congruent to

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \text{see e.g. [13]}.$$

By using the above argument K may be made into the required form.

Finally we note the following corollary to the above results.

THEOREM 3.5. *Let $U_1, V_1, \cdots, U_g, V_g$ be words in the free group on $x_1, \cdots, x_{2g}$. Then $\{x_1, \cdots, x_{2g} \mid [U_1, V_1] \cdots [U_g, V_g] = 1\}$ satisfies Poincaré duality with respect to $\boldsymbol{Z}$-coefficients if and only if, the group $\{x_1, \cdots, x_{2g} \mid U_1 = V_1 = \cdots = U_g = V_g = 1\}$ is perfect.*

Thus there exists a correspondence between presentations of perfect groups on an even number of generators with defect zero and group presentations satisfying Poincaré duality over $\boldsymbol{Z}$. For example the binary icosahedral group I^* has the defect zero presentation $\{x_1, x_2 \mid U = V = 1\}$ where $U = x_1x_2x_1x_2^{-4}$ and $V = x_1^{-2}x_2x_1x_2$. Therefore the group presentation

$$K = \{x_1, x_2 \mid x_1x_2x_1x_2^{-4}x_1^{-2}x_2x_1x_2^5x_1^{-1}x_2^{-1}x_1^{-1}x_2^{-1}x_1^{-1}x_2^{-1}x_1^2\}$$

of the group G satisfies Poincaré duality with $\boldsymbol{Z}$ coefficients. Notice that K cannot possibly satisfy duality for twisted coefficients since this would force G to be isomorphic to $\boldsymbol{Z} \oplus \boldsymbol{Z}$ and there is a homomorphism of G onto the binary icosahedral group.

4. Poincaré duality with twisted coefficients. As in the previous section $K = \{x_1, \cdots, x_n \mid V = 1\}$ will denote a presentation of the group G such that $V \in [F, F]$ is not a proper power.

The presenting homomorphism $\phi: F \to G$ induces a ring homomorphism $\phi: ZF \to ZG$ also denoted by ϕ.

In this section we will obtain necessary and sufficient conditions for G to satisfy Poincaré duality with respect to a fixed G module A. To do this we need the duality map on the chain level. Thus let $\Lambda = Z[G]$ and let C_* denote the usual chain complex associated to the Lyndon resolution, i.e., C_* is

$$0 \longrightarrow \Lambda \xrightarrow{d_2} \underbrace{\Lambda \oplus \cdots \oplus \Lambda}_{n \text{ copies}} \xrightarrow{d_1} \Lambda \longrightarrow 0 ,$$

where

$$d_2(\lambda) = (\lambda\phi(\partial_1 V), \cdots, \lambda\phi(\partial_n V))$$
$$d_1(\lambda_1, \cdots, \lambda_n) = \lambda_1(\phi(x_1) - 1) + \cdots + \lambda_n(\phi(x_n) - 1) .$$

Now define $D: \mathrm{Hom}_\Lambda(C_i, A) \to \bar{A} \otimes_\Lambda C_{2-i}$ as follows:

$i = 2$, $D: A \longrightarrow \bar{A}$ is $D: a \longrightarrow -a$

$i = 0$, $D: A \longrightarrow \bar{A}$ is $D: a \longrightarrow a$

$i = 1$, $D: A \oplus \cdots \oplus A \longrightarrow \bar{A} \oplus \cdots \oplus \bar{A}$ is given by the formula

$$D(a_1, \cdots, a_n) = (\cdots, \underbrace{-\sum_j \phi(\overline{\partial_i(\overline{\partial_j V})})a_j}_{i\text{th coordinate}}, \cdots) .$$

THEOREM 4.1. *$D: \mathrm{Hom}_\Lambda(C_*, A) \to \bar{A} \otimes_\Lambda C_*$ is a chain map.*

Proof. We must verify the commutativity of the diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathrm{Hom}_\Lambda(C_0, A) & \xrightarrow{d_1^*} & \mathrm{Hom}_\Lambda(C_1, A) & \xrightarrow{d_2^*} & \mathrm{Hom}_\Lambda(C_2, A) & \longrightarrow 0 \\ & \downarrow D & & \downarrow D & & \downarrow D & \\ 0 \longrightarrow & \bar{A} \otimes_\Lambda C_2 & \xrightarrow{d_2} & \bar{A} \otimes_\Lambda C_1 & \xrightarrow{d_1} & \bar{A} \otimes_\Lambda C_0 & \longrightarrow 0 . \end{array} \tag{4.2}$$

Thus

$$\begin{aligned}(d_1 \circ D)(a_1, \cdots, a_n) &= d_1(\cdots, -\sum_j \phi(\overline{\partial_i(\overline{\partial_j V})})a_j, \cdots) \\ &= -\sum_i \sum_j \phi(\overline{\partial_i(\overline{\partial_j V})})a_j(\phi(x_i) - 1) \\ &= -\sum_i \sum_j (\phi(x_i^{-1}) - 1)\phi(\overline{\partial_i(\overline{\partial_j V})})a_j .\end{aligned}$$

But

$$\begin{aligned}\sum_i (\phi(x_i^{-1}) - 1)\phi(\overline{\partial_i(\overline{\partial_j V})}) &= \phi \sum_i (x_i^{-1} - 1)\overline{\partial_i(\overline{\partial_j V})} = \phi \sum_i \overline{\partial_i(\overline{\partial_j V})(x_i - 1)} \\ &= \phi(\overline{\overline{\partial_j V} - \varepsilon(\overline{\partial_j V})}) = \phi(\partial_j V) .\end{aligned}$$

Therefore

$$(d_1 \circ D)(a_1, \cdots, a_n) = -\sum_j \phi(\partial_j V)a_j = (D \circ d_2^*)(a_1, \cdots, a_n) .$$

On the other hand

$$\begin{aligned}(D \circ d_1^*)(a) &= D((\phi(x_1) - 1)a, \cdots, (\phi(x_n) - 1)a) \\ &= (\cdots, -\sum_j \phi(\overline{\partial_i(\overline{\overline{\partial_j V}})})(\phi(x_j) - 1)a, \cdots) .\end{aligned}$$

However

$$\begin{aligned}\sum_j \phi(\overline{\partial_i(\overline{\overline{\partial_j V}})})(\phi(x_j) - 1) &= \phi \sum_j \overline{\partial_i(\overline{\overline{\partial_j V}})}(x_j - 1) \\ &= \phi \overline{\sum_j (x_j^{-1} - 1)\partial_i(\overline{\partial_j V})} = \phi \overline{\sum_j \partial_i[(x_j^{-1} - 1)\overline{\partial_i V}]}\end{aligned}$$

since

$$\begin{aligned}\partial_i[(x_j^{-1} - 1)\overline{\partial_j V}] &= \partial_i(x_j^{-1} - 1)\varepsilon(\overline{\partial_j V}) + (x_j^{-1} - 1)\partial_i(\overline{\partial_j V}) \\ &= (x_j^{-1} - 1)\partial_i(\overline{\partial_j V})\end{aligned}$$

(recall that $\varepsilon(\overline{\partial_j V}) = \varepsilon(\partial_j V) = \varepsilon_j(V) = 0$ because $V \in [F, F]$). Hence

$$\begin{aligned}\sum_j \phi(\overline{\partial_i(\overline{\overline{\partial_j V}})})(\phi(x_j) - 1) &= \phi\overline{\partial_i(\sum_j (x_j^{-1} - 1)\overline{\partial_j V})} = \phi\partial_i(\overline{\sum_j \partial_j(V)(x_j - 1)}) \\ &= \phi\overline{\partial_i(\overline{V} - 1)} = \phi\overline{\partial_i(\overline{V})} = \phi(\overline{\partial_i(V^{-1})}) \\ &= \phi(\overline{-V^{-1}\partial_i(V)}) = -\phi(\overline{\partial_i(V)}) \text{ since } \phi(V) = 1 .\end{aligned}$$

This shows that $(Dd_1^*)(a) = (\cdots, \phi(\overline{\partial_i V})a, \cdots) = (d_2 D)(a)$. □

The chain transformation $D\colon \mathrm{Hom}_\Lambda(C_*, A) \to \bar{A} \otimes_\Lambda C_*$ is clearly natural in A and so the induced map in homology $D_*\colon H^*(G; A) \to H_*(G; \bar{A})$ is functional in A. The cap product homomorphism $[G] \cap \cdot\colon H^*(G; A) \to H_*(G; \bar{A})$ is also functorial in A. In the next theorem we prove that $D_* = [G] \cap \cdot$, but first we compare D_*, $[G] \cap \cdot$ for the special case $H^1(G) \to H_1(G)$. We have

$$\begin{aligned}D_*(x_k^*) &= D_*(0, \cdots, 0, 1, 0, \cdots, 0) = (\cdots, -\sum_j \phi(\overline{\partial_i(\overline{\overline{\partial_j V}})})\delta_{jk}, \cdots) \\ &= -\sum_i \phi(\overline{\partial_i(\overline{\overline{\partial_k V}})})\bar{x}_i = -\sum_i \varepsilon(\overline{\partial_i(\overline{\overline{\partial_k V}})})\bar{x}_i\end{aligned}$$

(since the module structure on the coefficients is given by augmentation). Now $-\varepsilon(\overline{\partial_i(\overline{\overline{\partial_k V}})}) = -\varepsilon\, \partial_i(\overline{\partial_k V}) = \varepsilon\, \partial_i\partial_k(V)$ because $\varepsilon\, \partial_i(\bar{f}) = -\varepsilon\, \partial_i(f)$ for $f \in F$. Therefore

$$D_*(x_k^*) = \sum_i \varepsilon_{ik}(V)\bar{x}_i = \sum_i \langle x_i^* \cup x_k^*, [G]\rangle \bar{x}_i$$

according to (3.2). But we also have

$$[G] \cap x_k^* = \sum_i \langle x_i^*, [G] \cap x_k^* \rangle \bar{x}_i = \sum_i \langle x_i^* \cup x_k^*, [G] \rangle \bar{x}_i \,.$$

Thus we proved that

$$D_* = [G] \cap \cdot : H^1(G; Z) \longrightarrow H_1(G; Z) \,.$$

THEOREM 4.3. *$D_* = [G] \cap \cdot : H^*(G; A) \to H_*(G; \bar{A})$ for any A.*

Proof. The method of proof is modelled on some of the proofs in [1, 2]. For any A the homomorphism $D_*: H^2(G; A) \to H_0(G; \bar{A})$ is induced by the chain map $D: \mathrm{Hom}_\Lambda(C_2, A) \to \bar{A} \otimes C_0$, $D: a \to -a$. Thus $D_*: H^2(G; A) \to H_0(G; \bar{A})$ is the homomorphism

$$A/(\sum \lambda_i \phi(\partial_i V)) \longrightarrow A/(\sum \lambda_i(\phi(x_i) - 1)) \quad \text{induced by} \quad a \longrightarrow -a \,.$$

It follows that $D_*: H^2(G; Z) \to H_0(G; Z)$ is an isomorphism. Since both of these groups are infinite cyclic and $[G] \cap \cdot : H^2(G; Z) \to H_0(G; Z)$ is also an isomorphism we must have

$$D_* = e \cap \cdot : H^2(G; Z) \longrightarrow H_0(G; Z) \,, \quad \text{where} \quad e = \pm[G] \,.$$

Now consider the coefficient sequence $0 \to I[G] \to \Lambda \xrightarrow{\varepsilon} Z \to 0$ of left Λ modules. Conjugating we get the exact sequence $0 \to I[G] \to \bar{\Lambda} \xrightarrow{\varepsilon} Z \to 0$ of right Λ modules. Then the functoriality of D_* and $e \cap \cdot$ gives the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^2(G; I[G]) \arrow[r] \arrow[d, shift right=2ex, "D_*"'] \arrow[d, shift left=2ex, "e \cap \cdot"] & H^2(G; \Lambda) \arrow[r, "\varepsilon_*"] \arrow[d, shift right=2ex, "D_*"'] \arrow[d, shift left=2ex, "e \cap \cdot"] & H^2(G; Z) \arrow[r] \arrow[d, "D_* = e \cap \cdot"] & 0 \\
\cdots \arrow[r] & H_0(G; I[G]) \arrow[r] & H_0(G; \bar{\Lambda}) \arrow[r, "\varepsilon_*"] & H_0(G; Z) \arrow[r] & 0 \,.
\end{tikzcd}$$

But $\varepsilon_*; H_0(G; \bar{\Lambda}) \to H_0(G; Z)$ is a monomorphism since the homomorphism $H_0(G; I[G]) \to H_0(G; \bar{\Lambda})$ may be identified with the homomorphism

$$I[G]/I[G] \cdot I[G] \longrightarrow \Lambda/\Lambda \cdot I[G] \quad \text{induced by} \quad I[G] \subseteqq \Lambda \,.$$

Chasing around the second square in the diagram now gives

$$D_* = e \cap \cdot : H^2(G; \Lambda) \longrightarrow H_0(G; \bar{\Lambda}) \,.$$

The group G admits a finite resolution of Z by finitely generated free Λ modules and hence the functor $H^*(G; \cdot)$ commutes with direct sums. From this fact it follows that

$$D_* = e \cap \cdot : H^2(G; M) \longrightarrow H_0(G; \bar{M}) \quad \text{for any free module } M \,.$$

Given any module A we choose some presentation $0 \to N \to M \xrightarrow{\phi} A \to 0$. By naturality there is a commutative diagram

$$\begin{array}{ccccc} H^2(G; M) & \xrightarrow{\psi_*} & H^2(G; A) & \longrightarrow & 0 \\ \Big\downarrow D_* = e \cap \cdot & D_* \Big\downarrow & \Big\downarrow e \cap \cdot & & \\ H_0(G; \bar{M}) & \xrightarrow{\bar{\psi}_*} & H_0(G; \bar{A}) & \longrightarrow & 0\,. \end{array}$$

Note that $\psi_*\colon H^2(G; M) \to H^2(G; A)$ is an epimorphism since G has cohomological dimension 2. Commutativity of this diagram now implies that

$$D_* = e \cap \cdot\colon H^2(G; A) \longrightarrow H_0(G; \bar{A}) \quad \text{for any module} \quad A\,.$$

Now consider the commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H^1(G; M) & \longrightarrow & H^1(G; A) & \longrightarrow & H^2(G; N) & \longrightarrow & \cdots \\ & D_*\Big\downarrow & \Big\downarrow e \cap \cdot & D_*\Big\downarrow & \Big\downarrow e \cap \cdot & & \Big\downarrow D_* = e \cap \cdot & & \\ \cdots & \longrightarrow & H_1(G; \bar{M}) & \longrightarrow & H_1(G; \bar{A}) & \longrightarrow & H_0(G; \bar{N}) & \longrightarrow & \cdots\,. \end{array}$$

$\bar{M}$ is a free right module and so $H_1(G; \bar{M}) = 0$. Therefore $H_1(G; \bar{A}) \to H_0(G; \bar{N})$ is a monomorphism, and this implies that

$$D_* = e_* \cap \cdot\colon H^1(G; A) \longrightarrow H_1(G; \bar{A}) \quad \text{for all} \quad A\,.$$

Finally we look at the commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H^0(G; M) & \longrightarrow & H^0(G; A) & \longrightarrow & H^1(G; N) & \longrightarrow & \cdots \\ & D_*\Big\downarrow & \Big\downarrow e \cap \cdot & D_*\Big\downarrow & \Big\downarrow e \cap \cdot & & \Big\downarrow D_* = e \cap \cdot & & \\ \cdots & \longrightarrow & H_2(G; \bar{M}) & \longrightarrow & H_2(G; \bar{A}) & \longrightarrow & H_1(G; \bar{N}) & \longrightarrow & \cdots\,. \end{array}$$

$H_2(G; \bar{M}) = 0$ as $\bar{M}$ is free and therefore

$$D_* = e \cap \cdot\colon H^0(G; A) \longrightarrow H_2(G; \bar{A}) \quad \text{for all} \quad A\,.$$

To prove that $e = [G]$ we use the functoriality of D_* and $[G] \cap \cdot$ with respect to the variable G, while keeping the coefficients fixed at Z. If G has the presentation $\{x_1, \cdots, x_n \mid V = [U_1, V_1] \cdots [U_g, V_g] = 1\}$ let π be the surface group $\{y_1, \cdots, y_{2g} \mid [y_1, y_2] \cdots [y_{2g-1}, y_{2g}] = 1\}$. We also have the obvious degree 1 map $\phi\colon \pi \to G$. Then there are classes $e_G \in H_2(G)$, $e_\pi \in H_2(\pi)$ and a commutative diagram

$$\begin{array}{ccc} H^2(G) & \xrightarrow{D_* = e_G \cap \cdot} & H_0(G) \\ \Big\downarrow \phi^* & & \Big\uparrow \phi_* \\ H^2(\pi) & \xrightarrow{D_* = e_\pi \cap \cdot} & H_0(\pi)\,. \end{array}$$

It has already been noted that $D_* = [\pi] \cap \cdot\colon H^1(\pi) \to H_1(\pi)$. This coupled with the fact that $D_*\colon H^1(\pi) \to H_1(\pi)$ is an isomorphism implies that $e_\pi = [\pi]$. If $[G]^*$, $[\pi]^*$ are the cohomology classes dual to $[G]$, $[\pi]$ respectively then

$$\varepsilon D_*([G]^*) = \varepsilon\phi_* D_* \phi^*([G]^*) = \varepsilon\phi_* D_*([\pi]^*) \quad (\text{as } \phi^*([G]^*) = [\pi]^*)$$

where $\varepsilon\colon H_0(\cdot) \to Z$ is the augmentation. Hence

$$\varepsilon D_*([G]^*) = \varepsilon\phi_*([\pi] \cap [\pi]^*) = \langle[\pi]^*, [\pi]\rangle = 1$$

and therefore $\langle [G]^*, e_G\rangle = \varepsilon e_G \cap [G]^* = \varepsilon D_*([G]^*) = 1$. This proves that $e_G = [G]$.

By chasing around diagram 4.2 we prove the following theorem.

THEOREM 4.4. *With the notation above, G satisfies Poincaré duality with respect to A if, and only if, $D\colon \bigoplus_1^n A \to \bigoplus_1^n \bar{A}$ is an isomorphism.*

As an example of this theorem consider the case $A = \boldsymbol{Z}$ with the trivial module structure. Then

$$\phi(\overline{\partial_i(\overline{\partial_j V})})a = \varepsilon(\overline{\partial_i(\overline{\partial_j V})})a = \varepsilon(\partial_i(\overline{\partial_j V}))a .$$

But for any $f \in F$ we have

$$\varepsilon\partial_i(\bar{f}) = \varepsilon\partial_i(f^{-1}) = \varepsilon(-f^{-1}\partial_i(f)) = -\varepsilon\,\partial_i(f) .$$

Therefore $-\phi(\overline{\partial_i(\overline{\partial_j V})})a = \varepsilon\,\partial_i\partial_j(V)a = \varepsilon_{ij}(V)a$. This means that the cap product map $D\colon \mathrm{Hom}_\Lambda(C_1, \boldsymbol{Z}) \to \boldsymbol{Z} \otimes_\Lambda C_1$, that is $D\colon \boldsymbol{Z} \oplus \cdots \oplus \boldsymbol{Z} \to \boldsymbol{Z} \oplus \cdots \oplus \boldsymbol{Z}$, becomes

$$D(a_1, \cdots, a_n) = (\cdots, \sum_j \varepsilon_{ij}(V)a_j, \cdots) .$$

In other words D is the $n \times n$ matrix $[\varepsilon_{ij}(V)]$, a result in agreement with 3.2.

As another example consider the Λ module $\boldsymbol{Z}[G_{ab}]$, where the Λ module structure is induced by the abelianization homomorphism $\alpha\colon G \to G_{ab}$. For convenience set $t_i = \alpha\phi(x_i)$, $1 \leqq i \leqq n$. Then $\boldsymbol{Z}[G_{ab}]$ is the Laurent polynomial ring on the variables $t_1, \cdots, t_n$. If $p(t_1, \cdots, t_n)$ is a Laurent polynomial then the module structure is given by

$$\phi(x_i^{\pm 1})\cdot p(t_1, \cdots, t_n) = t_i^{\pm 1} p(t_1, \cdots, t_n) , \quad 1 \leqq i \leqq n .$$

THEOREM 4.5. *G satisfies duality for $\boldsymbol{Z}[G_{ab}]$ coefficients if, and only if, the matrix $[\alpha\,\partial_i(\overline{\partial_j V})]$ is invertible over $\boldsymbol{Z}[G_{ab}]$.*

Proof. Since $\phi\colon F \to G$ induces an isomorphism $F_{ab} \cong G_{ab}$ we have

$$-\phi(\overline{\partial_i(\overline{\partial_j V})})p(t_1, \cdots, t_n) = -\alpha(\overline{\partial_i(\overline{\partial_j V})})p(t_1, \cdots, t_n)$$

where $\alpha: F \to F_{ab}$ again denotes abelianization. But $\alpha(\bar{f}) = -\alpha(f)$ and so the duality map $D: Z[G_{ab}] \oplus \cdots \oplus Z[G_{ab}] \to Z[G_{ab}] \oplus \cdots \oplus Z[G_{ab}]$ may be identified with the matrix $[\alpha\,\partial_i(\overline{\partial_j V})]$. $\square$

We can generalize this result by replacing G_{ab} by an abelian group J and letting $\alpha: G \to J$ be some homomorphism. Then G satisfies duality for $Z[J]$ coefficients if, and only if, the $n \times n$ matrix $[\beta\,\partial_i(\overline{\partial_j V})]$ is invertible over $Z[J]$, where $\beta = \alpha\phi: F \to J$.

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