PACIFIC JOURNAL OF MATHEMATICS
Vol. 108, No. 1, 1983

TOPOLOGICAL EXTENSIONS OF PRODUCT SPACES

FRANK KOST

The Stone-Čech compactification βX and the Hewitt real-compactification υX [6] of a completely regular T_1-space X can be obtained as certain spaces of ultrafilters from the collection of zero sets of members of $C^*(X)$ [4]. With the appropriate structure βX is the space of all ultrafilters and υX those with the countable intersection property. In this framework we give a necessary and sufficient condition for $\beta X \times \beta Y \approx \beta(X \times Y)$.

Glicksberg [5], and then Frolík [3], established for infinite spaces X and Y that $\beta X \times \beta Y \approx \beta(X \times Y)$ if and only if $X \times Y$ is pseudocompact. Our condition is in terms of the zero sets of $X \times Y$ and we do not insist that X and Y be infinite. This result extends to arbitrary products. We give some sufficient conditions for $\upsilon X \times \upsilon Y \approx \upsilon(X \times Y)$ and in case $\upsilon X \times \upsilon Y$ (or $\upsilon(X \times Y)$) is Lindelöf give a condition that is both sufficient and necessary.

1. For Z a normal base [2] for the closed sets of X and $F \in Z$ define $F^* \equiv \{$ultrafilters from Z that contain $F\}$. $\{F^*: F \in Z\}$ is a base for the closed sets of the ultrafilter space $\omega(Z)$ which is a Hausdorff compactification of X. The normality property of Z is not needed to construct the T_1-compact space $\omega(Z)$. However, $\omega(Z)$ is a Hausdorff space if and only if Z is a normal family. If Z is the zero sets from X then $\omega(Z) \approx \beta X$. Extensions of this kind are called Wallman-type. Say a base Z_1 *separates* a base Z_2 if disjoint members of Z_2 are contained in disjoint members of Z_1.

THEOREM 1.1. *Let $Z_1 \subset Z_2$ be normal bases for X. Then $\omega(Z_1) \approx \omega(Z_2)$ if and only if Z_1 separates Z_2.*

Let Z_1 and Z_2 be normal bases for the closed sets of X and Y.

THEOREM 1.2. *$\omega(Z_1) \times \omega(Z_2)$ is a Wallman-type compactification of $X \times Y$.*

Proof (Sketch). Let $Z_1 \times Z_2 = \{F \times G: F \in Z_1, G \in Z_2\}$ and $Z_1 \times Z_{2_\Sigma}$ be all finite unions from $Z_1 \times Z_2$. $Z_1 \times Z_{2_\Sigma}$ is the needed normal base, i.e., $\omega(Z_1) \times \omega(Z_2) \approx \omega(Z_1 \times Z_{2_\Sigma})$. The mapping $(\mathcal{A}, \mathcal{B}) \to \mathcal{A} \times \mathcal{B}$

is one-one from $\omega(Z_1) \times \omega(Z_2)$ onto the ultrafilters from $Z_1 \times Z_2$ which are in one-one correspondence with those from $Z_1 \times Z_{2_\Sigma}$. We take $(\mathcal{A}, \mathcal{B}) \to$ the ultrafilter from $Z_1 \times Z_{2_\Sigma}$ that contains $\mathcal{A} \times \mathcal{B}$. This is a homeomorphism. $\beta X \times \beta Y$ is, then, a Wallman-compactification of $X \times Y$.

Let Z_1 be the zero sets from X and Z_2 those from Y. Denote the zero sets from $X \times Y$ by $Z(X \times Y)$. It is evident that $Z_1 \times Z_{2_\Sigma} \subset Z(X \times Y)$.

Our main result is

THEOREM 1.3. *$\beta X \times \beta Y \approx \beta(X \times Y)$ if and only if $Z_1 \times Z_{2_\Sigma}$ separates the zero sets of $X \times Y$.*

Proof. Assume that $Z_1 \times Z_{2_\Sigma}$ separates $Z(X \times Y)$. By Theorem 1.1, $\omega(Z_1 \times Z_{2_\Sigma}) \approx \beta(X \times Y)$. Using Theorem 1.2 we have $\beta X \times \beta Y \approx \beta(X \times Y)$.

If $\beta X \times \beta Y \approx \beta(X \times Y)$ then Theorem 1.2 implies that $\omega(Z_1 \times Z_{2_\Sigma}) \approx \omega(Z(X \times Y))$ and by Theorem 1.1, $Z_1 \times Z_{2_\Sigma}$ separates $Z(X \times Y)$.

Let N be the positive integers with the discrete topology. In $N \times N$, $F_1 =$ all points below the diagonal and $F_2 =$ all points above the diagonal belong to $Z(N \times N)$ but cannot be separated by $Z_1 \times Z_{2_\Sigma}$. In $R \times R$, where R is the real line, $Z_1 \times Z_{2_\Sigma}$ fails to separate the y-axis and $y = 1/x$.

REMARK. From Theorem 1.3 and Theorem 1 of [5] it is seen that, for X and Y infinite spaces, $X \times Y$ is pseudocompact if and only if $Z_1 \times Z_{2_\Sigma}$ separates $Z(X \times Y)$.

Let $\{X_\alpha\}$ be a collection of completely regular T_1-spaces and Z_α the zero sets from X_α.

THEOREM 1.4. *$\prod \beta X_\alpha$ is a Wallman compactification of $\prod X_\alpha$.*

Proof (Sketch). Let $\prod Z_\alpha \equiv \{\prod F_\alpha : F_\alpha \in Z_\alpha, F_\alpha = X_\alpha$ for all but finitely many $\alpha\}$ and Z be all finite unions from $\prod Z_\alpha$. Z has sufficient properties to construct the compact T_1-space $\omega(Z)$. We show $\prod \beta X_\alpha \approx \omega(Z)$ and it follows that $\omega(Z)$ is a Hausdorff space and that Z is a normal base.

REMARK. The Tychonoff Product Theorem can be obtained as a corollary to Theorem 1.4. In this case $\beta X_\alpha \approx X_\alpha$ and the homeomorphism gives $\prod X_\alpha$ compact.

Let Z be as above. Using Theorems 1.1 and 1.4 we arrive at an extension of our main result.

THEOREM 1.5. *$\Pi\, \beta X_\alpha \approx \beta(\Pi\, X_\alpha)$ if and only if Z separates the zero sets of $\Pi\, X_\alpha$.*

2. For Z a normal base for X let $p(Z)$ be the subspace of $\omega(Z)$ consisting of those points that have the countable intersection property (C.I.P.). $p(Z)$ is called a *real*-extension of X. Again, if Z is the zero sets from X then $p(X) \approx vX$. If Z is a normal base, the family of countable intersections from Z, denoted $Z_\cap$, is a normal base and $p(Z) \approx p(Z_\cap)$. Although $Z_\cap$ may introduce "new" ultrafilters none of these will have the C.I.P. e.g. $Z = \{F \subset N\colon F \text{ or } N\setminus F \text{ is finite}\}$. $Z_\cap$ is all subsets of N and $\omega(Z_\cap) \approx \beta N$. $\omega(Z)$ is the one-point compactification of N. Clearly $\omega(Z_\cap) \neq \omega(Z)$ yet $p(Z_\cap) \approx N \approx p(Z)$.

THEOREM 2.1. *Let $Z_1 \subset Z_2$ be normal bases for X each closed under formation of countable intersections. In case $p(Z_2)$ (or $p(Z_1)$) is Lindelöf it follows that $p(Z_1) \approx p(Z_2)$ if and only if Z_1 separates Z_2.*

REMARK. We insist on the Lindelöf property to show the condition is necessary.

Let Z_1 and Z_2 be normal bases for X and Y.

THEOREM 2.2. *$p(Z_1) \times p(Z_2)$ is a real extension of $X \times Y$.*

Proof (Sketch). $\omega(Z_1) \times \omega(Z_2) \approx \omega(Z_1 \times Z_{2_\Sigma})$ by Theorem 1.2. Under the mapping the image of $(\mathcal{A}, \mathcal{B})$ has the C.I.P. if and only if both $\mathcal{A}$ and $\mathcal{B}$ do. Therefore $p(Z_1) \times p(Z_2) \approx p(Z_1 \times Z_{2_\Sigma})$.

Let Z_1, Z_2 be the zero sets of X, Y.

THEOREM 2.3. *If $Z_1 \times Z_{2_\Sigma}$ separates $Z(X \times Y)$ then $vX \times vY \approx v(X \times Y)$.*

Proof. $\omega(Z_1 \times Z_{2_\Sigma}) \approx \omega(Z(X \times Y))$ by Theorem 1.1.

If follows that $p(Z_1 \times Z_{2_\Sigma}) \approx v(X \times Y)$. We have $vX \times vY \approx v(X \times Y)$ from Theorem 2.2.

THEOREM 2.4. *Assume that $vX \times vY$ (or $v(X \times Y)$) is Lindelöf. Then $vX \times vY \approx v(X \times Y)$ if and only if $Z_1 \times Z_{2_{\Sigma_\cap}}$ separates $Z(X \times Y)$.*

Proof. Note that $Z_1 \times Z_{2_{\Sigma_\cap}} \subset Z(X \times Y)$. Theorems 2.1 and 2.2 establish sufficiency.

If $\upsilon X \times \upsilon Y \approx \upsilon(X \times Y)$ then $p(Z_1 \times Z_{2_{\Sigma_\cap}}) \approx p(Z(X \times Y))$ by Theorem 2.2 and the remarks preceding Theorem 2.1. From Theorem 2.1 we have $Z_1 \times Z_{2_{\Sigma_\cap}}$ separates $Z(X \times Y)$.

There certainly are spaces X, Y with $\upsilon X \times \upsilon Y$ Lindelöf and $\upsilon X \times \upsilon Y \not\approx (X \times Y)$. Take a pseudocompact space X [**4**] with $X \times X$ not pseudocompact. $\upsilon X \times \upsilon X$ is compact, hence Lindelöf. However $\upsilon(X \times X)$ is not compact.

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Received February 8, 1982.

SUNY
ONEONTA, NY 13820