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AMPLENESS IN COMPLEX HOMOGENEOUS SPACES AND A SECOND LEFSCHETZ THEOREM

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This paper investigates how ampleness of the normal bundle of a smooth subvariety Y of a complex homogeneous space $Z = G/H$ influences the intersection of Y with other subvarieties of Z.

We consider a class of homogeneous spaces, rigged spaces, that includes Grassmannians, quadrics and $\mathbf{P}^r \backslash \mathbf{P}^k$ (the compliment in $\mathbf{P}^r$ of a linear subspace $\mathbf{P}^k$). A result of Corollary 4.5.2 is:

Let Z be a rigged homogeneous space with group G. Let Y be a compact smooth subvariety of Z possessing an ample normal bundle NY. (See [**10**] for the definition of ample.) Then the map

$$\phi_Y\colon \mathbf{P}(N^*Y) \to \mathbf{P}^a$$

determined by the G-sections of TZ is generically 1-1 (see 2.2 for the definition of ϕ_Y).

Corollary 4.5.2 and Theorem 5.2 imply that if X and Y are both smooth and compact subvarieties of Z with ample normal bundles, then for all $g \in G$, except for a closed codimension 2 subvariety of G, $X \cap g^{-1}(Y)$ is either a transverse intersection, or has precisely one singular point and it is non-degenerate quadratic.

In §5 these results are used to prove a generalized "second Lefschetz theorem on hyperplane sections", in analogy to the author's previous paper [**6**], and following the generalized first Lefschetz theorems of Barth [**2, 2A**] and Sommese [**19, 20**].

I expand, now, the outline of the paper.

Section 1 begins by considering a holomorphic bundle map $\psi\colon E \to F$ of holomorphic vector bundles over a complex space W, i.e. $\psi_x\colon E_x \to F_x$ is linear for all $x \in W$. The linear fibre space $\mathcal{E}$ (see 4.1) is of central importance to the paper, and is defined as the kernel $\ker(g^*) := g^{*-1}$ (zero section of F) for a certain bundle map g^*. (The confusing notation "g^*" for the bundle map does not refer, of course, to any one element $g \in G$!) The map g^* fits into a commutative diagram of vector bundles (4.2.3) and the results of Lemma 1.4 allow us to conclude, by a vector bundle "diagram chase", that $\mathfrak{S}$ is isomorphic to ker(p^*), another linear fibre space.

Section 2 describes the ampleness map

$$\phi\colon \mathbf{P}(T^*Z) \to \mathbf{P}(T_e^*G),$$

and the orbit decomposition $\mathbf{P}(T^*Z) = \bigcup_\rho O_\rho$ under the action of G.

Section 3 defines "rigged spaces", and discusses why Grassmannians and quadrics are rigged.

Sections 4 and 5, as already described, analyse the linear fibre space $\mathfrak{S}$ and prove the Lefschetz theorem.

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1. Preliminaries. More background material is in [**6**, §0]. All mappings in this paper are regular, i.e., everywhere defined.

Let X be a complex space. The reduction of X is X_{red}, and we denote

$$\text{redim}(X) = \dim(X_{\text{red}}). \tag{1.1}$$

Let $\pi\colon E \to X$ be a complex vector bundle over the space X. We denote, also by "X", the zero section of E.

Let $\phi\colon E \to F$ be a linear map of vector bundles over X. Then $\phi^{-1}(X) = \operatorname{Ker}\phi$ is a linear fibre space over X cf [**5**] (1.6).

Note. The variety Ker(ϕ) is smooth at e precisely when ϕ meets the zero section of F transversely from e.

(1.1.2) DEFINITION. Let $\pi\colon E \to X$ be a vector bundle over X. A local projection on E is a map

$$\nu\colon \pi^{-1}(U) \to \mathbf{C}^b$$

where U is a neighbourhood of $x \in X$, $\pi^{-1}(U) \simeq U \times \mathbf{C}^b$, and ν is the corresponding natural projection.

I omit the elementary proofs of the following four lemmas.

(1.2) LEMMA. *Let ϕ: $E \to F$ be a map of vector bundles over X, and suppose that $\phi(e) \in X$.*

Then ϕ meets the zero section, X, of F transversely from $e \Leftrightarrow \nu \circ \phi$ is a submersion for a local projection, ν, on F near $\phi(e)$.

(1.3) LEMMA. *Let ϕ: $E \to F$ and ψ: $G \to F$ be maps of vector bundles over X. Suppose that $\phi(e) = \psi(g)$.*

Then ϕ and ψ meet transversely, from e and g, respectively $\Leftrightarrow$ the map

$$\phi \oplus (-\psi): E \oplus G \to F$$

meets X transversely from (e, g).

(1.4) LEMMA (i). *Let ϕ: $E \to F$ be a map of vector bundles over X. Then*

$$\phi: X \to X$$

is an isomorphism.

(ii) *Let $E \overset{\phi}{\to} F \overset{\psi}{\to} G$ be an exact sequence of vector bundles over X. Then $\phi(E) \subset \mathrm{Ker}(\psi)$, and*

$$\phi: E \to \mathrm{Ker}(\psi) \text{ is a submersion},$$

i.e. the sequence is "exact" when viewed as a map of varieties.

(1.5) LEMMA. *Let $\mathcal{E}$ be a linear fibre space over X, and $\mathbf{P}(\mathcal{E}) = (\mathcal{E} \setminus X)/\mathbf{C}^*$ its projectivization. Let $e \in \mathcal{E}$ be non-zero.*
Then $\mathcal{E}$ is smooth at e

$$\Leftrightarrow \mathbf{P}(\mathcal{E}) \text{ is smooth at } [e].$$

(1.6) LEMMA. *Let f: $X^m \to Y^n$ be a map of complex manifolds, and*

$$f^*: f^{-1}(T^*Y) \to T^*X$$

the codifferential of f.

Then, as a map of varieties, the natural map

$$f': \mathrm{Ker}(f^*) \to T^*Y$$

has constant rank n. (See example below.)

Proof. Locally,

$$T^*X = X \times \mathbf{C}^m,$$
$$T^*Y = Y \times \mathbf{C}^n,$$

$$f^{-1}(T^*Y) = X \times \mathbf{C}^n,$$

$$f^*(x, w) = \left(x, \left(\frac{\partial f}{\partial x}(x)\right)^t w\right), \quad \text{and}$$

$$\operatorname{Ker} f^* = \left\{(x, w) \in X \times \mathbf{C}^n = w^t \frac{\partial f}{\partial x}(x) = 0\right\}.$$

Let A be the $m \times (m + n)$ matrix

$$\left(\frac{\partial^2(w^t f)}{\partial x^2}(x) \,\vdots\, \left(\frac{\partial f}{\partial x}(x)\right)^t \right)$$

which is the Jacobean of the equations describing $\operatorname{Ker} f^*$ in $X \times \mathbf{C}^n$. Then

$$\begin{aligned} T_{(x,w)}(\operatorname{Ker} f^*) &= \ker A \\ &= \left\{(u, v) \in \mathbf{C}^m \times \mathbf{C}^n\colon \frac{\partial^2(w^t f)}{\partial x^2}(x) u + \left(\frac{\partial f}{\partial x}(x)\right)^t v = 0\right\}. \end{aligned}$$

Now, $f'(x, w) = (f(x), w)$. It follows that

$$\ker f'_* = (\ker A^t) \oplus \{0\}.$$

Thus

$$\begin{aligned} \operatorname{rk} f' &= \dim(\ker A) - \dim(\ker A^t) \\ &= m + n - \operatorname{rk} A - (m - \operatorname{rk} A) = n. \end{aligned}$$ □

EXAMPLE. The above lemma may be illustrated with the map

$$\begin{aligned} f\colon X = \mathbf{C} &\to Y = \mathbf{C}, \\ x &\mapsto x^2; \\ f^*\colon f^{-1}(T^*Y) = \mathbf{C}^2 &\to T^*X = \mathbf{C}^2, \\ (x, w) &\mapsto (x, 2xw); \end{aligned}$$

$$\operatorname{Ker} f^* = \{(x, w) \in \mathbf{C}^2\colon xw = 0\};$$

$$\begin{aligned} f'\colon \operatorname{Ker} f^* &\to T^*Y = \mathbf{C}^2, \\ (x, w) &\mapsto (x^2, w). \end{aligned}$$

It is easy to compute that the differential of f' has constant rank 1.

(1.7) LEMMA. *Let*

$$f_i\colon X_i \to Y, \qquad i = 1, \ldots, a,$$

be maps of complex algebraic varieties, and let $Y_k = \{y \in Y\colon \exists x_i \in X_i,$ $i = 1, \ldots, a$ *with* $f_i(x_i) = y$ *and* $\dim(\bigcap_{i=1} f_{i*}(T_{x_i}(X_i))) \leq k\}$.

Then Y_k is a constructable subset of Y of dimension at most k.

Proof (sketch). Let $\mathcal{X}$ be the fibre product of the f_i, and

$$f: \mathcal{X} \to Y$$

the natural projection.

Then

$$f_*\left(T_{(x_1,\dots,x_a)}\mathcal{X}\right) = \bigcap_{i=1}^{a} f_{i*}\left(T_{x_i}X_i\right).$$

Thus,

$$Y_k = f(\{rk(f) \le k\})$$

and the lemma follows. □

(1.7.1) Corollary. *Let f: $X \to Y$ be a map of complex algebraic varieties. Let*

$$Y_k = \left\{ y \in Y: \dim\left(\bigcap_{x \in f^{-1}(y)} f_*(T_x X)\right) \le k \right\}.$$

Then Y_k is a constructable subset of Y of dimension at most k.

Proof. Let a be the local embedding dimension of Y, and apply the lemma to a copies of the map f. □

Let Z be a complex algebraic homogeneous space with algebraic group G. The evaluation map

(1.8) $$Z \times G \to Z, \quad (z, g) \to gz$$

is necessarily algebraic (cf [3], Proposition 7).

Let X and Y be Zariski locally closed smooth subvarieties of Z. The main tool that we use is the family of intersections

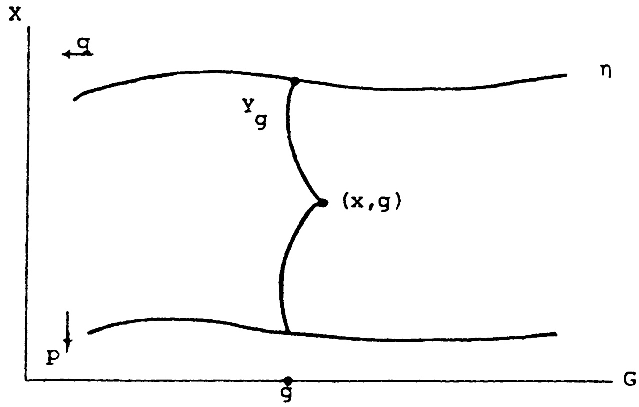
$$\{X \cap g^{-1}(Y)\}_{g \in G}.$$

Let

(1.9) $$\mu: X \times G \to Z$$

be the restriction of the evaluation (1.8) to $X \times G$. Then $\eta = \mu^{-1}(Y)$ is a Zariski locally closed smooth subvariety of $X \times G$.

(1.10)



Let p and q be the restrictions to η of the natural projections on $X \times G$, as in Figure (1.10). Let Y_g denote the variety $X \cap g^{-1}(Y)$. Then

$$Y_g \simeq p^{-1}(g) \quad \text{via } q.$$

For simplicity, we assume that Z, G, X and Y are connected. Let $\dim Z = r$, $\dim G = N$, $\dim X = n$ and $\dim Y = r - s$. Let

$$\operatorname{cork}_x(Y_g) = \dim(T_x(Y_g)) - (n - s)$$

be the corank of Y_g at x. Then Y_g is smooth at x (i.e. X meets $g^{-1}(Y)$ transversely at x) precisely when $\operatorname{cork}_x(Y_g) = 0$. Each smooth Y_g has pure dimension $n - s$. Moreover

$$\operatorname{rank}_{(x,g)} p = N - \operatorname{cork}_x(Y_g).$$

Let

(1.11) $$\mathcal{S} = \{(x, g) \in \eta \colon x \text{ is a singular point of } Y_g\}.$$

2. Ampleness and group actions. In [17] §1, Sommese defines the notion of k-ampleness:

Let A be a reduced compact complex space. A line bundle $L \to A$ is k-ample if the sections of some tensor power of L induce a map of A to projective space, whose fibre dimensions are at most k. Ample is the same as 0-ample. A vector bundle $E \to A$ is k-ample if the tautological line bundle

$$O_{\mathbf{P}(E^*)}(1) \to \mathbf{P}(E^*)$$

is k-ample. Here, $\mathbf{P}(E^*) = (E^* \setminus A)/\mathbf{C}^*$.

If E is spanned by sections $\{s_0,\ldots,s_a\}$, it follows that E is k-ample precisely when the map

$$\mathbf{P}(E^*) \to \mathbf{P}^a$$

defined by $\{s_0,\ldots,s_a\}$ has maximal fibre dimension at most k.

Let Z and G be as in (1.8), and let e be the identity element of G. Each $v \in T_eG$ determines a section

$$\begin{aligned} S_v\colon Z &\to TZ, \\ z &\mapsto z^{\#}_*(v) \in T_zZ, \end{aligned}$$

where

$$\begin{aligned} z^{\#}\colon G &\to Z, \\ g &\mapsto gz. \end{aligned}$$

These sections span TZ since $z^{\#}$ is a submersion. Let $v_1,\ldots,v_N$ be a basis of T_eG. The sections $\{S_{v_1},\ldots,S_{v_N}\}$ induce the map

$$\begin{aligned} &\phi\colon T^*Z \to \mathbf{C}^N \\ &T^*_zZ \ni \alpha \mapsto \big(\alpha\big(S_{v_1}(z)\big),\ldots,\alpha\big(S_{v_N}(z)\big)\big) \\ &\qquad\qquad = \big(z^{\#*}(\alpha)(v_1),\ldots,z^{\#*}(\alpha)(v_N)\big). \end{aligned}$$

Thus, ϕ may be viewed in the coordinate free way

$$\begin{aligned} &\phi\colon T^*Z \to T^*_eG, \\ &T^*_zZ \ni \alpha \mapsto z^{\#*}(\alpha). \end{aligned}$$

We denote, also by "ϕ", the map

$$\phi\colon \mathbf{P}(T^*Z) \to \mathbf{P}(T^*_eG). \tag{2.1}$$

Let Y be a smooth subvariety of Z, and NY the normal bundle of Y in Z. Then, $S_{v_1},\ldots,S_{v_N}$ induce sections which span NY, and

$$\phi_Y = \phi\,|_{N^*Y}\colon N^*Y \to T^*_eG$$

is the map that they define. If Y is compact, then NY is k-ample precisely when

$$\phi_Y\colon \mathbf{P}(N^*Y) \to \mathbf{P}(T^*_eG) \tag{2.2}$$

has fibre dimensions at most k.

Group Actions. Let Z be a complex algebraic homogeneous space, as in (1.8). The action of G on Z induces an action of G on T^*Z, through linear maps, and hence an algebraic action

$$\mathbf{P}(T^*Z) \times G \to \mathbf{P}(T^*Z). \tag{2.3}$$

Let

$$\mathbf{P}(T^*Z) = \bigcup_\rho O_\rho$$

be the orbit decomposition of the action (2.3). Each orbit O_ρ is an irreducible, Zariski locally closed, smooth subvariety of $\mathbf{P}(T^*Z)$. The natural projection $\mathbf{P}(T^*Z) \to Z$ is G-equivariant. Since the action of G on Z is transitive, there is a 1-1 correspondence between orbits of G in $\mathbf{P}(T^*Z)$ and orbits of G_z in $\mathbf{P}(T_z^*Z)$, where G_z is the isotropy group of an element $z \in Z$.

Let Y be a Zariski locally closed, smooth subvariety of Z. Then, $\mathbf{P}(N^*Y) \subset \mathbf{P}(T^*Z)$, and we define

$$\mathbf{P}(N^*Y)_\rho = \mathbf{P}(N^*Y) \cap O_\rho, \tag{2.4}$$

$$d_\rho^Y = \operatorname{redim}(\mathbf{P}(N^*Y)_\rho) \qquad (1.1).$$

In particular, let

$$d_\rho = d_\rho^{\{z\}} = \text{dimension of the } \rho\text{th orbit}$$

of the action of the isotropy group G_z on $\mathbf{P}(T_z^*Z)$.

It is easily seen that

$$\dim(O_\rho) = r + d_\rho \tag{2.5}$$

where $\dim Z = r$.

3. **Rigged spaces.** This paper is concerned, ostensibly, with Grassmannians and quadrics. In this section, we define a class of homogeneous spaces, rigged spaces, which includes the above two families. For simplicity, in the remainder of the paper, we will be assuming that the ambient homogeneous space, Z, is rigged, although it will be clear that many of the results are true under less stringent hypotheses.

(3.1) Definition. Let Z be a complex algebraic homogeneous space with connected group G. We say that Z is rigged if

(i) The universal covering map of G is algebraic. As is well-known, this is equivalent to $\pi_1(G)$ being finite, and this is the case when G is semisimple.

(ii) There are finitely many orbits $O_1, \ldots, O_m$ for the action of G on $\mathbf{P}(T^*Z)$ cf (2.3). In particular, there is a (unique) open orbit, say, O_m.

(iii) The isotropy group of an element of O_m is connected.

(iv) The map

$$\phi: \mathbf{P}(T^*Z) \to \mathbf{P}(T_e^*G)$$

is 1-1 on O_m cf (2.1). This condition is equivalent to:

(iv′) The isotropy group of an element α of O_m is equal to the isotropy group of $\phi(\alpha)$ with respect to the co-adjoint action of G on $\mathbf{P}(T_e^*(G))$.

(v) For each orbit O_ρ, there is a Zariski locally closed submanifold X_ρ of Z such that $\mathbf{P}(N^*(X_\rho))$ is generically contained in O_ρ i.e. $d_\rho^{X_\rho} = r - 1$, cf. (2.4).

We discuss, now, briefly, why Grassmannians and quadrics are rigged, and mention only that $\mathbf{P}^r \backslash \mathbf{P}^k$ is rigged where $\mathbf{P}^k$ is a linear projective k-subspace of $\mathbf{P}^r$, and the group G is $\{g \in \mathrm{PGL}(r, \mathbf{C}): g(\mathbf{P}^k) = \mathbf{P}^k\}$.

Grassmannians. Let $Z = \mathrm{Gr}(t, \mathbf{C}^{\tau+t})$ be the Grassmann manifold of $\mathbf{C}^t$'s in $\mathbf{C}^{\tau+t}$. The group G is the projective linear group $\mathrm{PGL}(\tau + t - 1, \mathbf{C})$. The universal cover of G is the algebraic projection

$$\mathrm{SL}(\tau + t, \mathbf{C}) \to G$$

where $\mathrm{SL}(\tau + t, \mathbf{C})$ is the special linear group in $\mathbf{C}^{\tau+t}$.

Over Z, there is a well known tautological sequence

$$0 \to E \xrightarrow{i} O^{\tau+t} \xrightarrow{\pi} F \to 0$$

where E and F are, respectively, rank t and rank τ vector bundles. Moreover,

$$T^*Z = \mathrm{Hom}(F, E)$$

so that each element $\alpha \in T_z^*Z$ has a rank between 0 and $m = \min(\tau, t)$ its rank as a linear map

$$\alpha: F_z \to E_z.$$

We can write

$$\mathbf{P}(T^*Z) = \bigcup_{\rho=1}^{m} O_\rho$$

where O_ρ is the projectivization of the elements of rank ρ. This is also the orbit decomposition of the action of G on $\mathbf{P}(T^*Z)$. The orbit O_m of the maximal rank vectors is the open orbit. A calculation shows that the isotropy groups are connected, and that

$$d_\rho = t\tau - (t-\rho)(\tau-\rho) - 1, \quad \text{where } t\tau = r = \dim Z.$$

Consider now the map

$$\phi\colon \mathbf{P}(T^*Z) \to \mathbf{P}(T_e^*G)$$
$$\phi(\alpha) = \text{projectivization of } i_z \circ \alpha \circ \pi_z.$$

Here, T_e^*G is identified with $\mathrm{sl}(\tau + t, \mathbf{C})$, i_z is the inclusion $E_z \to \mathbf{C}^{\tau+t}$ and π_z is the projection $\mathbf{C}^{\tau+t} \to F_z$. From this, one sees that $\mathrm{rk}\alpha = \mathrm{rk}\phi(\alpha)$. Let, now, $A = \phi(\alpha)$ have rank ρ. Then $\phi^{-1}(A) \simeq \{z \in Z\colon \operatorname{im} A \subset z \subset \ker A\}$ $\simeq \mathrm{Gr}(t-\rho, \mathbf{C}^{\tau+t-2\rho})$. In particular, ϕ is 1-1 on O_m.

Finally, let $\mathbf{C}^{\tau+t-\rho}$ be a codimension ρ subspace of $\mathbf{C}^{\tau+t}$, and put

$$X_\rho = \{z \in Z\colon z \subset \mathbf{C}^{t+\tau-\rho}\}.$$

Then $\mathbf{P}(N^*X_\rho)$ is generically contained in O_ρ.

Quadrics. Let $Z = \mathcal{Q}^r \subset \mathbf{P}^{r+1}$ be the r-dimensional quadric. The group G is the projectivization of the special complex orthogonal group $\mathbf{P}(\mathrm{SO}(r+2, \mathbf{C}))$ and has a finite fundamental group (of order 2 for r odd and order 4 for r even).

There is a commutative diagram

$$\begin{array}{ll} \mathbf{P}(T^*Z) \xrightarrow{\phi} \mathbf{P}(T_e^*G) & \\ \Gamma \downarrow \qquad\qquad \nearrow \text{Plücker embedding} & \\ \mathrm{Gr}(1, \mathbf{P}^{r+1}) & \end{array}$$

The image of Γ consists of all $\mathbf{P}^1$'s that are tangent to Z. If $\alpha \in \mathbf{P}(T^*Z)$, then

$$\Gamma^{-1}(\Gamma(\alpha)) \simeq \Gamma(\alpha) \cap Z = \begin{cases} \mathbf{P}^1, & \Gamma(\alpha) \subset Z, \\ \text{point}, & \Gamma(\alpha) \not\subset Z. \end{cases}$$

Now, $\mathbf{P}(T^*Z) = O_1 \cup O_m$ is the orbit decomposition where

$$O_1 = \{\alpha \in \mathbf{P}(T^*Z)\colon \Gamma(\alpha) \subset Z\} \quad \text{and}$$
$$O_m = \{\alpha \in \mathbf{P}(T^*Z)\colon \Gamma(\alpha) \not\subset Z\}.$$

In particular, ϕ is 1-1 on O_m. Again, a computation shows that the isotropy groups are connected, and $d_1 = r - 2$.

Finally, let X_1 be the smooth points of a singular hyperplane section of Z. Then, $\mathbf{P}(N^*(X_1))$ is contained in O_1.

4. Analytical results on rigged spaces. Throughout this section, we study the relationship between two smooth subvarieties X and Y of the homogeneous space Z. The main tool that we use is a linear fibre space $\mathfrak{S}$ which contains information of how singular are the intersections of X with the translates of Y by the group G.

(4.1) *The Construction of* $\mathfrak{S}$. Recall the maps

$$\mu\colon \eta \to Y, \qquad q\colon \eta \to X,$$
$$(x, g) \mapsto gx, \quad (x, g) \mapsto x.$$

Define a map (with abuse of notation)

$$g^*\colon \mu^{-1}(N^*Y) \to q^{-1}(T^*X),$$
$$N^*_{gx}Y \ni \alpha \mapsto g^*(\alpha) \in T^*_x X$$

and let

$$\mathfrak{S} = \operatorname{Ker}(g^*), \tag{4.1.1}$$

cf. §1. We note that

$$\mathfrak{S}_{(x,g)} = g^{*-1}(N^*_x X) \cap N^*_{gx}Y = \big(g_*(T_x X) + T_{gx}Y\big)^{\perp} \subset T^*_{gx}Z. \tag{4.1.2}$$

In particular

$$\dim\big(\mathfrak{S}_{(x,g)}\big) = \operatorname{cork}_x(Y_g). \tag{4.1.3}$$

(4.2) *Bounding* $\mathfrak{S}$. Let $\mathbf{P}(\mathfrak{S}) = (\mathfrak{S}\backslash\eta)/\mathbf{C}^*$ be the projectivization of $\mathfrak{S}$. The map $\mathfrak{S} \to \eta$ induces the map $\mathbf{P}(\mathfrak{S}) \to \mathfrak{S}$. Also, the map $\phi_Y\colon N^*Y \to T^*_e G$ induces the map

$$\begin{aligned} \phi_{\mathfrak{S}}\colon \mathfrak{S} &\to G \times T^*_e G, \\ \mathfrak{S}_{(x,g)} \ni \alpha &\mapsto \big(g, \phi_Y(\alpha)\big). \end{aligned} \tag{4.2.1}$$

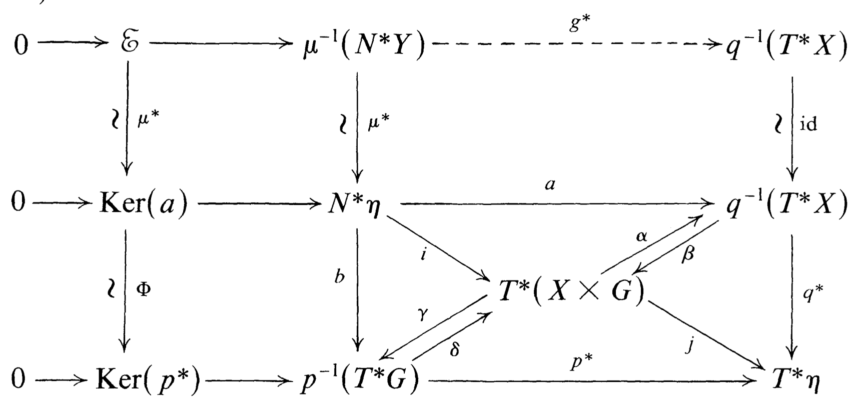
LEMMA. *There is a commutative diagram*

$$\begin{array}{ccc} \mathfrak{S} & \stackrel{\phi_{\mathfrak{S}}}{\to} & G \times T^*_e G \\ \wr\downarrow & & \wr\downarrow \\ \operatorname{Ker}(p^*) & \stackrel{p'}{\to} & T^*G \end{array} \tag{4.2.2}$$

(*See* (1.6) *for the definition of* p'.) *The vertical maps are isomorphisms.*

Proof.

(4.2.3)



In Figure (4.2.3), one of the diagonal exact sequences is for the conormal bundle of η in $X \times G$. The other expresses the direct sum decomposition of $T^*(X \times G)$. The maps a and b are defined, respectively, as αi and γi. It is clear that $q^* = j\beta$ and $p^* = j\delta$. Since

$$\beta\alpha + \delta\gamma = \mathrm{id}_{T^*(X \times G)}$$

we have

$$q^*a + p^*b = ji = 0.$$

This, Lemma (1.4) and a diagram chase show that

$$\Phi\colon \mathrm{Ker}(a) \to \mathrm{Ker}(p^*)$$

is well-defined and an isomorphism of varieties.

As for the top two rows of Figure (4.2.3), the map $\mu\colon X \times G \to Z$ is a submersion and $\eta = \mu^{-1}(Y)$. Thus, by a dimension count, the codifferential of μ induces an isomorphism

$$\mu^*\colon \mu^{-1}(N^*Y) \to N^*\eta.$$

The composition $a \circ \mu^*$ is just "the partial of μ with respect to x" viz. g^*. This fills in the top row of Figure (4.2.3), inducing, also, the isomorphism

$$\mu^*\colon \mathcal{E} \to \mathrm{Ker}(a).$$

To complete the proof of the lemma, the two left columns of Figure (4.2.3) give the diagram

$$(4.2.4)\qquad \begin{array}{ccccc} \mathcal{E}_{(x,g)} & \longrightarrow & N^*_{gx}Y & \xrightarrow{\phi_Y} & T^*_eG \\ \wr\downarrow & & \downarrow x^{\#*} & \swarrow \wr\ R^*_{g^{-1}} & \\ \operatorname{Ker}(p^*_{(x,g)}) & \longrightarrow & T^*_gG & & \end{array}$$

Here, we have replaced $b \circ \mu^*$ with $x^{\#*}$ i.e. "the partial of μ with respect to g". Now, $x^{\#} = (gx)^{\#} \circ R_g^{-1}$, so that $x^{\#*} = R^*_{g^{-1}} \circ \phi_Y$, cf. (2.2). This fills in the right half of Figure (4.2.4), and proves the lemma. □

(4.3) Proposition. *Let X be a locally closed smooth subvariety of the complex algebraic homogeneous space Z. If Y is a compact smooth subvariety of Z and NY is k-ample, then*

$$\operatorname{redim}(\mathbf{P}(\mathcal{E})) \leq N - 1 + k.$$

Proof. Projectivize diagram (4.2.2).

$$(4.3.1)\qquad \begin{array}{ccc} \mathbf{P}(\mathcal{E}) & \overset{\phi_{\mathcal{E}}}{\to} & G \times \mathbf{P}(T^*_eG) \\ \wr\downarrow & & \wr\downarrow \\ \mathbf{P}(\operatorname{Ker} p^*) & \overset{p'}{\to} & \mathbf{P}(T^*G) \end{array}$$

By Lemma (1.6), $\operatorname{rk} p' \equiv N - 1$.

Thus, $\operatorname{redim}(\operatorname{im}\phi_{\mathcal{E}}) \leq N - 1$. But, Y is compact and NY k-ample, so the fibres of ϕ_Y are at most k-dimensional. The same is true of $\phi_{\mathcal{E}}$, and it follows that

$$\operatorname{redim}(\mathbf{P}(\mathcal{E})) \leq N - 1 + k. \qquad \square$$

(4.4) Definition. We define the decomposition

$$\mathbf{P}(\mathcal{E}) = \bigcup_{\rho=1}^{m} \mathbf{P}(\mathcal{E})_\rho$$

$$\text{by } \mathbf{P}(\mathcal{E}) = \mathbf{P}(\mathcal{E}) \cap \mu^{-1}(\mathbf{P}(N^*Y)_\rho)$$
$$= \{\alpha \in \mathbf{P}(\mathcal{E}) : \alpha \text{ is in the } \rho\text{th orbit } O_\rho\}.$$

(4.5) Proposition. *Let Y be a smooth compact subvariety of the rigged homogeneous space Z. If NY is k-ample then the reduced dimension of each component of $\mathbf{P}(N^*Y)_\rho$ is between d_ρ and $k + d_\rho$, cf.* (2.4).

Proof. A lower bound for the dimension of each component of $\mathbf{P}(N^*Y) \cap O_\rho$ is

$$\dim(\mathbf{P}(N^*Y)) - \operatorname{cod}_{\mathbf{P}(T^*Z)} O_\rho$$
$$= r - 1 - \big(2r - 1 - (r + d_\rho)\big) = d_\rho.$$

To determine an upper bound, let X be any locally closed smooth subvariety of Z, and construct the space $\mathfrak{S}$ for X and Y. Define a map (cf. (4.1.2))

$$\theta \colon \mathbf{P}(\mathfrak{S}) \to \mathbf{P}(N^*X) \times \mathbf{P}(N^*Y),$$
$$\mathbf{P}\big(\mathfrak{S}_{(x,g)}\big) \ni \beta \mapsto (g^*\beta, \beta).$$

For each ρ,

$$\mathbf{P}(\mathfrak{S})_\rho = \theta^{-1}(\mathbf{P}(N^*X)_\rho \times \mathbf{P}(N^*Y)_\rho).$$

Thus,

$$\theta \colon \mathbf{P}(\mathfrak{S})_\rho \to \mathbf{P}(N^*X)_\rho \times \mathbf{P}(N^*Y)_\rho \tag{4.5.1}$$

and this restriction has fibres isomorphic to the isotropy group of an element of O_ρ, i.e. the fibres are $N - r - d_\rho$ dimensional.

By (4.3),

$$k + N - 1 \geq \operatorname{redim} \mathbf{P}(\mathfrak{S})_\rho = d_\rho^X + d_\rho^Y + N - r - d_\rho,$$

so $d_\rho^X + d_\rho^Y \leq r - 1 + d_\rho + k$. But, since Z is rigged, there is an X with $d_\rho^X = r - 1$.

Hence $d_\rho^Y \leq d_\rho + k$. □

(4.5.2) Corollary. *Let* $\kappa = r - 2 - \sup_{\rho \neq m}\{d_\rho\}$. *If Z is rigged, Y compact, and NY is κ-ample, then $\mathbf{P}(N^*Y)$ meets the open orbit O_m, and*

$$\phi_Y \colon \mathbf{P}(N^*Y) \to \mathbf{P}(T_e^*G)$$

is generically 1-1. *For the Grassmannian* $Z = \mathrm{Gr}(t, \mathbf{C}^{\tau+t})$ *we have* $\kappa = |t - \tau|$, *and for quadrics* $\kappa = 0$.

Proof. The hypotheses ensure that $d_\rho^Y \leq r - 2$ for $\rho \neq m$, so that $\mathbf{P}(N^*Y)_m$ must be non-empty. The definition of rigged implies that ϕ_Y is 1-1 on this Zariski open set. The specific values of κ may be computed (cf. §3). □

(4.5.3) Corollary. *Let Z be rigged, Y compact, NY ample, and assume that* $\mathbf{P}(N^*X)$ *meets the open orbit* O_m. *Then* $\operatorname{redim}(\mathbf{P}(\mathfrak{S})_\rho) \leq N - 2$ *for* $\rho \neq m$.

Proof. Using the map (4.5.1) we have

$$\begin{aligned}\operatorname{redim}(\mathbf{P}(\mathfrak{S})_\rho) &= d_\rho^X + d_\rho^Y + N - r - d_\rho \\ &= d_\rho^X + N - r \leq N - 2 \quad \text{for } \rho \neq m \text{ since } \mathbf{P}(N^*X)\end{aligned}$$

meets the open orbit O_m. □

We fix, now, an element z_0 of Z.

(4.6) *Note*. Let σ denote a local cross section of the bundle

$$z_0^\#: G \to Z, \quad g \mapsto gz_0.$$

So, $\sigma_z \in G$, $\sigma_z(z_O) = z \ \forall z \in U =$ domain of σ, and σ determines a local projection (cf. (1.1.2))

$$\nu\colon (T^*Z)|_U \to T^*_{z_O}Z, \quad T_z^*Z \ni \alpha \mapsto \sigma_z^*(\alpha).$$

(4.7) PROPOSITION. *Let X and Y be smooth connected subvarieties of the rigged homogeneous space Z.*

Then $\mathbf{P}(\mathfrak{S})_m$ is a smooth ($N-1$ dimensional) connected open subvariety of $\mathbf{P}(\mathfrak{S})$.

Proof. Certainly, $\mathbf{P}(\mathfrak{S})_m$ is open, and the fibration (4.5.1) shows that $\mathbf{P}(\mathfrak{S})_m$ is connected. By Lemma (1.5), we need only verify the smoothness of $\mathfrak{S}$.

$$(4.7.1)\qquad \begin{array}{ccccc} 0 \to \mathfrak{S} \to & \mu^{-1}(N^*Y) & \overset{g^*}{\to} & q^{-1}(T^*X) \\ \downarrow & \downarrow g^* & & \wr\!\downarrow \mathrm{id} \\ 0 \to q^{-1}(N^*X) \to & q^{-1}(T^*Z) & \to & q^{-1}(T^*X) \end{array}$$

The top row of (4.7.1) defines $\mathfrak{S}$ as $\operatorname{Ker}(g^*)$ cf. (4.1.1). In particular, $\mathfrak{S}$ is smooth (i.e. N-dimensional) when the horizontal g^* meets the zero section of $q^{-1}(T^*X)$ transversely. Now, by Lemma (1.4), $\mathfrak{S}$ is smooth precisely when the vertical g^* meets $q^{-1}(N^*X)$ transversely. By Lemma (1.3), $\mathfrak{S}$ is smooth precisely when

$$\mu^{-1}(N^*Y) \oplus q^{-1}(N^*X) \to q^{-1}(T^*Z), \quad (\beta, \alpha)_{(x,g)} \mapsto (g^*(\beta) - \alpha)_{(x,g)} \in T_x^*Z$$

meets the zero section transversely.

Let ν be the local projection of $q^{-1}(T^*Z)$ defined in Note (4.6). Now, by Lemma (1.2), $\mathfrak{S}$ is smooth precisely when

$$\text{(4.7.2)} \qquad \begin{aligned} \mu^{-1}(N^*Y) \oplus q^{-1}(T^*X) &\to T^*_{z_0}Z, \\ (\beta, \alpha)_{(x,g)} &\mapsto \sigma_x^*(g^*(\beta) - \alpha) \end{aligned}$$

is a submersion.

Fix α, β, x and y, and consider g's of the form $g = \sigma_y h \sigma_x^{-1}$ where $h \in G_{z_0}$, the isotropy group of z_0. For (4.7.2) to be a submersion, it suffices that

$$\begin{aligned} \mathbf{C}^* \times G_{z_0} &\to T^*_{z_0}Z, \\ (\lambda, h) &\to h^*\big(\sigma_y^*(\lambda\beta)\big) = \lambda h^* \sigma^* y(\beta) \end{aligned}$$

be a submersion, and this is equivalent to

$$\begin{aligned} G_{z_0} &\to \mathbf{P}\big(T^*_{z_0}Z\big), \\ h &\to h^*\sigma_y^*(\beta) \end{aligned}$$

being a submersion, which is certainly the case when β is in the open orbit O_m. □

5. A second Lefschetz theorem. In [**6**], a Second Lefschetz Theorem is discussed and proved in the context of complex projective space. This section proves an analogous theorem for rigged homogeneous spaces (cf. §3). The version of the First Leftschetz Theorem that we use is due to Sommese [**18**, **19**, **20**]:

(5.0) Let Z be a complex algebraic homogeneous space, Y a smooth connected compact subvariety of Z with an ample normal bundle, and X a smooth connected closed subvariety of Z. Let

$$\dim Z = r, \quad \dim X = n \quad \text{and} \quad \dim Y = r - s.$$

Assume that $2s \leq r$. Then

$$\pi_i(X, X \cap Y, y) = 0 \quad \text{for } i \leq \min(n - s, r - 2s + 1).$$

In particular, if $X = Z$ then

$$\text{(5.0.1)} \qquad \pi_i(Z, Y, y) = 0 \quad \text{for } i \leq r - 2s + 1,$$

or if $n + s \leq r + 1$ then

$$\text{(5.0.2)} \qquad \pi_i(X, X \cap Y, y) = 0 \quad \text{for } i \leq n - s.$$

(5.1) With the notation of (5.0), we recall (1.11) that

$$\mathfrak{S} = \big\{(x, g) \in \eta \colon x \text{ is a singular point of } Y_g\big\}$$

is a closed subvariety of η.

Let $\mathcal{S}_\alpha = \{(x, g) \in \eta\colon \operatorname{cork}_x(Y_g) = \alpha\}$ (cf. §1) so that $\mathcal{S} = \bigcup_{\alpha \geq 1} \mathcal{S}_\alpha$, and for each k $\bigcup_{\alpha \geq k} \mathcal{S}_\alpha$ is a closed subvariety of $\mathcal{S}$. The space $\mathcal{S}_1$ possesses a natural variety structure, as defined in [**6**], (3.3). (The same is true of all the $\mathcal{S}_\alpha$, but we will not be using them.)

Let $\Delta = p(\mathcal{S})$ be the discriminant of the family, i.e.,

$$\Delta = \{g \in G\colon Y_g \text{ is singular}\}.$$

The hypotheses on X and Y ensure that p is proper, so that Δ is a proper closed algebraic subvariety of G.

Fix a base point $o \in G\backslash\Delta$, and let

$$\mathfrak{M} \subset \operatorname{Aut}(H_{n-s}(Y_0, \mathbf{C}))$$

be the monodromy group associated to the fibre bundle

$$p\colon \eta\backslash p^{-1}(\Delta) \to G\backslash\Delta.$$

We define the invariant cycles

(5.1.1) $$\operatorname{inv} = \{v \in H_{n-s}(Y_0, \mathbf{C})\colon Tv = v \forall T \in \mathfrak{M}\}.$$

If $g_0 \in \Delta$ and if Y_{g_0} has but one singular point and if it is non-degenerate quadratic, then there is associated to it special monodromies (Picard-Lefschetz transformations)

(5.1.2) $$Tv = v - (v, a)a$$

where $a \in H_{n-s}(Y_0, \mathbf{C})$ is called a vanishing cycle, and (,) is the middle homology pairing.

We define

van = span of the vanishing cycles determined by the family η.

The next theorem is a main ingredient of the Second Lefschetz Theorem.

(5.2) THEOREM. *Let Z^r be a rigged homogeneous space with group G^N (cf. §3). Let Y^{r-s} be a compact smooth connected subvariety of Z with an ample normal bundle, and let X be a smooth connected closed subvariety of Z such that* $\mathbf{P}(N^*X)$ *meets the open orbit O_m.* cf (4.5.2).

Then there is a smooth connected open subvariety, D, of the discriminant Δ^{N-1} such that

(i) $\dim(\Delta\backslash D) \leq N - 2$, and

(ii) if $g \in D$ then Y_g has only one singular point and it is nondegenerate quadratic.

REMARK 1. If $\dim \Delta \leq N-2$ the conclusion is trivially satisfied by taking D to be the empty set.

REMARK 2. The hypothesis that $\mathbf{P}(N^*X)$ should meet the open orbit O_m is stronger than necessary. For example, if $Z = \mathrm{Gr}(2, \mathbf{C}^{2k})$ and Y is a smooth hyperplane section of Z, then $\mathbf{P}(N^*Y) \subset O_m$. Thus, if $\mathbf{P}(N^*X) \cap O_m = \varnothing$ then the discriminant Δ is empty.

Proof of theorem. Construct the linear fibre space $\mathfrak{E}$, as in (4.1).

There is a diagram

$$\begin{array}{ccccc} & \mathfrak{E} \to & \eta & \overset{p}{\to} & G \\ & & \cup & & \cup \\ (5.2.1) & \mathbf{P}(\mathfrak{E}) \to & \mathfrak{S} & \overset{p^1}{\to} & \Delta \end{array}$$
$$\mathbf{P}(\mathfrak{E}) \xrightarrow{\ \pi\ } \Delta$$

(The notation is described in (5.1).)

Recall (4.4) that $\mathbf{P}(\mathfrak{E}) = \bigcup_\rho \mathbf{P}(\mathfrak{E})_\rho$. Let

$$E_1 = \pi\left(\bigcup_{\rho \neq m} \mathbf{P}(\mathfrak{E})_\rho \right),$$

$$E_2 = p^1\{(x, g) \colon \mathrm{rk}_{(x,g)} p^1 \leq N-2\},$$

$$E_3 = \left\{ g \in \Delta \colon \dim\left(\bigcap_{x \in Y_g} \mathrm{im}\left(p_{*_{(x,g)}} \right) \right) \leq N-2 \right\}.$$

Put $D = \Delta \backslash (E_1 \cup \bar{E}_2 \cup \bar{E}_3 \cup \mathrm{sing}(\Delta))$.

Certainly, D is a smooth open subset of Δ, and is also connected since D is an open subset of the irreducible variety $\pi(\overline{\mathbf{P}(\mathfrak{E})_m})$ cf. (4.7).

By Corollary (4.5.3), $\dim(E_1) \leq N-2$, and by Corollary (1.7.1) we also have

$$\dim(E_2) \leq N-2 \quad \text{and} \quad \dim(E_3) \leq N-2.$$

Thus, $\dim(\Delta \backslash D) \leq N-2$.

The restricted mapping

$$\mathbf{P}(\mathfrak{E})\,|_{\mathfrak{S}_1} \to \mathfrak{S}_1$$

is a bijection of sets, as is easily seen from (4.1.3). In fact, it is an isomorphism of varieties. This is a local calculation using the special coordinates of [**6**] Lemma (3.3). From Proposition (4.7) it, now, follows that $\mathfrak{S}_1$ is smooth ($N - 1$ dimensional) at each point over D.

Fix, now, a point $g \in D$.

As in [**6**] Proposition (3.3.3), each singular point of Y_g is nondegenerate quadratic since $g \notin E_2$.

Moreover, since $g \notin E_3$, we have

$$\dim\left(\sum_{x \in Y_g} \ker p^*_{(x,g)}\right) = 1.$$

We remark that $\ker p^* = (\operatorname{im} p_*)^\perp$.

Now, if x_1 and x_2 are two singular points of Y_g then

$$\ker p^*_{(x_1,g)} = \ker p^*_{(x_2,g)}$$

as elements of $\mathbf{P}(T^*_g(G))$. Referring to diagram (4.3.1), we see that $x_1 = x_2$ since $\pi^{-1}(D) \subset \mathbf{P}(\mathfrak{S})_m$ and $\phi_{\mathfrak{S}}$ is 1-1 on $\mathbf{P}(\mathfrak{S})_m$ (cf. (3.1)iv). □

(5.3) THEOREM (*Second Lefschetz Theorem*). *Let Z be a Kaehler complex algebraic rigged homogeneous space* (e.g. *Grassmanian*, *quadric*, $\mathbf{P}^r \backslash \mathbf{P}^k$ cf. §3). *Let Y be a connected smooth compact subvariety of Z with an ample normal bundle NY. Let X be a connected smooth closed subvariety of Z such that* $\mathbf{P}(N^*X)$ *meets the open orbit of* $\mathbf{P}(T^*Z)$ cf. (2.4) *and Corollary* (4.5.2).

Let $\dim Z = r$, $\dim X = n$ and $\dim Y = r - s$. Assume that $n + s \leq r + 1$, $2s \leq r$ and $\pi_i(Z, Y) = 0$ for $i \leq n - s + 1$. (This last assumption is satisfied, for example, when $n + s \leq r$ (5.0.1).)

Let $Y_0 = X \cap Y$ and assume that X meets Y transversely in Z. In particular, Y_0 is a smooth $n - s$ dimensional subvariety of X. Let

ker = kernel of $H_{n-s}(Y_0, \mathbf{C}) \to H_{n-s}(X, \mathbf{C})$,

van = span of vanishing cycles in $H_{n-s}(Y_0, \mathbf{C})$, and

inv = cycles in $H_{n-s}(Y_0, \mathbf{C})$ invariant under the monodromy cf. (5.1.1).

Then

$H_{n-s}(Y_0, \mathbf{C}) = \ker \oplus \operatorname{inv}$,

ker = van,

the monodromy acts transitively, up to sign, on the set of vanishing cycles, and

van is irreducible under the monodromy action.

There is a corresponding conclusion in cohomology.

Proof. The proof is essentially the same as that of **[6]** (6.1). In the latter, replace "$\mathbf{P}^r$" by "Z" and **[6]** Theorem (3.5) by Theorem (5.2) of this paper.

We need, now, only two lemmas to modify the proof of **[6]** (6.1)ii.

Let $\nu: G^1 \to G$ be the universal cover of G, which is an algebraic map (3.1i). Put $\eta^1 = \nu^{-1}(\eta)$, cf. (1.10).

LEMMA. *$H^{n-s}(X \times G^1, \mathbf{C}) \to H^{n-s}(\eta^1, \mathbf{C})$ is onto.*

Proof. The map

$$X \times G^1 \to Z,$$
$$(x, g) \to \nu(g)x$$

is a fibration, so the proof of **[6]** (6.1.3) applies. □

LEMMA. van$^\perp \subset$ inv.

Proof. We show, in fact, that van$^\perp \subset$ im. Since $\pi_1(G^1) = 0$, the monodromy subgroup, $\mathfrak{M}^1$, of Aut$(H_{n-s}(Y_0, \mathbf{C}))$ determined by the family η^1 is generated by the Picard-Lefschetz transformations cf. (5.1.2). This implies that van$^\perp =$ inv^1, where

$$\text{inv}^1\{\alpha \in H_{n-s}(Y_0, \mathbf{C}) \colon T\alpha = \alpha \forall T \in \mathfrak{M}^1\}.$$

But inv^1 = im, as in **[6]**, (6.1)iii. □

6. Concluding remarks. The question in **[6]** §7 asks whether there is a smooth hypersurface Y in Gr(2, 4) such that $\mathbf{P}(N^*Y)$ is contained in the lower dimensional orbit of rank 1 vectors. According to Proposition (6.5) this is impossible, since NY is necessarily ample. Actually, Andrew Sommese had answered the above question using a line bundle argument, and this was encouragement enough to work out a general solution.

Note added in proof. In Definition 3.1, condition (ii) may be replaced by the weaker condition "The action of G on $\mathbf{P}(T^*Z)$ has an open orbit, O_m" i.e. one need not require that there be finitely many orbits. In "Finding the nondegenerate quadratic singularities", Proc. Symp. Pure Math., **40** (1982), the author indicates why this is so.

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