

SOME RANDOM FIXED POINT THEOREMS FOR CONTINUOUS RANDOM OPERATORS

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In this paper some random fixed point theorems for continuous random operators are obtained. These theorems generalize and unify some recent results.

1. Introduction. Random fixed point theorems are of fundamental importance in probabilistic functional analysis. In Polish spaces, i.e. separable complete metric spaces, random fixed point theorems were proved by Špaček, Hanš, Bharucha-Reid, Itoh, Engl and others (cf. [1-6, 8-11, 14]).

In this paper, we continue to consider this problem, and present several random fixed point theorems for abstract continuous random operators. These results generalize and unify some recent results.

2. Preliminaries. Throughout this paper, $(\Omega, \mathfrak{B}, P)$ denotes a complete probability measure space, X is a separable complete metric space with metric d , and (X, \mathfrak{X}) is a measurable space, where \mathfrak{X} is the σ -algebra of all Borel subsets of X .

A mapping $x: \Omega \rightarrow X$ is said to be a random variable with value in X , if the inverse image under the mapping x of every Borel set of X belongs to \mathfrak{B} . A mapping $T(\cdot, \cdot): \Omega \times X \rightarrow X$ is said to be a random operator, if $\{\omega \in \Omega: T(\omega, x) \in B\} \in \mathfrak{B}$ for all $x \in X, B \in \mathfrak{X}$. A random operator $T(\cdot, \cdot): \Omega \times X \rightarrow X$ is called continuous, if for each $\omega \in \Omega, T(\omega, \cdot)$ is continuous. A random variable $\xi: \Omega \rightarrow X$ is called a random fixed point of a random operator $T(\cdot, \cdot): \Omega \times X \rightarrow X$, if $T(\omega, \xi(\omega)) = \xi(\omega)$ a.s.

LEMMA 1. ([12]). *Let $x(\omega)$ be an X -valued random variable. Let $T: \Omega \times X \rightarrow X$ be a continuous random operator. Then $T(\omega, x(\omega))$ is also an X -valued random variable.*

Let $T: \Omega \times X \rightarrow X$ be a continuous random operator. For any X -valued random variable $x(\omega)$ let $O_T(x(\omega), 0, \infty)$ denote the random orbit of $x(\omega) \in X$ under T , i.e.

$$O_T(x(\omega), 0, \infty) = \{x_0(\omega) = x(\omega), x_1(\omega) = T(\omega, x_0(\omega)), \dots \\ x_n(\omega) = T(\omega, x_{n-1}(\omega)) = T^n(\omega, x_0(\omega)), \dots\}.$$

For any positive integers $i, j, j \geq i$, we put

$$O_T(x(\omega), i, j) = \{x_i(\omega), x_{i+1}(\omega), \dots, x_j(\omega)\}.$$

For any pair of X -valued random variables $x(\omega)$, $y(\omega)$, let $O_T(x(\omega), y(\omega), 0, \infty)$ denote the random orbit of $x(\omega), y(\omega)$ under T , i.e.

$$\begin{aligned} O_T(x(\omega), y(\omega), 0, \infty) \\ = \{x_0(\omega) = x(\omega), y_0(\omega) = y(\omega), x_1(\omega) = T(\omega, x_0(\omega)), \\ y_1(\omega) = T(\omega, y_0(\omega)), \dots, x_n(\omega) = T(\omega, x_{n-1}(\omega)) = T^n(\omega, x_0(\omega)), \\ y_n(\omega) = T(\omega, y_{n-1}(\omega)) = T^n(\omega, y_0(\omega)), \dots\}. \end{aligned}$$

For any positive integer $i, j, j \geq i$, we denote

$$\begin{aligned} O_T(x(\omega), y(\omega), i, j) \\ = \{x_i(\omega), y_i(\omega), x_{i+1}(\omega), y_{i+1}(\omega), \dots, x_j(\omega), y_j(\omega)\}. \end{aligned}$$

For any subset $A \subset X$, let $\delta(A)$ be the diameter of A , i.e.

$$\delta(A) = \sup_{x, y \in A} d(x, y).$$

3. Main results.

THEOREM 1. *Let $T: \Omega \times X \rightarrow X$ be a continuous random operator. Suppose that there exists a positive integer p such that for any $x \in X$ the following holds:*

$$(3.1) \quad P\{\omega \in \Omega: \delta(O_T(x, P, \infty)) \leq \Phi(\omega, \delta(O_T(x, 0, \infty)))\} = P(E_x) = 1,$$

where $\Phi(\omega, t): \Omega \times [0, \infty) \rightarrow [0, \infty)$ is a random function satisfying the following conditions

(Φ 1) For any fixed $\omega \in \Omega$, $\Phi(\omega, t)$ is non-decreasing and right continuous with respect to t .

(Φ 2) $P\{\omega \in \Omega: \lim_{t \rightarrow \infty} (t - \Phi(\omega, t)) = \infty\} = P(F) = 1$.

(Φ 3) For any $t > 0$,

$$P\{\omega \in \Omega: \lim_{n \rightarrow \infty} \Phi^n(\omega, t) = 0\} = P(G_t) = 1,$$

where $\Phi^n(\omega, t)$ is the n th iterate of $\Phi(\omega, t)$.

Then for any random variable $x_0(\omega): \Omega \rightarrow X$ the sequence of the X -valued random variable $\{x_n(\omega)\}_{n=0}^\infty$ defined by

$$x_n(\omega) = T(\omega, x_{n-1}(\omega)), \quad n = 1, 2, \dots,$$

converges almost surely to a random fixed point of T .

To prove Theorem 1, we first verify the following lemma.

LEMMA 2. Let the random functions $\Phi(\omega, t)$ satisfy conditions $(\Phi 1)$ and $(\Phi 3)$. Then

(i) for any sequence $\{t_n(\omega)\}$ satisfying the following conditions:

(a) $t_n(\omega): \Omega \rightarrow [0, \infty)$, $n = 1, 2, \dots$,

(b) $t_n(\omega) \leq t_{n-1}(\omega)$, $n = 2, 3, \dots$,

(c) $P\{\omega \in \Omega: t_{n+1}(\omega) \leq \Phi(\omega, t_n(\omega))\} = P(W_n) = 1$, $n = 1, 2, \dots$,

we have

$$\lim_{n \rightarrow \infty} t_n(\omega) = 0 \quad \text{a. s.};$$

(ii) Particularly, if $t(\omega): \Omega \rightarrow [0, \infty)$ satisfies

$$t(\omega) \leq \Phi(\omega, t(\omega)) \quad \text{a. s.},$$

then we have $t(\omega) = 0$ a. s.; and

(iii) for any $t > 0$, we have $\Phi(\omega, t) < t$ a. s.

Proof. (i). By the right continuity of $\Phi(\omega, t)$ with respect to t and condition (c), it is easy to see that the set

$$W = \left(\bigcap_{n=1}^{\infty} W_n \right) \cap \left(\bigcap_{t>0} G_t \right) \in \mathfrak{B} \quad \text{and} \quad P(W) = 1,$$

where the sets W_n , $n = 1, 2, \dots$, and G_t , respectively, are defined by condition (c) and $(\Phi 3)$.

For an arbitrary, but fixed, $\tilde{\omega} \in W$ it follows from (c) that

$$\begin{aligned} 0 \leq t_{n+1}(\tilde{\omega}) &\leq \Phi(\tilde{\omega}, t_n(\tilde{\omega})) \leq \Phi^2(\tilde{\omega}, t_{n-1}(\tilde{\omega})) \\ &\leq \dots \leq \Phi^n(\tilde{\omega}, t_1(\tilde{\omega})). \end{aligned}$$

Using conditions $(\Phi 3)$ we have

$$0 \leq \lim_{n \rightarrow \infty} t_{n+1}(\tilde{\omega}) \leq \lim_{n \rightarrow \infty} \Phi^n(\tilde{\omega}, t_1(\tilde{\omega})) = 0.$$

Hence

$$\lim_{n \rightarrow \infty} t_n(\omega) = 0 \quad \text{a. s.}$$

(ii) Putting $t_n(\omega) = t(\omega)$, $n = 1, 2, \dots$ in (i), we obtain immediately $t(\omega) = 0$ a. s.

(iii) Suppose that there exists some $t_0 > 0$ such that $t_0 \leq \Phi(\omega, t_0)$ a. s. From (ii) we have $t_0 = 0$. This yields a contradiction. Hence the conclusion of (iii) holds.

This completes the proof of Lemma 2.

Now we return to the proof of Theorem 1.

For any X -valued random variable $x_0(\omega)$, by the continuity of random operator T , it follows from Lemma 1 that the random orbit of $x_0(\omega)$ under T i.e., $O_T(x_0(\omega), 0, \infty) = \{x_n(\omega)\}_{n=0}^\infty$,

$$(3.2) \quad x_n(\omega) = T(\omega, x_{n-1}(\omega)), \quad n = 1, 2, \dots$$

is a sequence of X -valued random variables. By the separability of X , the continuity of $T(\omega, \cdot)$, and the right continuity of $\Phi(\omega, \cdot)$, it follows that the set

$$H = \left\{ \left(\bigcap_{x \in X} E_x \right) \cap F \left(\bigcap_{t > 0} G_t \right) \right\} \in \mathfrak{B} \quad \text{and} \quad P(H) = 1.$$

By the assumptions of Theorem 1, for the random variable $x_0(\omega) \in X$ we have

$$(3.3) \quad \delta(O_T(x_0(\omega), p, \infty)) \leq \Phi(\omega, \delta(O_T(x_0(\omega), 0, \infty))) \quad \forall \omega \in H.$$

It follows from (3.1) and (3.3) that

$$(3.4) \quad \begin{aligned} \delta(O_T(x_0(\omega), np, \infty)) &= \delta(O_T(x_{(n-1)p}(\omega), p, \infty)) \\ &\leq \Phi(\omega, \delta(O_T(x_{(n-1)p}(\omega), 0, \infty))) \\ &= \Phi(\omega, \delta(O_T(x_0(\omega), (n-1)p, \infty))), \quad \omega \in H, n = 1, 2, \dots \end{aligned}$$

However

$$(3.5) \quad \begin{aligned} &\delta(O_T(x_0(\omega), (n-1)p, \infty)) \\ &= \max \left\{ \delta(O_T(x_0(\omega), (n-1)p, np)), \right. \\ &\quad \left. \delta(O_T(x_0(\omega), np+1, \infty)), \sup_{\substack{(n-1)p \leq r \leq np \\ s \geq np+1}} d(x_r(\omega), x_s(\omega)) \right\} \\ &\leq \max \left\{ \delta(O_T(x_0(\omega), (n-1)p, np)), \delta(O_T(x_0(\omega), np, \infty)), \right. \\ &\quad \left. \sup_{(n-1)p \leq r \leq np} d(x_r(\omega), x_{np}(\omega)) + \sup_{s \geq np+1} d(x_{np}(\omega), x_s(\omega)) \right\} \\ &\leq \delta(O_T(x_0(\omega), (n-1)p, np)) + \delta(O_T(x_0(\omega), np, \infty)). \end{aligned}$$

From (3.4) and (3.5) we obtain

$$(3.6) \quad \begin{aligned} &\delta(O_T(x_0(\omega), (n-1)p, \infty)) \\ &\leq \delta(O_T(x_0(\omega), (n-1)p, np)) \\ &\quad + \Phi(\omega, \delta(O_T(x_0(\omega), (n-1)p, \infty))), \quad \omega \in H. \end{aligned}$$

Taking $n = 1$ in (3.6) we have

$$(3.7) \quad \begin{aligned} \delta(O_T(x_0(\omega), 0, \infty)) \\ \leq \delta(O_T(x_0(\omega), 0, p)) \\ + \Phi(\omega, \delta(O_T(x_0(\omega), 0, \infty))), \quad \omega \in H. \end{aligned}$$

Now we prove that for $\omega \in H$

$$\delta(O_T(x_0(\omega), 0, \infty)) < \infty.$$

Suppose this is not the case, hence there exists some $\tilde{\omega} \in H$ such that

$$\delta(O_T(x_0(\omega), 0, \infty))|_{\omega=\tilde{\omega}} = \infty.$$

Letting

$$A_n(\tilde{\omega}) = \delta(O_T(x_0(\omega), 0, n))|_{\omega=\tilde{\omega}}, \quad n = 1, 2, \dots,$$

we know that $\{A_n(\tilde{\omega})\}$ is an increasing sequence of positive numbers and

$$\lim_{n \rightarrow \infty} A_n(\tilde{\omega}) = \delta(O_T(x_0(\omega), 0, \infty))|_{\omega=\tilde{\omega}} = \infty.$$

By condition $(\Phi 2)$ and (3.7) we have

$$\begin{aligned} \infty &= \lim_{n \rightarrow \infty} (A_n(\tilde{\omega}) - \Phi(\tilde{\omega}, A_n(\tilde{\omega}))) \\ &\leq \delta(O_T(x_0(\omega), 0, p))|_{\omega=\tilde{\omega}} < \infty. \end{aligned}$$

This yields a contradiction. Hence we obtain

$$\delta(O_T(x_0(\omega), 0, \infty)) < \infty \quad \forall \omega \in H.$$

Putting

$$t_n(\omega) = \begin{cases} \delta(O_T(x_0(\omega), np, \infty)) & \text{for } \omega \in H, \\ 0 & \text{for } \omega \in \Omega \setminus H, \end{cases} \quad n = 0, 1, 2, \dots$$

from (3.4) we know that $\{t_n(\omega)\}$ satisfies all conditions of Lemma 2 (i). Therefore we have

$$\lim_{n \rightarrow \infty} \delta(O_T(x_0(\omega), np, \infty)) = 0 \quad \text{a. s.}$$

This implies that $\{x_n(\omega)\}$ is a Cauchy sequence of X -valued random variables. Suppose $x_n(\omega) \rightarrow x_*(\omega)$ a. s. Hence $x_*(\omega)$ is also a X -valued random variable. Letting $n \rightarrow \infty$ in the following equality

$$x_n(\omega) = T(\omega, x_{n-1}(\omega)),$$

and invoking the continuity of T , we have

$$x_*(\omega) = T(\omega, x_*(\omega)) \quad \text{a. s.}$$

This completes the proof of Theorem 1.

THEOREM 2. *Let $\Phi(\omega, t)$ be the same as in Theorem 1, and let T be a continuous random operator from $\Omega \times X \rightarrow X$. Suppose that there exists a positive integer p such that for any $k \in N$ (N denotes the set of all non-negative integers) and any $x \in X$ the following holds:*

$$(3.8) \quad P\{\omega \in \Omega: d(T^p(\omega, x), T^{p+k}(\omega, x)) \leq \Phi(\omega, \delta(O_T(x, 0, \infty)))\} = P(E_{xk}) = 1.$$

Then the conclusion of Theorem 1 still holds.

Proof. By the assumptions on X, T and Φ , it is easy to see that the set

$$E = \bigcap_{x \in X} \bigcap_{k=0}^{\infty} E_{xk} \in \mathfrak{B} \quad \text{and} \quad P(E) = 1.$$

Hence for any pair $r, s \in N$ (with no loss of generality we can assume $r \leq s$), and any $x \in X$, it follows from (3.8) that for any $\omega \in E$ the following holds:

$$\begin{aligned} & d(T^{p+r}(\omega, x), T^{p+s}(\omega, x)) \\ &= d(T^p(\omega, T^r(\omega, x)), T^{p+s-r}(\omega, T^r(\omega, x))) \\ &\leq \Phi(\omega, \delta(O_T(T^r(\omega, x), 0, \infty))) \\ &= \Phi(\omega, \delta(O_T(x, r, \infty))) \\ &\leq \Phi(\omega, \delta(O_T(x, 0, \infty))). \end{aligned}$$

Consequently

$$\begin{aligned} \delta(O_T(x, p, \infty)) &= \sup_{r, s \geq 0} d(T^{p+r}(\omega, x), T^{p+s}(\omega, x)) \\ &\leq \Phi(\omega, \delta(O_T(x, 0, \infty))), \quad \omega \in E, \end{aligned}$$

i. e.

$$\delta(O_T(x, p, \infty)) \leq \Phi(\omega, \delta(O_T(x, 0, \infty))) \quad \text{a. s.}$$

This implies that from (3.8) we can deduce (3.1). Therefore the desired conclusion follows from Theorem 1.

This completes the proof of Theorem 2.

REMARK 1. In the deterministic case, the main result in Pal and Maiti [13] is a special case of Theorem 2.

As a corollary of Theorem 1 and Theorem 2 we have the following result.

COROLLARY 1. *Let $T: \Omega \times X \rightarrow X$ be a continuous random operator. Suppose that there exists a positive integer p such that for any $x \in X$ one of the following conditions holds:*

$$(3.9) \quad P\{\omega \in \Omega: \delta(O_T(x, p, \infty)) \leq \mathcal{Q}(\omega)\delta(O_T(x, 0, \infty))\} = 1;$$

$$(3.10) \quad \text{for any } k \in \mathbb{N}$$

$$P\{\omega \in \Omega: d(T^p(\omega, x), T^{p+k}(\omega, x)) \leq \mathcal{Q}(\omega)\delta(O_T(x, 0, \infty))\} = 1,$$

where $\mathcal{Q}(\omega): \Omega \rightarrow [0, \infty)$ is a random variable satisfying

$$P(\omega \in \Omega: 0 < \mathcal{Q}(\omega) < 1) = 1.$$

Then the conclusion of Theorem 1 holds.

Proof. Taking $\Phi(\omega, t) = \mathcal{Q}(\omega) \cdot t$, $t \geq 0$, it is easy to see that in this case conditions $(\Phi 1)$, $(\Phi 2)$, $(\Phi 3)$ are all satisfied, and the conclusion follows from Theorem 1 and Theorem 2, respectively.

THEOREM 3. *Let $T: \Omega \times X \rightarrow X$ be a continuous random operator. Let Φ be the same as in Theorem 1. Suppose that there exist positive integers p, q such that for any pair $x, y \in X$*

$$(3.11) \quad P\{\omega \in \Omega: d(T^p(\omega, x), T^q(\omega, y)) \leq \Phi(\omega, \delta(O_T(x, y, 0, \infty)))\} = P(E_{xy}) = 1.$$

Then T has a unique X -valued random fixed point $x_(\omega)$; and for any X -valued random variable $x_0(\omega)$ the sequence $\{x_n(\omega)\}$, where*

$$x_n(\omega) = T(\omega, x_{n-1}(\omega)), \quad n = 1, 2, \dots,$$

converges almost surely to $x_(\omega)$.*

Proof. With no loss of generality, we can assume $p \geq q$.

Considering the set $\tilde{G} = \bigcap_{x \in X} \bigcap_{y \in X} (E_{xy})$, it is easy to see that $\tilde{G} \in \mathfrak{B}$ and $P(\tilde{G}) = 1$.

For any $x \in X$, any $k \in N$ and any $\omega \in \tilde{G}$, we denote $y = T^{p-q+k}(\omega, x)$. From (3.11) we have

$$\begin{aligned} d(T^p(\omega, x), T^q(\omega, T^{p-q+k}(\omega, x))) \\ &= d(T^p(\omega, x), T^{p+k}(\omega, x)) \\ &\leq \Phi(\omega, \delta(O_T(x, T^{p-q+k}(\omega, x), 0, \infty))) \\ &= \Phi(\omega, \delta(O_T(x, 0, \infty))). \end{aligned}$$

By Theorem 1, for any random variable $x_0(\omega)$, the sequence of X -valued random variables $\{x_n(\omega)\}$ defined by

$$x_n(\omega) = T(\omega, x_{n-1}(\omega)), \quad n = 1, 2, \dots,$$

converges almost surely to a random fixed point of T .

To prove T has a unique X -valued random fixed point, we proceed as follows:

Let $x_*(\omega), \tilde{x}(\omega)$ be two X -valued random fixed points of T ; therefore

$$\begin{aligned} (3.12) \quad d(x_*(\omega), \tilde{x}(\omega)) &= d(T^p(\omega, x_*(\omega)), T^q(\omega, \tilde{x}(\omega))) \\ &\leq \Phi(\omega, \delta(O_T(x_*(\omega), \tilde{x}(\omega), 0, \infty))) \\ &= \Phi(\omega, d(x_*(\omega), \tilde{x}(\omega))) \quad \text{a. s.} \end{aligned}$$

Using Lemma 2(ii) we get $d(x_*(\omega), \tilde{x}(\omega)) = 0$ a.s., i.e. $x_*(\omega) = \tilde{x}(\omega)$ a.s.

This completes the proof of Theorem 3.

From Theorem 3, we can easily deduce the following corollaries.

COROLLARY 2. *Let $T: \Omega \times X \rightarrow X$ be a continuous random operator. Suppose that there exist positive integers p and q such that for any pair $x, y \in X$*

$$P\{\omega \in \Omega, d(T^p(\omega, x), T^q(\omega, y)) \leq \mathcal{Q}(\omega)\delta(O_T(x, y, 0, \infty))\} = 1,$$

where $\mathcal{Q}(\omega): \Omega \rightarrow [0, \infty)$ is a random variable such that

$$P(\omega \in \Omega: 0 < \mathcal{Q}(\omega) < 1) = 1.$$

Then the conclusion of Theorem 3 holds.

REMARK 2. In the deterministic case, Corollary 2 is a generalization of Fisher [7, Theorem 2].

COROLLARY 3. Let $T: \Omega \times X \rightarrow X$ be a continuous random operator. Suppose that there exist positive integers p, q such that for any pair $x, y \in X$

$$(3.13) \quad P\{\omega \in \Omega: d(T^p(\omega, x), T^q(\omega, y)) \leq \Phi(\omega, \max[d(x, y), d(x, T^p(\omega, x)), d(y, T^q(\omega, y)), d(x, T^q(\omega, y)), d(y, T^p(\omega, x))])\} = 1.$$

Then the conclusion of Theorem 3 holds.

REMARK 3. Corollary 3 is a generalization of Theorem 4 in Chang [3] and Theorem 1 in [4].

In particular, from Theorem 3 we can obtain a deterministic result as follows:

COROLLARY 4. Let (X, d) be a complete metric space, $T: X \rightarrow X$ a continuous mapping and Φ a function satisfying the following conditions:

- (i) $\Phi: [0, \infty) \rightarrow [0, \infty)$ is nondecreasing and right continuous;
- (ii) $\lim_{t \rightarrow \infty} (t - \Phi(t)) = \infty$;
- (iii) For each $t > 0$

$$\lim_{n \rightarrow \infty} \Phi^n(t) = 0.$$

Suppose that there exist positive integers p, q such that for any pair $x, y \in X$

$$d(T^p x, T^q y) \leq \Phi(\delta(O_T(x, y, 0, \infty))).$$

Then T has a unique fixed point in X and for any $x_0 \in X$ the sequence $\{x_n = T^n x_0\}$ converges to this fixed point of T .

THEOREM 4. Let T be a continuous random operator satisfying the following conditions:

$$(3.14) \quad P\left\{\bigcup_{p=1}^{\infty} \bigcup_{q=1}^{\infty} \bigcap_{x \in X} \bigcap_{y \in X} [\omega \in \Omega: d(T^p(\omega, x), T^q(\omega, y)) \leq \Phi(\omega, \delta(O_T(x, y, 0, \infty)))]\right\} = P(Q) = 1,$$

where $\Phi(\omega, t)$ is the same as in Theorem 1.

Then the conclusion of Theorem 3 holds.

Proof. By the assumptions of Theorem 4, it is clear that the set

$$H_1 = Q \cap F \cap \left(\bigcap_{t>0} G_t \right) \in \mathfrak{B} \quad \text{and} \quad P(H_1) = 1,$$

where the sets Q , F , G_t , respectively, are defined in (3.14), $(\Phi 2)$ and $(\Phi 3)$.

For an arbitrary, but fixed $\tilde{\omega} \in H_1$, hence $\tilde{\omega} \in Q$, there exist positive integers p_0, q_0 such that for any $x, y \in X$

$$d(T^{p_0}(\tilde{\omega}, x), T^{q_0}(\tilde{\omega}, y)) \leq \Phi(\tilde{\omega}, \delta(O_T(x, y, 0, \infty))).$$

By Corollary 4 there exists a unique fixed point $x_*(\tilde{\omega})$ of $T(\tilde{\omega}, \cdot)$. Taking any X -valued random variable $x_0(\omega)$, we define a sequence of X -valued random variables $\{x_n(\omega)\}$:

$$x_n(\omega) = T(\omega, x_{n-1}(\omega)), \quad n = 1, 2, \dots$$

As $\omega = \tilde{\omega}$ by Corollary 4 we know that $x_n(\tilde{\omega})$ converges to $x_*(\tilde{\omega})$. This means that $x_n(\omega)$ converges almost surely to $x_*(\omega)$, hence $x_*(\omega)$ is also a X -valued random variable and

$$x_*(\omega) = T(\omega, x_*(\omega)) \quad \text{a. s.}$$

This completes the proof of Theorem 4.

REMARK 4. Theorem 6 of [3] is a special case of Theorem 4.

Especially, taking $p = q$ and $\Phi(\omega, t) = (1 - 1/m)t$ in Theorem 4, where m is some positive integer, we obtain a generalization of an important random fixed point theorem of Bharucha-Reid (cf. [1, Theorem 3, 4] or [2, Theorem 6]). The main results in Hanš [8, 9] and Spaček [14] are also special cases of Theorem 4.

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