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NON-LINEAR REPRESENTATIONS OF POINCARÉ GROUP AND GLOBAL SOLUTIONS OF RELATIVISTIC WAVE EQUATIONS

JACQUES C. H. SIMON

Non-linear massive representations of the Poincaré group are proved to be equivalent, on certain sectors, to massive linear representations with an energy of definite sign. As a consequence (for small initial data in these sectors), the existence of global solutions for massive wave equations is proved.

1. Introduction. The aim of this article is to study a family of non-linear representations of the Poincaré group $P = \mathrm{SL}(2, \mathbf{C}) \cdot T_4$, the universal covering of the inhomogeneous Lorentz group. By non-linear representation of a real Lie group one means a non-linear local action, in a vector space, which has a fixed point (say the origin).

Non-linear representations of the Poincaré group appear in a natural way in the study of relativistic wave equations, where the one parameter group of evolution is imbedded in a non-linear representation of the Poincaré group.

A formal study of this aspect of relativistic wave equations can be found in references [**2**–**5**]. The main results, there, are that the evolution of a massless wave equation is intertwined by a formal power series with the evolution of the corresponding free wave equation, and that this is also true for the evolution of a massive equation in some sectors of the space of initial conditions.

One now proves that, for massive fields, this intertwining series is convergent in some domains which will be explicated later. The proof steps will seem natural to those who are familiar with linearization, without small denominators, of vector fields [9]:

1. Check that there is no cohomological obstruction (no resonance condition on the eigenvalues of the linear part, in the language of vector fields) in order to prove the existence of the intertwining formal power series.

In the present situation this is obtained by extending the calculus to the enveloping algebra of P where the existence of a resolvent for the mass operator in a tensor product of representations permits to trivialize the cohomology. The study of the resolvent of the mass operator is done in part 3 and the construction of the formal intertwining in part 4.

2. Prove the convergence of the intertwining series. For vector fields this is easily obtained if there are no small denominators. In the present situation, the analogous condition is a bound for the norm of resolvent of the mass operator in a tensor product of representations. Convergence of the intertwining series is proved in part 5.

The intertwining power series between the time evolution of the non-linear wave equation and the evolution of the corresponding free field can be viewed as an abstract wave operator the existence of which, in some sectors, implies the existence of global solutions.

The wave equation $\square\varphi + m^2\varphi = J(\varphi)$, J analytic around the origin and $J(0) = J'(0) = 0$, is given in part 6 as an example of a wave equation with (non-zero) global solutions for small initial data in some sectors. Though the formulation is given in a four-dimensional space-time, the results are not sensitive to space-time dimension.

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2. Notations. Suppose given a Lorentz basis (X_0, X_1, X_2, X_3), with X_0 time-like, in the dual space T_4^* of T_4. If $p \in T_4^*$, $p = \sum_{\mu=0}^3 p^\mu X_\mu$ one uses the notation $p = (p^0, \vec{p})$ with $\vec{p} = \sum_{j=1}^3 p^j X_j$.

If $m \geq 0$ and $\varepsilon = \pm 1$ one defines the surface

$$M(m, \varepsilon) = \left\{ p = \sum_{\mu=0}^{3} p^\mu X_\mu \,|\, (p)^2 = (p^0)^2 - |\vec{p}|^2 = m^2, \varepsilon p^0 > 0 \right\}.$$

One denotes by $d\nu(p) = d\vec{p}/|p^0|$ the invariant measure on $M(m, \varepsilon)$. Given a unitary representation (V, Σ) of the stabilizer H of a point $p \in M(m, \varepsilon)$ in a Hilbert space Σ, one denotes by $U^{m,\varepsilon}$ the representation of P induced by V on the space $E(M(m, \varepsilon)) = L^q_{d\nu}(M(m, \varepsilon), \Sigma)$. In what follows $q = 1$ or 2. On T_4 the representation writes

$$(U_g^{m,\varepsilon} f)(p) = e^{i\langle g, p\rangle} f(p) \quad \text{and} \quad (dU_x^{m,\varepsilon} f)(p) = i\langle x, p\rangle f(p),$$

where $f \in E(M(m, \varepsilon))$, $p \in M(m, \varepsilon)$, $g \in T_4$, $x \in \mathfrak{t}_4$ (the Lie algebra of T_4). $U^{m,\varepsilon}$ is norm preserving on $E(M(m, \varepsilon))$. One denotes by $E_\infty(M(m, \varepsilon))$ (resp. $E_\tau(M(m, \varepsilon))$) the space of C^∞ vectors of $U^{m,\varepsilon}$ (resp. $U^{m,\varepsilon}|_{T_4}$). Given a compact set $K \subset M(m, \varepsilon)$, one denotes by $E(K)$ the space of functions in $L^q_{d\nu}(M(m, \varepsilon), \Sigma)$ with support included in K; obviously $E(K) \subset E_\tau(M(m, \varepsilon))$. Denote by $E_c(M(m, \varepsilon)) = \bigsqcup_K E(K)$ the union being taken over all K, compact, in $M(m, \varepsilon)$.

Suppose that $m > 0$ and $p \in M(m, \varepsilon)$; one denotes by $\Lambda(p)$ the set of the Lorentz bases for which $\vec{p} = 0$; if K is compact in $M(m, \varepsilon)$, define $\Lambda(K) \cup_{p \in K} \Lambda(p)$ which can be identified with a compact subset of $SO(1,3)$.

Suppose that $m_1 > 0$ and $m_2 \geq 0$; K_1 (resp. K_2) being a compact subset of $M(m_1, \varepsilon)$ (resp. $M_2(m_2, \varepsilon)$), define

$$c(K_1, K_2) = \inf_{p \in K_2, L \in \Lambda(K_1)} (\varepsilon p^0)$$

and

$$d(K_1, K_2) = \inf_{p \in K_2, L \in \Lambda(K_1)} (\varepsilon p^0 - |\vec{p}|).$$

Given two topological vector spaces Y, Z, one denotes by $\mathfrak{L}_n(Y, Z)$ the space of continuous n-linear symmetric mappings from Y^n to Z and by $\mathfrak{F}(Y, Z)$ the space of formal power series from Y to Z of the form $f = \sum_{n \geq 1} f^n, f^n \in \mathfrak{L}_n(Y, Z)$. When Y and Z are Banach spaces and $\lambda > 0$, $\mathfrak{H}_\lambda(Y, Z)$ is the Banach space of formal power series $f \in \mathfrak{F}(Y, Z)$ such that $\|f\| = \sum_{n \geq 1} \lambda^n \|f^n\| < +\infty$.

Given two topological vector spaces Y, Z, $A \in \mathfrak{F}(Y, Z)$, and $B \in \mathfrak{F}(Y, Y)$, one defines $A * B \in \mathfrak{F}(Y, Z)$ by

$$A * B = \sum_{n \geq 1} \left(\sum_{1 \leq p \leq n} A^p \left(\sum_{0 \leq q \leq p-1} I_q \otimes B^{n-p+1} \otimes I_{p-q-1} \right) \sigma_n \right)$$

where I_q is the identity mapping on $X \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X$ (q times) and σ_n is the symmetrization operator on $X \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X$ (n times):

$$\sigma_n(\varphi_1 \otimes \cdots \otimes \varphi_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \varphi_{\sigma(1)} \otimes \cdots \otimes \varphi_{\sigma(n)},$$

$\mathfrak{S}_n$ being the group of permutations of n elements. Whenever $Y = Z$ one defines $[A, B]_* = A * B - B * A$. Given two Banach spaces X and Y, the norm on $X \otimes_\pi Y$ is defined by $\|z\| = \inf \sum_{i,j} |a_{ij}|$, the infimum being taken over all the ways of writing $z = \sum_{i,j} a_{ij} x_i y_j$ with $\|x_i\| = \|y_j\| = 1$. In general we denote by $X \hat{\otimes}_\pi Y$ the completed projective tensor product of two locally convex topological vector spaces X and Y.

3. Resolvent of the mass operator. Denote by $(U, \mathfrak{X})$ the linear representation of the Poincaré group defined on the Banach space $\mathfrak{X} = E(M(m_1, \varepsilon)) \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E(M(m_n, \varepsilon))$ by $U = U^{m_1, \varepsilon} \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi U^{m_n, \varepsilon}$. Choose a Lorentz basis $L = (X_0, X_1, X_2, X_3)$ of $\mathfrak{t}_4$. The mass operator $Q(m_1, \ldots, m_n, \varepsilon)$ is defined on the space $D_Q = E_\infty(M(m_1, \varepsilon)) \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_\infty(M(m_n, \varepsilon))$ by

$$Q(m_1, \ldots, m_n, \varepsilon) = -\left(dU_{X_0}\right)^2 + \sum_{j=1}^{3} \left(dU_{X_j}\right)^2$$

D_Q is a space of C^∞ functions from $\prod_{j=1}^n M(m_j, \varepsilon)$ to $\Sigma_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \Sigma_n$. Therefore if $f \in D_Q$, we have

$$(Q(m_1,\dots,m_n, \varepsilon)f)(p_1,\dots,p_n) = (p_1 + \cdots + p_n)^2 f(p_1,\dots,p_n)$$

where $p_j \in M(m_j, \varepsilon)$. The operator $Q(m_1,\dots,m_n, \varepsilon)$ is obviously independent of the choice of the Lorentz basis.

PROPOSITION 3.1. *Suppose that* $1 \le k \le n$, $m_1 \ne 0,\dots,m_k \ne 0$, $m_{k+1} = \cdots = 0$. *Given* K_r, *compact, in* $M(m_r, \varepsilon)$ $(r = 1,\dots,n)$, *define* $E_\infty(K_r) = E(K_r) \cap E_\infty(M(m_r, \varepsilon))$. *The restriction* $Q'(m_1,\dots,m_n, \varepsilon)$ *of* $Q(m_1,\dots,m_n, \varepsilon)$ *to* $D_{Q'} = E_\infty(K_1) \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_\infty(K_n)$ *is closable and has a resolvent* $R_\lambda = (\overline{Q}'(m_1,\dots,m_n, \varepsilon) - \lambda)^{-1}$ *in the Banach space*

$$E(K_1) \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E(K_n)$$

$$\textit{for } \lambda < 2\Big(\sum_{1\le i\le k} d(K_1, K_i)\Big)\Big(\sum_{k+1\le j\le n} c(K_1, K_j)\Big) + (m_1 + \cdots + m_n)^2.$$

Moreover

$$\|R_\lambda\| \le \left(2\Big(\sum_{1\le i\le k} d(K_1, K_i)\Big)\Big(\sum_{k+1\le j\le n} c(K_1, K_j)\Big) + (m_1 + \cdots + m_n)^2 - \lambda\right)^{-1} \tag{3.1}$$

Proof. Suppose, say, that $\varepsilon = +1$. Write for short $Q' = Q'(m_1,\dots,m_n, \varepsilon)$.

(a) Suppose that $E(M(m_i, \varepsilon)) = L^1_{dv}(M(m_i, \varepsilon), \Sigma_i)$. Take $p_r \in K_r$ $(r = 1,\dots,n)$. Then $(p_1 + \cdots + p_n)^2 - \lambda = 2\sum_{i<j} p_i \cdot p_j + \sum_{i=1}^n m_i^2 - \lambda$.

Since $p_i \cdot p_j \ge m_i m_j$ we have

$$(p_1 + \cdots + p_n)^2 - \lambda \ge 2\Big(\sum_{1\le i\le k} p_i\Big)\Big(\sum_{k+1\le j\le n} p_j\Big) + (m_1 + \cdots + m_n)^2 - \lambda.$$

If $m_j = 0$, expressing the vectors in a Lorentz basis in $\Lambda(K_1)$, we have

$$p_i \cdot p_j \ge (p_i^0 - |\vec{p}_i|)p_j^0 \ge d(K_1, K_i)c(K_1, K_j).$$

Therefore

$$(p_1 + \cdots + p_n)^2 - \lambda \ge 2\Big(\sum_{1\le i\le k} d(K_1, K_i)\Big)\Big(\sum_{k+1\le j\le n} c(K_1, K_j)\Big) + (m_1 + \cdots + m_n)^2 - \lambda.$$

Denote by $Z = E(K_1) \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E(K_n)$. It is known [**6**, §2] that Z is isomorphic to $L^1(K_1 \times \cdots \times K_n, \Sigma_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \Sigma_n)$ (the measure on $K_1 \times \cdots \times K_n$ being the tensor product measure).

If one defines R_λ on Z by

$$(R_\lambda f)(p_1,\ldots,p_n) = \left((p_1 + \cdots + p_n)^2 - \lambda\right)^{-1} f(p_1,\ldots,p_n),$$

inequality (3.1) is satisfied.

Take $\varphi \in D_{Q'}$; we have $R_\lambda(Q' - \lambda)\varphi = \varphi$. Take a sequence $\varphi_k \in D_{Q'}$ converging to 0 and such that $(Q' - \lambda)\varphi_k$ converges to a limit ψ in Z. Then, $\varphi_i = R_\lambda(Q' - \lambda)\varphi_i$ converges to $R_\lambda\psi$. Therefore $R_\lambda\psi = 0$ and $\psi = 0$. This means that $Q' - \lambda$ is closable and so is Q'.

Take now any $\psi \in Z$ and $\varphi = R_\lambda\psi$ and choose a sequence $\psi_k \in D_{Q'}$ which converges to ψ. The sequence $R_\lambda\psi_k$ converges to φ; the relation $\psi_i = (Q' - \lambda)R_\lambda\psi_i$ implies that φ is in the domain of $\bar{Q}'$ and that $(\bar{Q}' - \lambda)R_\lambda = \mathrm{Id}_Z$.

(b) Suppose that $E(M(m_i, \varepsilon)) = L^2_{d\nu}(M(m_i, \varepsilon), \Sigma_i)$. The mapping R_λ defined above is continuous from $Y = Y^2$ to $Z = Y^1$, with

$$Y^q = L^q_{d\nu}(K_1, \Sigma_1) \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi L^q_{d\nu}(K_n, \Sigma_n).$$

In order to prove that R_λ is continuous from Y to itself one has to write the function $\mu(p_1,\ldots,p_n) = ((p_1 + \cdots + p_n)^2 - \lambda)^{-1}$ as a series of products of functions of one variable and then evaluate the norm in Y. Write $\sigma = \Sigma_{i=1}^n m_i^2 - \lambda$; then $\mu(p_1,\ldots,p_n) = (\sigma + 2\Sigma_{i<j} p_i p_j)^{-1}$. Choose a Lorentz basis $L \in \Lambda(K_1)$, then

$$\begin{aligned}(3.2)\quad & \mu(p_1,\ldots,p_n) \\ &= \frac{m_1}{p_1^0}\nu_L(p_2,\ldots,p_n)\left(1 - \xi_L(p_1,\ldots,p_n)\nu_L(p_2,\ldots,p_n)\right)^{-1},\end{aligned}$$

where

$$(3.3)\qquad \nu_L(p_2,\ldots,p_n) = \left(\sigma + 2m_1 \sum_{j>1} p_j^0 + 2\sum_{1<i<j} p_i \cdot p_j\right)^{-1}$$

and

$$\begin{aligned}(3.4)\qquad \xi_L(p_1,\ldots,p_n) = &\left(1 - \frac{m_1}{p_1^0}\right)\left(\sigma + 2\sum_{1<i<j} p_i \cdot p_j\right) \\ &+ 2m_1 \frac{\vec{p}_1}{p_1^0} \cdot \sum_{j>1} \vec{p}_j.\end{aligned}$$

Noticing that $|\xi_L(p_1,\ldots,p_n)| < (\nu_L(p_2,\ldots,p_n))^{-1}$, we can write

$$\mu(p_1,\ldots,p_n) = \sum_{k\geq 1} \mu_{k,L}(p_1,\ldots,p_n) \tag{3.5}$$

with

$$\mu_{k,L}(p_1,\ldots,p_n) = \frac{m_1}{p_1^0}(\xi_L(p_1,\ldots,p_n))^{k-1}(\nu_L(p_2,\ldots,p_n))^k. \tag{3.6}$$

Define

$$\rho = 2\Big(\sum_{1\leq i\leq k} d(K_1, K_i)\Big)\Big(\sum_{k+1\leq j\leq n} c(K_1, K_j)\Big) + (m_1 + \cdots + m_n)^2 - \lambda. \tag{3.7}$$

If $m_i \neq 0$ we define $m_i' = m_i$ and if $m_i = 0$ we define $m_i' = \inf_{p\in K_i, L\in\Lambda(K_1)}(p^0)$. Define then

$$\sigma' = 2\Big(\sum_{1\leq i\leq k} d(K_1, K_i)\Big)\Big(\sum_{k+1\leq j\leq n} c(K_1, K_j)\Big) + \sum_{i=1}^{n}(m_i')^2 - \lambda$$

and

$$\rho' = 2\Big(\sum_{1\leq i\leq k} d(K_1, K_i)\Big)\Big(\sum_{k+1\leq j\leq n} c(K_1, K_j)\Big) + (m_1' + \cdots + m_n')^2 - \lambda.$$

Now,

$$\nu_L(p_2,\ldots,p_n) = \frac{m_2'\cdots m_n'}{\rho' p_2^0\cdots p_n^0}(1 - \tau_L(p_2,\ldots,p_n))^{-1} \tag{3.8}$$

where

$$\begin{aligned}
&\tau_L(p_2,\ldots,p_n) \\
&\quad = \frac{1}{\rho' p_2^0\cdots p_n^0}\Big\{\sigma'\Big(\prod_{i>1} p_i^0 - \prod_{i>1} m_i'\Big) + (\sigma' - \sigma)\prod_{i>1} m_i' \\
&\qquad + 2m_1\sum_{j>1} m_j' p_j^0\Big(\prod_{i\neq 1,j} p_i^0 - \prod_{i\neq 1,j} m_i'\Big) \\
&\qquad + 2\sum_{1<i<j} m_i' m_j' p_i^0 p_j^0\Big(\prod_{l\neq 1,i,j} p_l^0 - \prod_{l\neq 1,i,j} m_l'\Big) \\
&\qquad + 2m_2'\cdots m_n'\sum_{1<i<j}\vec{p}_i\cdot\vec{p}_j\Big\}.
\end{aligned} \tag{3.9}$$

Moreover $\tau_L(p_2,\ldots,p_n) < 1$. We can therefore write (3.8) as a series

$$(3.10) \qquad \nu_L(p_2,\ldots,p_n) = \frac{m'_2 \cdots m'_n}{\rho' p_2^0 \cdots p_n^0} \sum_{k\geq 0} (\tau_L(p_2,\ldots,p_n))^k.$$

Introduce the variables

$$x_j = 1 - \frac{m'_j}{p_j^0}, \quad y_j = 1 + \frac{m'_j}{p_j^0}, \quad \text{and} \quad z = \frac{m'_2 \cdots m'_n}{\rho' p_2^0 \cdots p_n^0} \sum_{1<i<j} \vec{p}_i \cdot \vec{p}_j.$$

Note that $1 - m'_{i_1} \cdots m'_{i_k}/p_{i_1}^0 \cdots p_{i_k}^0$ is a polynomial with *positive* coefficients in the variables $x_{i_1},\ldots,x_{i_k}, y_{i_1},\ldots,y_{i_k}$. Therefore, it results from (3.9) that

$$\tau_L(p_2,\ldots,p_n) = \Omega(x_2,\ldots,x_n, y_2,\ldots,y_n, z)$$

Ω being a polynomial with positive coefficients in the variables $x_2,\ldots,x_n$, $y_2,\ldots,y_n$, z.

Before proceeding further one needs the following lemma.

LEMMA 3.2. *Suppose given $m_1 \geq 0,\ldots,m_r \geq 0$, positive integers $(k_{ij})_{1\leq i<j\leq r}$, K_i compact in $M(m_i, \varepsilon)$, and $f_i \in L^2_{d\nu}(K_i, \Sigma_i)$ $(i = 1,\ldots,r)$. Define*

$$h(p_1,\ldots,p_r) = \prod_{1\leq i<j\leq r} (\vec{p}_i \cdot \vec{p}_j)^{k_{ij}}$$

and

$$h'(p_1,\ldots,p_r) = \prod_{1\leq i<j\leq r} (|\vec{p}_i||\vec{p}_j|)^{k_{ij}}.$$

Then, in the Banach space $L^2(K_1, \Sigma_1) \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi L^2(K_r, \Sigma_r)$, we have

$$\|h \cdot f_1 \otimes \cdots \otimes f_r\| \leq \|h' \cdot f_1 \otimes \cdots \otimes f_r\|.$$

Proof. Suppose that $r = 2$; write $k = k_{12}$.

$$\|h \cdot f_1 \otimes f_2\| \leq \sum_{\alpha_1,\ldots,\alpha_k=1}^{3} \left(\int_{K_1} |p_1^{\alpha_1} \cdots p_1^{\alpha_k} f_1(p_1)|^2 \, d\nu(p_1)\right)^{1/2}$$
$$\times \left(\int_{K_2} |p_2^{\alpha_1} \cdots p_2^{\alpha_k} f_2(p_2)|^2 \, d\nu(p_2)\right)^{1/2}.$$

Therefore

$$\|h \cdot f_1 \otimes f_2\| \le \left(\int_{K_1} \sum_{\alpha_1, \ldots, \alpha_k = 1}^{3} |p_1^{\alpha_1} \cdots p_1^{\alpha_k} f_1(p_1)|^2 \, d\nu(p_1) \right)^{1/2}$$
$$\times \left(\int_{K_2} \sum_{\alpha_1, \ldots, \alpha_k = 1}^{3} |p_2^{\alpha_1} \cdots p_2^{\alpha_k} f_2(p_2)|^2 \, d\nu(p_2) \right)^{1/2}.$$

We get by induction

$$\|h \cdot f_1 \otimes \cdots \otimes f_r\|$$
$$\le \prod_{1 \le i < j \le r} \left(\int_{K_i} \sum_{\alpha_1, \ldots, \alpha_{k_{ij}} = 1}^{3} |p_i^{\alpha_1} \cdots p_i^{\alpha_{k_{ij}}} f_i(p_i)|^2 \, d\nu(p_i) \right)^{1/2}$$
$$\times \left(\int_{K_j} \sum_{\alpha_1, \ldots, \alpha_{k_{ij}} = 1}^{3} |p_j^{\alpha_1} \cdots p_j^{\alpha_{k_{ij}}} f_j(p_j)|^2 \, d\nu(p_j) \right)^{1/2}$$

So,

$$\|h \cdot f_1 \otimes \cdots \otimes f_r\| \le \prod_{1 \le i < j \le r} \left(\int_{K_i} \left| |\vec{p}_i|^{k_{ij}} f_i(p_i) \right|^2 d\nu(p_i) \right)^{1/2}$$
$$\times \left(\int_{K_j} \left| |\vec{p}_j|^{k_{ij}} f_j(p_j) \right|^2 d\nu(p_j) \right)^{1/2}.$$

Note that the right-hand side of the last inequality is equal to $\|h' f_1 \otimes \cdots \otimes f_r\|$. □

Coming back to the proof of Proposition 2.1, one introduces the following quantities

$$\nu_L'(p_2, \ldots, p_n) = \left(\sigma + 2m_1 \sum_{j>1} p_j^0 + 2 \sum_{1<i<j} \left(p_i^0 p_j^0 - |\vec{p}_i| \, |\vec{p}_j| \right) \right)^{-1}$$

and $\tau_L'(p_2, \ldots, p_n)$ defined as $\tau_L(p_2, \ldots, p_n)$ in expression (3.9) with $\vec{p}_i \cdot \vec{p}_j$ replaced by $|\vec{p}_i| \, |\vec{p}_j|$. We have $\tau_L'(p_2, \ldots, p_n) < 1$. Therefore

$$\nu_L'(p_2, \ldots, p_n) = \frac{m_2' \cdots m_n'}{\rho' p_2^0 \cdots p_n^0} (1 - \tau_L'(p_2, \ldots, p_n))^{-1}. \tag{3.11}$$

Choose $\eta > 0$. There exist *finite* partitions $(V^i_\beta)_{\beta \in B}$ of K_i, $i = 1,\ldots,n$, by measurable subsets such that:

1. For every $\beta \in B$ there exists $L_\beta \in \Lambda(K_1)$ such that, expressing $p = (p^0, \vec{p})$ in the basis L_β, we have $|\vec{p}| \leq \eta$ and $1 - m_1/p^0 \leq \eta$ for any $p \in V^1_\beta$.

2.

$$\Omega(\bar{x}_2,\ldots,\bar{x}_n,\bar{y}_2,\ldots,\bar{y}_n,\bar{z}) \leq (1+\eta) \sup_{V^2_{\beta_2}\times\cdots\times V^n_{\beta_n}} \tau'_{L_{\beta_1}}(p_2,\ldots,p_n), \tag{3.12}$$

where

$$\bar{x}_i = \sup_{V^i_{\beta_i}} x_i,\ \bar{y}_i = \sup_{V^i_{\beta_i}} y_i, \quad \text{and}$$

$$\bar{z} = \sup_{V^2_{\beta_2}\times\cdots\times V^n_{\beta_n}} \left(\frac{m'_2 \cdots m'_n}{\rho' p^0_2 \cdots p^0_n} \sum_{1<i<j} |\vec{p}_i||\vec{p}_j| \right)$$

for any $\beta_1,\ldots,\beta_n \in B$.

3.

$$(1+\eta) \inf_{V^2_{\beta_2}\times\cdots\times V^n_{\beta_n}} \tau'_{L_{\beta_1}}(p_2,\ldots,p_n) \geq \sup_{V^2_{\beta_2}\times\cdots\times V^n_{\beta_n}} \tau'_{L_{\beta_1}}(p_2,\ldots,p_n) \tag{3.13}$$

for any $\beta_1,\ldots,\beta_n \in B$.

4.

$$(1+\eta) \inf_{V^i_{\beta_i}} p^0_i \geq \sup_{V^i_{\beta_i}} p^0_i \tag{3.14}$$

for $\beta_2,\ldots,\beta_n \in B$ and any basis $L \in \Lambda(K_1)$.

Denote by χ^i_β the characteristic function of V^i_β. Take $f_j \in L^2_{d\nu}(K_j, \Sigma_j)$, $j = 1,\ldots,n$. Taking into account (3.6), (3.10), (3.11), and Lemma 3.2, we have

$$\begin{aligned} &\|\mu_{k,L_{\beta_1}} \chi^1_{\beta_1} f_1 \otimes \cdots \otimes \chi^n_{\beta_n} f_n\| \\ &\qquad \leq C(k,\beta_1,\ldots,\beta_n) \|\chi_{\beta_1} f_1\| \cdots \|\chi_{\beta_n} f_n\|, \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} &C(k,\beta_1,\ldots,\beta_n) \\ &= \eta^{k-1} \left(\sigma + 2 \sum_{1<i<j} \left(\left(\sup_{K_i} p^0_i \right) \left(\sup_{K_j} p^0_j \right) \right. \right. \\ &\qquad \left. \left. + \left(\sup_{K_i} |\vec{p}_i| \right) \left(\sup_{K_j} |\vec{p}_j| \right) + 2 \sum_{j>1} \sup_{K_j} |\vec{p}_j| \right)^{k-1} (K(\beta_1,\ldots,\beta_n))^k \right) \end{aligned}$$

and

$$K(\beta_1,\ldots,\beta_n) = \frac{m_2' \cdots m_n'}{\rho' \prod_{i=2}^n \inf_{V_{\beta_i}^i}(p_i^0)} \sum_{l\geq 0} (\Omega(\bar{x}_2,\ldots,\bar{x}_n, \bar{y}_2,\ldots,\bar{y}_n, \bar{z}))^l.$$

Take now $(q_2,\ldots,q_n) \in V_{\beta_2}^2 \times \cdots \times V_{\beta_n}^n$; it results from (3.12), (3.13) and (3.14) that

$$K(\beta_1,\ldots,\beta_n) \leq \frac{(1+\eta)^{n-1} m_2' \cdots m_n'}{\rho' q_2^0 \cdots q_n^0} \sum_{l\geq 0} \left((1+\eta)^2 \tau'_{L_{\beta_1}}(q_2,\ldots,q_n)\right)^l.$$

So, if η is chosen small enough

$$K(\beta_1,\ldots,\beta_n) \leq \frac{(1+\eta)^{n-1} m_2' \cdots m_n'}{\rho' q_2^0 \cdots q_n^0} \left(1 - (1+\eta)^2 \tau'_{L_{\beta_1}}(q_2,\ldots,q_n)\right)^{-1}.$$

Define

$$R = \sup_{(L_\beta)_{\beta\in B}} \left(\sigma + 2 \sum_{1<i<j} \left(\left(\sup_{K_i} p_i^0\right)\left(\sup_{K_j} p_j^0\right)\right.\right.$$
$$\left.\left. + \left(\sup_{K_i} |\vec{p}_i|\right)\left(\sup_{K_j} |\vec{p}_j|\right) + 2\sum_{j>1} \sup_{K_j} |\vec{p}_j|\right)\right).$$

We have

$$(3.16)\quad C(k,\beta_1,\ldots,\beta_n)$$
$$\leq (R\eta)^{k-1} \left((1+\eta)^{n-1} \frac{m_2' \cdots m_n'}{\rho' q_2^0 \cdots q_n^0}\right.$$
$$\left. \times \left(1 - (1+\eta)^2 \tau'_{L_{\beta_1}}(q_2,\ldots,q_n)\right)^{-1}\right)^k.$$

Choose $\eta' > 0$. If $\eta > 0$ is small enough one has:

$$(3.17)\quad \frac{R\eta(1+\eta)^{n-1} m_2' \cdots m_n'}{\rho' q_2^0 \cdots q_n^0} \left(1 - (1+\eta)^2 \tau'_{L_{\beta_1}}(q_2,\ldots,q_n)\right)^{-1} \leq \eta'$$

and

$$(3.18)\quad \frac{(1+\eta)^{n-1} m_2' \cdots m_n'}{\rho' q_2^0 \cdots q_n^0} \left(1 - (1+\eta)^2 \tau'_{L_{\beta_1}}(q_2,\ldots,q_n)\right)^{-1}$$
$$\leq (1+\eta') \nu'_{L_{\beta_1}}(q_2,\ldots,q_n)$$

independently of $\beta_1,\ldots,\beta_n$ and $(q_2,\ldots,q_n) \in V_{\beta_2}^2 \times \cdots \times V_{\beta_n}^n$.

From (3.5), (3.15), (3.17) and (3.18) one gets the inequality

$$\|\mu \cdot \chi^1_{\beta_1} f_1 \otimes \cdots \otimes \chi^n_{\beta_n} f_n\|$$
$$\leq (1+\eta')\nu'_{L_{\beta_1}}(q_2,\ldots,q_n)\left(1+\frac{\eta'}{1-\eta'}\right)\|\chi^1_{\beta_1} f_1\| \cdots \|\chi^n_{\beta_n} f_n\|.$$

Since $\nu'_{L_{\beta_1}}(q_2,\ldots,q_n) \leq \rho^{-1}$, we finally have

$$\|\mu \cdot \chi^1_{\beta_1} f_1 \otimes \cdots \otimes \chi^n_{\beta_n} f_n\|$$
$$\leq (1+\eta')\rho^{-1}\left(1+\frac{\eta'}{1-\eta'}\right)\|\chi^1_{\beta_1} f_1\| \cdots \|\chi^n_{\beta_n} f_n\|.$$

Since the functions $\{\chi^i_{\beta_i} f\}_{\beta_i \in B}$ are orthogonal for each $i \in \{1,\ldots,n\}$ and since $f_i = \sum_{\beta_i \in B} \chi^i_{\beta_i} f$, we have

$$\|\mu \cdot f_1 \otimes \cdots \otimes f_n\| \leq (1+\eta')\rho^{-1}\left(1+\frac{\eta'}{1-\eta'}\right)\|f_1\| \cdots \|f_n\|.$$

Since η' is arbitrarily small

$$\|\mu \cdot f_1 \otimes \cdots \otimes f_n\| \leq \rho^{-1}\|f_1\| \cdots \|f_n\|$$

which proves that R_λ maps Y to itself and that its norm, as an operator on Y, is smaller than ρ^{-1}.

The proof that Q' is closable in Y and that R_λ is its resolvent is the same as in part (a). This completes the proof of Proposition 3.1.

PROPOSITION 3.3. *Suppose that* $m_1 \neq 0,\ldots,m_n \neq 0$ *and that* $\lambda < (m_1 + \cdots + m_n)^2$. *The operator* $Q(m_1,\ldots,m_n,\varepsilon)$ *is closable in* $\mathfrak{X}$ *and has a resolvent* $R_\lambda = (\overline{Q}(m_1,\ldots,m_n,\varepsilon) - \lambda)^{-1}$. *Moreover*

$$\|R_\lambda\| \leq \left((m_1 + \cdots + m_n)^2 - \lambda\right)^{-1}. \tag{3.19}$$

Proof. The norm of the operator R_λ defined in Proposition 3.1 on $E(K_1)\,\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E(K_n)$ satisfies inequality (3.19). Functions with compact support form a dense set in $E(M(m_i, \varepsilon))$. Therefore R_λ has a unique extension (denoted again by R_λ) to $\mathfrak{X}$, as a continuous linear mapping, satisfying (3.19). One then proves as in part (a) of Proposition 3.1 that Q is closable in $\mathfrak{X}$ and that $R_\lambda = (\overline{Q} - \lambda)^{-1}$. □

The Fréchet space $E_\tau(M(m,\varepsilon))$ is the set of functions f from $M(m,\varepsilon)$ to Σ such that the function $p \to |p^\mu|^n f(p)$, $\mu = 0,1,2,3$, $n \in \mathbf{N}$, belongs to $L^q(M(m,\varepsilon),\Sigma)$.

On D_Q the operators Q and P_i^μ (defined by $P_i^\mu f(p_1,\ldots,p_n) = p_i^\mu f(p_1,\ldots,p_n)$, $i = 1,\ldots,n$) commute.

Therefore, from the definition of the topology on $E_\tau(M(m_1, \varepsilon)) \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_\tau(M(m_n, \varepsilon))$, Q has a unique continuous extension (denoted again by Q) to this space and, using Proposition 3.3, we have:

PROPOSITION 3.4. *Suppose that* $m_1 \neq 0, \ldots, m_n \neq 0$ *and that* $\lambda < (m_1 + \cdots + m_n)^2$. *On* $E_\tau(M(m_1, \varepsilon)) \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_\tau(M(m_n, \varepsilon))$ *the operator* $Q(m_1, \ldots, m_n, \varepsilon) - \lambda$ *has a continuous inverse*

$$R_\lambda = (Q(m_1, \ldots, m_n, \varepsilon) - \lambda)^{-1}.$$

4. Formal properties of some non linear representations. Suppose given integers $n_1 \geq 0$, $n_2 \geq 0$, $n_3 \geq 0$ and $n_4 \geq 0$; put $n = n_1 + n_2 + n_3 + n_4$. Suppose also given a continuous linear representation (U, E) of the Poincaré group, $U = \oplus_{i=1}^n U^{m_i, \varepsilon_i}$ on $E = \oplus_{i=1}^n E(M(m_i, \varepsilon_i))$ such that

1. $m_i > 0$ and $\varepsilon_i = -1$ if $i = 1, \ldots, n_1$,
2. $m_i > 0$ and $\varepsilon_i = +1$ if $i = n_1 + 1, \ldots, n_1 + n_2$,
3. $m_i = 0$ and $\varepsilon_i = -1$ if $i = n_1 + n_2 + 1, \ldots, n_1 + n_2 + n_3$,
4. $m_i = 0$ and $\varepsilon_i = +1$ if $i = n_1 + n_2 + n_3 + 1, \ldots, n$.

Denote by E^- (resp. E^+) the space $\oplus_{i=1}^{n_1} E(m_i, \varepsilon_i)$ (resp. $\oplus_{i=n_1+1}^{n_1+n_2} E(m_i, \varepsilon_i)$), and by P^- (resp. P^+, resp. P^{m_i, ε_i}) the projector on E^- (resp. E^+, resp. $E(M(m_i, \varepsilon_i))$). One labels with a subscript the corresponding projectors (P_τ^-, P_τ^+, and $P_\tau^{m_i, \varepsilon_i}$) restricted to E_τ.

Define $E_c = \oplus_{i=1}^n E_c(M(m_i, \varepsilon_i))$. As a topological vector space, $E_c(M(m_i, \varepsilon_i)) = \lim\operatorname{ind} E(K)$, where $K \subset M(m_i, \varepsilon_i)$ is compact.

Given a topological vector space Y, denote by $\mathfrak{t}(E_\tau, Y)$ the set of power series $f \in \mathfrak{F}(E_\tau, Y)$ of the form $f = \sum_{k \geq 2} f^k$, $f^k \in \mathfrak{L}_k(E_\tau, Y)$, such that $f^k(P_\tau^{m_{i_1}, \varepsilon_{i_1}} \otimes \cdots \times P_\tau^{m_{i_k}, \varepsilon_{i_k}}) = 0$ whenever at least one of the following conditions is satisfied

(C_1) $\varepsilon_{i_1}, \ldots, \varepsilon_{i_k}$ are not all equal, or

(C_2) there exists $j \in \{i_1, \ldots, i_k\}$ such that $m_j = 0$.

For short one writes $\mathfrak{L}_n(X) = \mathfrak{L}_n(X, X)$, $\mathfrak{F}(X) = \mathfrak{F}(X, X)$ (X being a topological vector space).

PROPOSITION 4.1. *Suppose that G is either P or* T_4 *and that* (S, E) *is a smooth representation of G in E (for this notion see* [**1**, *Definition* 6]), $S_g = \sum_{n \geq 1} S_g^n$, $S_g^n \in \mathfrak{L}_n(E)$, *such that*

(1) $S^1 = U|_G$.

(2) *For any* i, j *such that* $m_i m_j \neq 0$, *then* $m_i + m_j > m_k$ *for any* k; $i, j, k \in \{1, \ldots, n\}$.

Then, there exists a unique $A \in \mathfrak{t}(E_\tau, E_\tau)$ *such that*:

$$((I + A)S_g^1 - S_g(I + A))P_\tau^\pm = 0. \tag{4.1}$$

Proof. One denotes by E_∞ the space of differentiable vectors of $U|_G$. (S, E) being smooth, the mapping $g \to S^1_{g^{-1}} S^n_g$ is C^∞ from G to $\mathfrak{L}_n(E)$ and $S^n_g \in \mathfrak{L}_n(E_\infty)$ for any $g \in G$. Since T_4 is an invariant subgroup of G, and $S^1_x S^1_g \varphi = S^1_g S^1_{g^{-1}xg}\varphi$, $x \in T_4$, $g \in G$, the mapping $x \to S^1_x S^1_g \varphi$ is C^∞ from T_4 to E whenever $\varphi \in E_\tau$. Therefore, $S^1_g \in \mathfrak{L}_1(E_\tau)$ for any $g \in G$. Using the relation $S_x S_g \varphi = S_g S_{g^{-1}xg}\varphi$ one sees by induction that $S^n_g \in \mathfrak{L}_n(E_\tau)$ for any $g \in G$. Take $l \geq 2$ and suppose that there exists a unique polynomial $A_{l-1} = A^2 + \cdots + A^{l-1} \in \mathfrak{t}(E_\tau, E_\tau)$ such that

$$((I + A_{l-1})S^1_g - S_g(1 + A_{l-1}))^k P^\pm_\tau = 0 \tag{4.2}$$

for $1 \leq k \leq l - 1$. Then,

$$\begin{aligned}(S_{gg'}(I + A_{l-1}))^l P^\pm_\tau = (S_g S_{g'}(I + A_{l-1}))^l P^\pm_\tau = S^1_g(S_{g'}(I + A_{l-1}))^l P^\pm_\tau \\ + \left(\left(\sum_{n\geq 2} S^n_g\right) S_{g'}(I + A_{l-1})\right)^l P^\pm_\tau.\end{aligned}$$

Now, from (4.2), one has

$$\left(\left(\sum_{n\geq 2} S^n_g\right) S_{g'}(I + A_{l-1})\right)^l P^\pm_\tau = \left(\left(\sum_{n\geq 2} S^n_g\right)(I + A_{l-1})S^1_{g'}\right)^l P^\pm_\tau.$$

Consequently, defining $R^\pm_g = S^1_g(\Sigma_{n\geq 2} S^n_{g^{-1}}(1 + A_{l-1}))^l P^\pm_\tau$, we have

$$R^\pm_{gg'} = R^\pm_g + S^1_g R^\pm_{g'} S^1_{g^{-1}}. \tag{4.3}$$

This means that $R^\pm$ is a 1-cocycle on G with coefficients in $\mathfrak{L}_l(E_\tau)$, the action of G being defined by $g \to S^1_g Z S^1_{g^{-1}}$, $Z \in \mathfrak{L}_l(E_\tau)$.

Consider now the cocycle $dR^\pm$, on the Lie algebra $\mathfrak{t}_4$ of T_4, defined by

$$dR^\pm_X \varphi = \frac{d}{ds}\left(R^\pm_{\exp sX}\varphi\right)_{s=0}, \qquad X \in \mathfrak{t}_4, \varphi \in E_\tau.$$

Then $dR^\pm_X \in \mathfrak{L}_l(E_\tau)$. By [**8**, Lemma 6.3], $dR^\pm_X$ has a linear extension (again denoted by $dR^\pm$) to the universal enveloping algebra $\mathfrak{U}(\mathfrak{t}_4)$ of $\mathfrak{t}_4$ such that

$$dR^\pm_{XY} = dS^1_X dR^\pm_Y + dR^\pm_X * dS^1_Y, \qquad X, Y \in \mathfrak{U}(\mathfrak{t}_4)$$

(because $dR^\pm_Y * dS^1_X = dR^\pm_Y\, d(\hat{\otimes}^l_\pi S^1)_X$).

Given a Lorentz basis $\{X_0, X_1, X_2, X_3\}$ in $\mathfrak{t}_4$, we have the central element $Q = \Sigma^3_{\mu=0} \eta_\mu X^2_\mu$, where $\eta_0 = -1$, $\eta_1 = \eta_2 = \eta_3 = 1$. Therefore

$$dR^\pm_Q = \sum_{\mu=0}^{3} \eta_\mu\left(dS^1_{X_\mu} dR^\pm_{X_\mu} + dR^\pm_{X_\mu} * dS^1_{X_\mu}\right).$$

If $X \in \mathfrak{t}_4$, we have

$$dR^{\pm}_{XQ} = dS^1_X dR^{\pm}_Q + dR^{\pm}_X d(\otimes^l S^1)_Q, \tag{4.4}$$

$$dR^{\pm}_{QX} = dS^1_Q dR^{\pm}_X + dR^{\pm}_Q * dS^1_X. \tag{4.5}$$

Equalities $XQ = QX$, (4.4), (4.5) and $P_\tau^{m_j,\varepsilon_j} dS^1_Q = m_j^2 P_\tau^{m_j,\varepsilon_j}$ imply (for $\varepsilon_{i_1} = \cdots = \varepsilon_{i_l} = \varepsilon = \pm 1$, R^ε meaning R^+ or R^- according to the sign of ε) that

$$P^{m_j,\varepsilon_j}\Big(dR^\varepsilon_X\big(m_j^2 - Q(m_{i_1},\ldots,m_{i_l}, \varepsilon)\big) - \big(dS^1_X dR^\varepsilon_Q - dR^\varepsilon_Q * dS^1_X\big)\Big) \times \big(P_\tau^{m_{i_1},\varepsilon_{i_1}} \otimes \cdots \otimes P_\tau^{m_{i_l},\varepsilon_{i_l}}\big) = 0. \tag{4.6}$$

When $m_{i_1} \neq 0,\ldots,m_{i_l} \neq 0$ and $\varepsilon_{i_1} = \cdots = \varepsilon_{i_k} = \varepsilon = \pm 1$, we shall define

$$A^l\big(P_\tau^{m_{i_1},\varepsilon_{i_1}} \otimes \cdots \otimes P_\tau^{m_{i_l},\varepsilon_{i_l}}\big) = \sum_{j=1}^{n} P^{m_j,\varepsilon_j} dR^\varepsilon_Q\big(m_j^2 - \bar{Q}(m_{i_1},\ldots,m_{i_l}, \varepsilon)\big)^{-1} P_\tau^{m_{i_1},\varepsilon_{i_1}} \otimes \cdots \otimes P_\tau^{m_{i_l},\varepsilon_{i_l}} \tag{4.7}$$

and take $A^l(P_\tau^{m_{i_1},\varepsilon_{i_1}} \otimes \cdots \otimes P_\tau^{m_{i_l},\varepsilon_{i_l}}) = 0$ if $\varepsilon_{i_1},\ldots,\varepsilon_{i_l}$ are not all equal or if there exists $j \in \{i_1,\ldots,i_l\}$ such that $m_j = 0$. We have

$$dR^\varepsilon_X = dS^1_X A^l P^\varepsilon_\tau - \big(A^l P^\varepsilon_\tau\big) * dS^1_X, \qquad x \in \mathfrak{t}_4, \tag{4.8}$$

which means that dR^ε is the coboundary of $A^l P^\varepsilon_\tau$ on $\mathfrak{t}_4$. Consequently

$$R^\varepsilon_x = S^1_x\big(A^l P^\varepsilon_\tau\big)S^1_{x^{-1}} - A^l P^\varepsilon_\tau, \qquad x \in T_4.$$

Define $\tilde{R}^{\pm}_g = R^{\pm}_g - S^1_g(AP^{\pm}_\tau)S^1_{g^{-1}} - A^l P^{\pm}_\tau$, $g \in G$. We have $\tilde{R}^{\pm}_x = 0$ for $x \in T_4$. $\tilde{R}^{\pm}$ is a 1-cocycle on G with coefficients in $\mathfrak{L}_l(E_\tau)$. Writing $\tilde{R}^{\pm}_{g^{-1}xg} = 0$, $x \in T_4$, one gets

$$S^1_x \tilde{R}^{\pm}_g S^1_{x^{-1}} = -S^1_g \tilde{R}^{\pm}_{g^{-1}} S^1_{g^{-1}}, \qquad x \in T_4, g \in G. \tag{4.9}$$

The right-hand side of (4.9) being independent of X one obtains

$$dS^1_X \tilde{R}^{\pm}_g - \tilde{R}^{\pm}_g * dS^1_X = 0, \qquad X \in \mathfrak{t}_4.$$

Therefore

$$\sum_{\mu=0}^{3} \eta_\mu\Big(\big(dS^1_{X_\mu}\big)^2 \tilde{R}^{\pm}_g - \Big(\big(\tilde{R}^{\pm}_g * dS^1_{X_\mu}\big) * dS^1_{X_\mu}\Big)\Big) = 0$$

which implies that

$$P^{m_j,\varepsilon_j}\tilde{R}^{\pm}_g\big(m_j^2 - Q(m_{i_1},\ldots,m_{i_l}, \varepsilon_{i_1})\big)\big(P_\tau^{m_{i_1},\varepsilon_{i_1}} \otimes \cdots \otimes P_\tau^{m_{i_l},\varepsilon_{i_l}}\big) = 0.$$

Proposition 3.4 implies then that $\tilde{R}_g^{\pm} = 0$. Going back to the definition of $R_g^{\pm}$, this means that

$$S_g^1\left(\left(\sum_{n\geq 2} S_{g^{-1}}^n\right)(I + A_{l-1})\right)^l P_\tau^{\pm} = S_g^1 A^l P_\tau^{\pm} S_{g^{-1}}^1 - A^l P_\tau^{\pm},$$

which can be rewritten with $A_l = A_{l-1} + A^l$

$$(I + A_l)^l S_g^1 P_\tau^{\pm} = \big(S_g(I + A_l)\big)^l P_\tau^{\pm}. \tag{4.10}$$

One then defines $A = \sum_{l\geq 2} A^l$. There remains to prove the uniqueness of $A^l \in \mathfrak{t}(E_\tau, E_\tau)$. Suppose that there exists a second one, $\tilde{A}^l$. Equality (4.10) implies that

$$S_x^1(A^l - \tilde{A}^l) - (A^l - \tilde{A}^l)S_x^1 = 0, \qquad x \in T_4.$$

Therefore

$$dS_X^1(A^l - \tilde{A}^l) - (A^l - \tilde{A}^l) * dS_X^1 = 0, \qquad X \in \mathfrak{t}_4.$$

The same calculation as above proves then that $A^l = \tilde{A}^l$. □

REMARK 4.2. Consider the series $A \in \mathfrak{t}(E_\tau, E_\tau)$ satisfying the conclusions of Proposition 4.1. The formal power series $I + A$ has an inverse in $\mathfrak{F}(E_\tau)$. Consider, on E_τ, the formal representation (for this notion see [**1**, Definition 1]) $\tilde{S}_g = (I + A)^{-1}S_g(I + A)$. It satisfies $(S_g^1 - \tilde{S}_g)P_\tau^{\pm} = 0$.

5. Convergence of the intertwining power series. Given two topological vector spaces X and Y, one denotes by $\tilde{\mathfrak{F}}(X, Y)$ the subspace of $\mathfrak{F}(X, Y)$ of the series $f = \sum_{n\geq 2} f^n$, $f^n \in \mathfrak{L}_n(X, Y)$. If X and Y are Banach spaces and $r > 0$ one defines $\tilde{\mathfrak{H}}_r(X, Y) = \mathfrak{H}_r(X, Y) \cap \tilde{\mathfrak{F}}(X, Y)$. We shall write $\tilde{\mathfrak{F}}(E) = \tilde{\mathfrak{F}}(E, E)$, $\mathfrak{H}_r(E) = \mathfrak{H}_r(E, E)$ and $\tilde{\mathfrak{H}}_r(E) = \tilde{\mathfrak{H}}_r(E, E)$, and keep the hypotheses of part 4.

A linear operator W in $\tilde{\mathfrak{F}}(E_\tau, E)$ can be defined by:

(1) $W(f^k(P_\tau^{m_{i_1}, \varepsilon_{i_1}} \otimes \cdots \otimes P_\tau^{m_{i_k}, \varepsilon_{i_k}})) = 0$ if condition (C_1 or C_2) of part 4 is satisfied.

(2) If condition (C_1 or C_2) is not satisfied:

$$\begin{aligned} &W\big(f^k\big(P_\tau^{m_{i_1}, \varepsilon_{i_1}} \otimes \cdots \otimes P_\tau^{m_{i_k}, \varepsilon_{i_k}}\big)\big) \\ &\quad = \sum_{j=1}^{n} P^{m_j, \varepsilon_j} f^k\Big(\big(m_j^2 - \overline{Q}\big(m_{i_1} \cdots m_{i_k}, \varepsilon_{i_1}\big)\big)^{-1} \\ &\qquad\qquad \times P_\tau^{m_{i_1}, \varepsilon_{i_1}} \otimes \cdots \otimes P_\tau^{m_{i_k}, \varepsilon_{i_k}}\Big). \end{aligned}$$

Obviously $W(f) \in \mathfrak{t}(E_\tau, E)$. Moreover, it results from Proposition 3.3 that there exists a constant C such that

$$\left\|\left(m_j^2 - \bar{Q}\left(m_{i_1},\ldots,m_{i_k}, \varepsilon_{i_1}\right)\right)^{-1}\right\| \leq Ck^{-2}. \tag{5.1}$$

Consequently, given compact sets $K_i \subset M(m_i, \varepsilon_i)$, $i = 1,\ldots,n$, W is continuous from $\tilde{\mathfrak{H}}_r(\oplus_{i=1}^n E(K_i), E)$ to itself.

LEMMA 5.1. *Suppose that G is either P or T_4 and that (S, E) is a smooth representation of G in E satisfying the hypotheses of Proposition* 4.1. *Given K_i compact in $M(m_i, \varepsilon_i)$ $(i = 1,\ldots,n)$ there exists $r > 0$ such that the series $A \in \mathfrak{t}(E_\tau, E_\tau)$ defined by Proposition* 4.1 *belongs to $\tilde{\mathfrak{H}}_r(\oplus_{i=1}^n E(K_i), E)$.*

Proof. We shall write equality 4.1 in a form which will be more convenient to prove the convergence of A.

Take $x \in \mathfrak{t}_4$; equality (4.1) implies, by differentiation, that

$$\left((I + A) * dS_X^1 - dS_X(I + A)\right)P_\tau^{\pm} = 0. \tag{5.2}$$

Let $T_X = \sum_{n\geq 2} dS_X^n$. Equality (5.2) now writes

$$dS_X^1\left(AP_\tau^{\pm}\right) - \left(AP_\tau^{\pm}\right) * dS_X^1 = -T_X(I + A)P_\tau^{\pm} \tag{5.3}$$

which in turn implies that

$$\begin{aligned} dS_X^1 dS_X^1\left(AP_\tau^{\pm}\right) &- \left(\left(AP_\tau^{\pm}\right) * dS_X^1\right) * dS_X^1 \\ &= -\left(dS_X^1 T_X(I + A) + \left(T_X(I + A)\right) * dS_X^1\right)P_\tau^{\pm}. \end{aligned}$$

Therefore, if $\varepsilon_{i_1} = \cdots = \varepsilon_{i_k} = \pm 1$ and $m_{i_1} \neq 0,\ldots,m_{i_k} \neq 0$, we have

$$\begin{aligned} &P^{m_j,\varepsilon_j}A^k\left(m_j^2 - Q\left(m_{i_1},\ldots,m_{i_k}, \varepsilon_{i_1}\right)\right)P_\tau^{m_{i_1},\varepsilon_{i_1}} \otimes \cdots \otimes P_\tau^{m_{i_k},\varepsilon_{i_k}} \\ &\quad = -P^{m_j,\varepsilon_j}\left[\sum_{\mu=0}^{3} \eta_\mu\left(dS_{X_\mu}^1 T_{X_\mu}(I + A) + \left(T_{X_\mu}(I + A)\right) * dS_{X_\mu}^1\right)\right] \\ &\qquad \times P_\tau^{m_{i_1},\varepsilon_{i_1}} \otimes \cdots \otimes P_\tau^{m_{i_k},\varepsilon_{i_k}} \end{aligned}$$

which is equivalent to

$$A = -W\left(\sum_{\mu=0}^{3} \eta_\mu\left(dS_{X_\mu}^1 T_{X_\mu}(I + A) + \left(T_{X_\mu}(I + A)\right) * dS_{X_\mu}^1\right)\right). \tag{5.4}$$

Now, using the fact that A satisfies equality (5.3) one has

$$\begin{aligned} dS_{X_\mu}^1 T_{X_\mu}(I + A)P_\tau^{\pm} = \Big(\left[dS_{X_\mu}^1, T_{X_\mu}\right]_*(I + A) &- \left(T_{X_\mu} * T_{X_\mu}\right)(I + A) \\ &+ \left(T_{X_\mu}(1 + A)\right) * dS_{X_\mu}^1\Big)P_\tau^{\pm}. \end{aligned}$$

Consequently, since $A \in \mathfrak{t}(E_\tau, E_\tau)$, one gets (from (5.4)) the equation:

$$A = N(A) \tag{5.5}$$

with

$$N(A) = -W\left(\sum_{\mu=0}^{3} \eta_\mu\Big(\big[dS^1_{X_\mu}, T_{X_\mu}\big]_*(I+A) - \big(T_{X_\mu} * T_{X_\mu}\big)(I+A)\right.$$
$$\left. +2\big(T_{X_\mu}(I+A)\big) * dS^1_{X_\mu}\Big)\right). \tag{5.6}$$

Notice that the term of degree k in $N(A)$ depends only on $A^2,\ldots,A^{k-1}$. Therefore equation (5.5) has one and only one solution in $\tilde{\mathfrak{F}}(E_c, E)$. Since, as was just proved, the solution of (4.1) is solution of (5.5), it is sufficient to solve (5.5) in $\mathfrak{H}_r(\oplus_{i=1}^n E(K_i), E)$ for some $r > 0$.

(S, E) being a smooth representation, there exists $\lambda_1 > 0$ such that $(g, g') \to S^1_{g'g^{-1}}S_{gg'^{-1}}$ is C^∞ from a neighbourhood of the identity in $G \times G$ to $\mathfrak{H}_{\lambda_1}(E)$. Writing $g = \exp sX$ and $g' = \exp s'X$, $X \in \mathfrak{t}_4$, one gets

$$\frac{\partial^2}{\partial s \partial s'}\left(S^1_{g'g^{-1}}S_{gg'^{-1}}\right)_{s=s'=0} = dS^1_X T_X\varphi - T_X * dS_X\varphi \quad \text{with } \varphi \in E_\infty.$$

Therefore

$$\big[dS^1_X, T_X\big]_*\varphi = \frac{\partial^2}{\partial s\partial s'}\left(S^1_{g'g^{-1}}S_{gg'^{-1}}\right)_{s=s'=0}\varphi + T_X * T_X\varphi.$$

Thus $[dS^1_X, T_X]_* \in \mathfrak{H}_{\lambda_2}(E)$ for some $\lambda_2 > 0$. Take now

$$f = \sum_{k\geq 2} f^k \in \mathfrak{H}_\lambda(E), \quad h_1 = \sum_{k\geq 2} h_1^k \quad \text{and}$$

$$h_2 = \sum_{k\geq 2} h_2^k \quad \text{in } \mathfrak{H}_{\lambda'}\left(\bigoplus_{i=1}^n E(K_i), E\right)$$

with $0 < 4\lambda' < \lambda$. We have

$$\|f(I+h_1)\|_{\lambda'} \leq \sum_{k\geq 2} \|f^k\|(\lambda' + \|h_1\|_{\lambda'})^k \tag{5.7}$$

and

$$f(I+h_2) - f(I+h_1) = \int_0^1 \frac{d}{ds} f(a(s))\, ds$$

with $a(s) = I + h_1 + s(h_2 - h_1)$. Since

$$\frac{d}{ds} f(a(s)) = \sum_{k\geq 2} k f^k(a(s),\ldots,a(s), h_2 - h_1),$$

we have

$$\left\| \frac{d}{ds} f(a(s)) \right\|_{\lambda'} \leq \sum_{k \geq 1} \lambda'^k \sum_{2 \leq p \leq k} p \| f^p \|$$

$$\times \left(\sum_{i_1 + \cdots + i_p = k} \| a^{i_1}(s) \| \cdots \| a^{i_{p-1}}(s) \| \, \| h_2^{i_1} - h_1^{i_p} \| \right)$$

$$\leq \left(\sum_{p \geq 2} p \| a(s) \|_{\lambda'}^{p-1} \| f^p \| \right) \| h_2 - h_1 \|_{\lambda'}.$$

Now, if $\| h_1 \|_{\lambda'} \leq \lambda'$ and $\| h_2 \|_{\lambda'} \leq \lambda'$ then $\| a(s) \|_{\lambda'} \leq 4\lambda'$, and there exists a constant $C(\lambda')$ such that $\lim_{\lambda' \to 0} C(\lambda') = 0$ and

$$\| f(I + h_2) - f(I + h_1) \|_{\lambda'} \leq C(\lambda') \| h_2 - h_1 \|_{\lambda'}. \tag{5.8}$$

It results from (5.7), (5.8), and from the fact that W is continuous from $\tilde{\mathfrak{H}}_{\lambda}(\oplus_{i=1}^n E(K_i), E)$ to itself for any $\lambda > 0$, that for $r > 0$ small enough the mapping

$$A \to f_1(I + A)$$
$$= -W\left(\sum_{\mu=0}^{3} \eta_\mu\left(\left[dS_{X_\mu}^1, T_{X_\mu} \right]_* (I + A) \right) - \left(T_{X_\mu} * T_{X_\mu} \right)(I + A) \right)$$

maps the closed ball of radius r to the ball of radius $r/2$ in $\tilde{\mathfrak{H}}_r(\oplus_{i=1}^n E(K_i), E)$ and

$$\| f_1(I + A_1) - f_1(I + A_2) \|_r \leq \frac{\delta}{2} \| A_1 - A_2 \|_r,$$

with $\delta < 1$, and A_1, A_2 in the closed ball of radius r.

Now, using the fact that $dS_{X_\mu}^1$ is bounded on $E(K_i)$, from inequality (5.1) and again from (5.7) and (5.8) one sees that, for $r > 0$ small enough, the mapping

$$A \to f_2(I + A) = -2W\left(\sum_{\mu=0}^{3} \left(T_{X_\mu}(1 + A) \right) * dS_{X_\mu}^1 \right)$$

maps the closed ball of radius r to the ball of radius $r/2$ in $\tilde{\mathfrak{H}}_r(\oplus_{i=1}^n E(K_i), E)$ and,

$$\| f_2(I + A_1) - f_2(I + A_2) \|_r \leq \frac{\delta}{2} \| A_1 - A_2 \|_r,$$

for A_1, A_2 in the closed ball of radius r.

REMARK 5.2. This was the point where inequality (5.1) is important. In fact inequalities of the type $\|m_j^2 - \overline{Q}(m_{i_1},\dots,m_{i_k}, \varepsilon_{i_1})\|^{-1} \leq Ck^{-1}$ would be sufficient to be still in the situation of no "small denominators". Going back to the proof of Proposition 5.1, the mapping

$$A \to N(A) = f_1(I + A) + f_2(I + A)$$

sends the closed ball of radius r in $\tilde{\mathfrak{H}}_r(\oplus_{i=1}^n E(K_i), E)$ to itself and $\|N(A_1) - N(A_2)\|_r \leq \delta \|A_1 - A_2\|_r$, for A_1, A_2 in the ball of radius r. By the contraction mapping theorem, there exists A, unique in $\tilde{\mathfrak{H}}_r(\oplus_{i=1}^n E(K_i), E)$, which is solution of equation (5.5). □

THEOREM 5.3. *Suppose that G is either P or T_4 and that (S, E) is an analytic representation of G in E (for this notion see* [**1**, *Definition* 2]), *$S_g = \Sigma_{n\geq 1} S_g^n$, $S_g^n \in \mathfrak{L}_n(E)$, such that*

1. $S_g^1 = U|_G$.

2. For any i, j such that $m_i \neq 0$ and $m_j \neq 0$, then $m_i + m_j > m_k$, for any $k, i, j, k \in \{1,\dots,n\}$.

Then there exists a unique $A \in \mathfrak{t}(E_\tau, E)$ such that

$$\big((I + A)S_g^1 - S_g(1 + A)\big)P_\tau^{\pm} = 0 \tag{5.9}$$

and, given $K_i \subset M(m_i, \varepsilon_i)$ compact $(i = 1,\dots,n)$ there exists $r > 0$ such that $A \in \tilde{\mathfrak{H}}_r(\oplus_{i=1}^n E(K_i), E)$.

Proof. From [**1**, Proposition 5], there exists $B = \Sigma_{n\geq 2} B^n$ in $\mathfrak{H}_\lambda(E)$, for some $\lambda > 0$, such that $V_g = (I + B)^{-1}S_g(I + B)$ is a smooth representation of G in E.

By Proposition 5.1, there exists $C = \Sigma_{n\geq 2} C^n$ in $\mathfrak{t}(E_\tau, E_\tau)$ such that

$$\big((1 + C)S_g^1\big)P_\tau^{\pm} = V_g(I + C)P_\tau^{\pm}$$

and there exists $\lambda > 0$ such that $C \in \tilde{\mathfrak{H}}_\lambda(\oplus_{i=1}^n E(K_i), E)$. Define $(I + D) = (I + B)(I + C)$. Obviously $(I + D)S_g^1P_\tau^{\pm} = S_g(1 + D)P_\tau^{\pm}$. Define now $A^k \in \mathfrak{L}_k(E_\tau, E)$ by

$$A^kP_\tau^{m_{i_1},\varepsilon_{i_1}} \otimes \cdots \otimes P_\tau^{m_{i_k},\varepsilon_{i_k}} = D^kP_\tau^{m_{i_1},\varepsilon_{i_1}} \otimes \cdots \otimes P_\tau^{m_{i_k},\varepsilon_{i_k}} \quad \text{if } m_{i_1} \neq 0,\dots,m_{i_k} \neq 0$$

and $\varepsilon_{i_1} = \cdots = \varepsilon_{i_k}$ and by $A^kP_\tau^{m_{i_1},\varepsilon_{i_1}} \otimes \cdots \otimes P_\tau^{m_{i_k},\varepsilon_{i_k}} = 0$ otherwise. Then $A \in \mathfrak{t}(E_\tau, E)$ and satisfies equality (5.9).

Suppose now that there exists a second series $\tilde{A}$ satisfying the conclusion of Theorem 5.3. One gets

$$(A - \tilde{A})S_g^1P_\tau^{\pm} = (S_g(1 + A) - S_g(1 + \tilde{A}))P_\tau^{\pm}.$$

Identifying degree by degree in this equality, one gets inductively $(A - \tilde{A})P_\tau^{\pm} = 0$ and therefore, since $A, \tilde{A} \in \mathfrak{t}(E_\tau, E)$, $A = \tilde{A}$. □

6. Application to a family of wave equations. We shall now give an example describing a spinless field of mass $m > 0$. One denotes by $f \to \hat{f}$ the Fourier transform in the space $\mathcal{S}'(\mathbf{R}^3)$ of tempered distributions on $\mathbf{R}^3$. Choose a Lorentz basis (X_0, X_1, X_2, X_3) in $\mathfrak{t}_4$. One introduces the following spaces.

(1) H_0 is the space of tempered distributions, the Fourier transform of which are elements of $L^1(\mathbf{R}^3)$. If $f \in H_0$, define $\| f \|_0 = \| \hat{f} \|_{L^1(\mathbf{R}^3)}$.

(2) H_1 is the space of tempered distributions $f \in \mathcal{S}'(\mathbf{R}^3)$, the Fourier transform of which, $\hat{f}$, is a function such that $\vec{p} \to (m^2 + |\vec{p}|^2)^{-1/2}\hat{f}(\vec{p})$ is in $L^1(\mathbf{R}^3)$. If $f \in H_1$ define

$$\| f \|_1 = \int_{\mathbf{R}^3} \left(m^2 + |\vec{p}|^2\right)^{-1/2} |\hat{f}(\vec{p})| \, d\vec{p}.$$

(3) $\tilde{H}$ is the Fréchet space of functions f such that the function $\vec{p} \to (m^2 + |\vec{p}|^2)^n \hat{f}(\vec{p})$ is in $L^1(\mathbf{R}^3)$ for any $n \geq 0$.

Suppose now given an analytic mapping $J \in \tilde{\mathfrak{F}}_r(\mathbf{C})$ $(r > 0)$. If $\varphi \in H_0$, define $(J(\varphi))(\vec{x}) = J(\varphi(\vec{x}))$ and $\hat{J}(\hat{f}) = J(f)\hat{}$. Since $L^1(\mathbf{R}^3)$ is a convolution algebra, J can be considered as an element of $\tilde{\mathfrak{F}}_r(H_0)$.

Consider now the wave equation (where $J(0) = J'(0) = 0$ by definition)

$$\Box \varphi_t + m^2 \varphi_t = J(\varphi_t) \tag{6.1}$$

Write $\dot{\varphi}_t = d/dt\, \varphi_t$. One wishes to solve (6.1) for $\varphi_t \in H_0$, $\dot{\varphi}_t \in H_1$ (i.e. one takes $H_0 \oplus H_1$ as space of initial conditions). Define

$$a_t^+(\vec{p}) = \hat{\dot{\varphi}}_t + i\left(m^2 + |\vec{p}|^2\right)^{1/2} \hat{\varphi}_t \quad \text{and}$$
$$a_t^-(\vec{p}) = \hat{\dot{\varphi}}_t - i\left(m^2 + |\vec{p}|^2\right)^{1/2} \hat{\varphi}_t \qquad (i^2 = -1).$$

Equation (6.1) is then equivalent to the system

$$\begin{cases} \dfrac{da_t^+}{dt} = i\left(m^2 + |\vec{p}|^2\right)^{1/2} a_t^+ + G_1(a_t^+, a_t^-), \\ \dfrac{da_t^-}{dt} = -i\left(m^2 + |\vec{p}|^2\right)^{1/2} a_t^- + G_2(a_t^+, a_t^-), \end{cases} \tag{6.2}$$

with

$$G_1(a^+, a^-) = G_2(a^+, a^-) = \hat{J}\left(-\frac{i}{2}\left(m^2 + |\vec{p}|^2\right)^{-1/2}(a^+ - a^-)\right).$$

Put $G = (G_1, G_2)$. The functions $\vec{p} \to (m^2 + |\vec{p}|^2)^{-1/2} a_t^{\pm}(\vec{p})$ belong to $L^1(\mathbf{R}^3)$, therefore $G \in \tilde{\mathfrak{H}}_r(H_0 \times H_0)$. One now defines $E(M(m, \varepsilon)) = L^1_{d\nu}(M(m, \varepsilon))$. We can consider $a_t^{\pm}$ as an element of $E(M(m, \pm 1))$ by the identification $a^{\pm}(\vec{p}) = a^{\pm}(p)$ if $p = (p_0, \vec{p}) \in M(m, \pm 1)$. We now denote $S^1 = U^{m,-1} \oplus U^{m,+1}$ the representation of the Poincaré group P on $E = E(M(m, -1)) \oplus E(M(m, +1))$. The system (6.2) becomes

$$\frac{d}{dt}\psi_t = dS^1_{X_0}\psi_t + G(\psi_t)$$

where $\psi_t = (a_t^-, a_t^+)$.

If $X = \Sigma_{\mu=0}^3 \alpha^\mu X_\mu$, one defines

$$\theta_X \psi = dS^1_X \psi + \alpha^0 G(\psi)$$

which is an analytic representation of $\mathfrak{t}_4$ in E_τ compatible with S^1 (for this notion see [**1**, Definition 8]). By [**1**, Proposition 10], there exists a unique analytic representation (S, E) of T_4 in E such that $dS = \theta$. It results from Theorem 5.3 that there exists $A \in \tilde{\mathfrak{F}}(E_\tau, E)$ such that given K_- (resp. K_+) compact in $M(m, -1)$ (resp. $M(m, +1)$),

$$A \in \tilde{\mathfrak{H}}_\lambda(E(K_-) \oplus E(K_+), E) \quad \text{for some } \lambda > 0,$$

and such that

$$(I + A)S^1_g P_\tau^{\pm} = S_g(I + A)P_\tau^{\pm}.$$

Take $h \in E(K_-) \cup E(K_+)$ with $\|h\| < \lambda$. Define $\psi_t = (I + A)S^1_{\exp tX_0} h$. Since S^1_g is norm preserving and $S_g(I + A)h = (I + A)S^1_g h$, the mapping $t \to \psi_t$ is C^∞ from $\mathbf{R}$ to E and we have $d/dt\psi_t = \theta_{X_0}\psi_t$.

Coming back to equation (6.1) one gets in particular:

PROPOSITION 6.1. *There exists $A \in \tilde{\mathfrak{F}}(\tilde{H} \times \tilde{H}, H_0 \times H_1)$ with the following properties*:

(1) *Given any compact K in $\mathbf{R}^3$ there exists $\lambda > 0$ for which $A \in \tilde{\mathfrak{H}}_\lambda(H_K \times H_K, H_0 \times H_1)$, H_K being the space of such functions in H_0, the Fourier transform of which vanish in the complement of K.*

(2) *If $h_0 \in H_K$, $\|h_0\| < \lambda/2$ and $\hat{h}_1(\vec{p}) = \pm i(m^2 + |\vec{p}|^2)^{1/2}\hat{h}_0(p)$, equation* (6.1) *has a solution φ_t for all t, with initial condition $(\varphi_0, \dot{\varphi}_0) = (I + A)(h_0, h_1)$.*

The approach followed in this article can be applied to systems of relativistic evolution equations with arbitrary spin with an analytic interaction (provided that the masses satisfy the inequalities given at point 2 of Theorem 5.3.) to get global solutions for small data in some sectors.

Let us now compare the example given above with previous results. The problem of existence of global solutions for relativistic wave equations, seen from an abstract point of view, was initiated by I. Segal [**11**]. An example which has been studied extensively is the equation

$$\Box\varphi + m^2\varphi = \lambda\varphi^p \qquad (p \in \mathbf{N}) \tag{6.3}$$

for which global solutions exist for small initial data (in a suitable Sobolev space) [**10**, Theorem 21], when $p \geq 3$. Furthermore, (6.3) has global weak solutions for any initial data, with the restriction $\lambda < 0$, p odd [**12**].

Concerning the equation $\Box\varphi + m^2\varphi = J(\varphi)$ with φ real, J being a continuous real valued function satisfying $\varphi J(\varphi) \leq 0$, W. Strauss [**13**] proved that there exist real global weak solutions for any real initial data.

Global solutions which are C^∞ in space and time for equations of the type

$$\Box\varphi = G(\partial_i\varphi, \partial_i\partial_j\varphi), \tag{6.4}$$

(where $G(\xi_i, \eta_{ij})$ is a C^∞ function in ξ_i and η_{ij} and vanishing for $\xi = 0$ and $\eta = 0$, and where the space of initial data is the space of C^∞ functions on $\mathbf{R}^d$, $d \geq 6$) have been proved to exist by S. Klainerman [7] if the initial data is small enough (i.e. the L^1 and L^2 norms of the initial data and a certain number of its derrivatives are small enough).

The problem of the existence of global solutions for relativistic wave equations for initial data in some sectors is discussed in a review work of M. Reed [**10**].

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LABORATOIRE DE PHYSIQUE-MATHEMATIQUE
FACULTE DES SCIENCES MIRANDE-BP 138
21004-DIJON CEDEX, FRANCE