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A NOTE ON PRIMARY POWERS OF A PRIME IDEAL

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Let $X \subset P_k^n$ be an irreducible projective variety. Let $B = k[x_0,\ldots,x_n]$ and let $P \subset B$ be the homogeneous prime ideal of X generated by $\operatorname{ht}(p) + 1$ elements and let $A = B/P$ be the homogeneous coordinate ring of X. The following are equivalent: (1) $A_{(p)}$ is a complete intersection for all homogeneous prime ideals p in A of height 1; (2) P^2 is primary: (3) P^i is primary for all integers $i > 0$.

1. Introduction. In [**RV**, Theorem 3.3, p. 497], Robbiano and Valla proved the following: if $Y \subset X \subset P_k^n$ are projective schemes in the projective n-space over a field k, which are complete intersections in P_k^n and if Y is a positive dimensional irreducible, reduced normal subscheme of X with $(\operatorname{sing} X) \cap Y \subseteq \operatorname{sing} Y$, and if P is the prime ideal of Y in the homogeneous coordinate ring of X, then (a) P^2 is primary and (b) P^n is primary for every integer $n > 0$ if $\dim Y > \operatorname{codim} X$. [**RV**, Example 2, p. 560] gave an example of a projectively Gorenstein projective smooth curve in P_k^7 with homogeneous prime ideal P such that P^2 is not primary. It is known that an almost complete intersection is not Gorenstein. We prove the following theorem: Let X be a projective irreducible variety in P_k^n which is an almost complete intersection, i.e. the prime ideal P of X in the polynomial ring $k[x_0,\ldots,x_n]$ is generated by (codimen X) + 1 elements. Let $B = k[x_0,\ldots,x_n]$ and let $A = B/P = \oplus_{i\geq 0} A_i$ be the homogeneous coordinate ring of X, where A_i is the group of all homogeneous elements of degree i and $A_iA_j \subset A_{i+j}$. For a homogeneous ideal prime p in A let

$$A_{(p)} = \{a_i/s_i | a_i \in A_i,\, s_i \in A_i - p\}.$$

The following are equivalent:

(1) $A_{(p)}$ is a complete intersection for all homogeneous prime ideals p of height 1 in A.

(2) P^2 is primary.

(3) P^i is primary for all integers $i > 0$. Thus for an almost complete intersection X, "X is free of codim 1 singularities", is equivalent to "P^i is primary for all integers $i > 0$". Examples of projective varieties which are almost complete intersections are plentiful, for example, the Segre imbedding of $P_k^1 \times P_k^2$ into P_k^5 and the twist cubic in P_k^3 [**Sz**, p. 15–4]. The local case of (1) $\Leftrightarrow$ (2) was proved in [**K2**, Theorem, p. 1]: Let B be a regular

local ring, $P \subset B$ a prime ideal which is an almost complete intersection, and let $A = B/P$. Then P^2 is primary if and only if A_q is a complete intersection for all prime ideals q of height 1 in A. Recently [**H**, Theorem 3.1] proves that (1)—(3) are equivalent in the affine case: Let B be a Cohen-Macauley ring, and let $P \subset B$ be a prime ideal which is an almost complete intersection such that B_P is a regular local ring. Then the following are equivalent: (1) $P^2 = P^{(2)}$; (2) $P^n = P^{(n)}$ for $n > 0$ where $P^{(n)}$ is the nth symbolic power of P; (3) B_Q is a complete intersection for all $Q \in \operatorname{Spec}(B)$ with $Q \supset P$ and $\operatorname{ht}(Q/P) = 1$; (4) $P_Q^{(n)} = P_Q^n$ for every Q in (3); and (5) $\operatorname{gr}_P(B)$ is a domain, where $\operatorname{gr}_P(B) = B/P \oplus P/P^2 \oplus \cdots$. The purpose of this note is two-fold. One is that the projective case can be easily derived from the local case by introducing a simple but interesting lemma of Seidenberg [**S**, Lemma 1, p. 618] which says that if A is a graded domain and q is a nonhomogeneous prime of height 1, then A_q is normal. The other is that we make some further comparisons between $A_{(p)}$ and the usual localization A_p for the homogeneous prime ideals p of A and show that $A_{(p)}$ is an almost complete intersection (complete intersection) if and only if A_p is an almost complete intersection (complete intersection). Equivalences of these two local rings on the properties of being regular and normal were proved in [**K1**] and those on the Cohen-Macauley condition, the Gorenstein property, and the Buchsbaum singularity have been shown in [**DE**].

2. Notations and Definitions. Let $A = \oplus_{i>0} A_i$ be a graded domain. Let p be a homogeneous ideal of A contained in $\oplus_{i>0} A_i$, A_p is the usual localization of A with respect to the multiplicative set $A - p$, namely

$$A_p = \{a/s \mid a \in A,\, s \in A - p\},$$
$$A_{(p)} = \{a_i/s_i \mid a_i \in A_i \text{ and } s_i \in A_i - p\}.$$

If $I \subset A$ is a homogeneous ideal, then I_p will mean the extended ideal of I in A_p and

$$I_{(p)} = \{b_i/s_i \mid b_i \in I \cap A_i \text{ and } s_i \in A_i - p\}.$$

Let $f \in A_1$, and $f \neq 0$.

$$A_{(f)} = \{A_i/f^i \mid a_i \in A_i\} \text{ and } I_{(f)} = \{b_i/f^i \mid b_i \in I \cap A_i\}.$$

$\mu(I) \equiv$ minimal number of generators of I, $\mu(I_p) \equiv$ minimal number of generators of I_p in A_p and $\mu(I_{(p)}) \equiv$ minimal number of generators of $I_{(p)}$ in $A_{(p)}$. A_p ($A_{(p)}$) is a complete intersection if it is isomorphic to the quotient ring of a regular local ring B modulo and ideal $\mathfrak{A}$ generated by a regular B-sequence. A_p ($A_{(p)}$) is an almost complete intersection if it is isomorphic to the quotient ring of a regular local ring B modulo an ideal $\mathfrak{A}$ minimally generated by $\operatorname{ht}(\mathfrak{A}) + 1$ elements. In this case the first $\operatorname{ht}(\mathfrak{A})$ elements can be taken to be a regular B-sequence.

Let $k[x_0,\dots,x_n]$ be a polynomial ring in indeterminates $x_0,\dots,x_n$ and let $P \subset k[x_0,\dots,x_n]$ be an ideal. P is an almost complete intersection if $\mu(P) = \operatorname{ht}(P) + 1$.

3. Notes.

LEMMA 1. *Let $R = \bigoplus_{i\geq 0} R_i$ be a Noetherian graded ring with R_0 a field, and let $m = \bigoplus_{i>0} R_i$ be the irrelevant maximal ideal. Let $\mathfrak{A} \subset R$ be a homogeneous ideal. Then* (1) $\mathfrak{A} R_{\mathfrak{m}} \cap R = \mathfrak{A}$; (2) $\mu(\mathfrak{A}) = \mu(\mathfrak{A} R_{\mathfrak{m}})$; *and* (3) *The minimal generating set of $\mathfrak{A}$ can be chosen to consist of homogeneous elements.*

Proof 1. It suffices to show that $\mathfrak{A} \cdot R_{\mathfrak{m}} \cap R \subset \mathfrak{A}$. Let $f \in \mathfrak{A} \cdot R_{\mathfrak{m}} \cap R$ and write $f = g/t$, where $g \in \mathfrak{A}$, $t \in R - \mathfrak{m}$. There exists $s \in R - \mathfrak{m}$ such that $s(ft - g) = 0$. Let $st = a_0 + a_1 + \cdots + a_u$, $sg = g_c + g_{c+1} + \cdots + g_d$ and $f = f_r + \cdots + f_v$, where a_i, g_i, $f_1 \in R_i$ and $a_0 \neq 0$, for some nonnegative integers c, d, r, u, v. Then $a_0 \cdot f_r = g_c \in \mathfrak{A}$ and $r = c$ because $\mathfrak{A}$ is a homogeneous ideal. Thus $f_r \in \mathfrak{A}$. Also $a_0 f_{r+1} + a_1 f_r = g_{c+1} \in \mathfrak{A}$. It follows that $f_{r+1} \in \mathfrak{A}$. Inductively, we get $f_i \in \mathfrak{A}$ for $i = r,\dots,v$, and hence $f \in \mathfrak{A}$. Thus $\mathfrak{A} \cdot R_m \cap R = \mathfrak{A}$.

For (2). Since R is noetherian $\mathfrak{A}$ is of finite type. It is known [**B**, Cor. 2, p. 248] that $\mu(\mathfrak{A}) = \dim_{R_0} \mathfrak{A}/\mathfrak{m}\mathfrak{A}$. As

$$\mathfrak{A} R_{\mathfrak{m}}/\mathfrak{A} \cdot \mathfrak{m} R_{\mathfrak{m}} \cong \mathfrak{A} \cdot R_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} R_{\mathfrak{m}}/\mathfrak{m} R_{\mathfrak{m}} \cong (\mathfrak{A} \otimes_R R_{\mathfrak{m}}) \otimes_{R_{\mathfrak{m}}} R_0$$
$$\cong \mathfrak{A} \otimes_R (R_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} R_0) \cong \mathfrak{A} \otimes_R R_0 \cong \mathfrak{A}/\mathfrak{m}\mathfrak{A},$$

it follows that

$$\mu(\mathfrak{A} R_{\mathfrak{m}}) = \dim_{R_0}(\mathfrak{A} R_{\mathfrak{m}}/\mathfrak{A} \cdot \mathfrak{m} R_{\mathfrak{m}}) = \dim_{R_0}(\mathfrak{A}/\mathfrak{A}_{\mathfrak{m}}) = \mu(\mathfrak{A}).$$

For (3). $\mathfrak{A}$ is homogeneous. Therefore $\mathfrak{A}$ has a set of generators consisting of homogeneous elements. Let $\{h_1,\dots,h_r\}$ be a minimal set of homogeneous generators. Then $r \geq \mu(\mathfrak{A})$. Let $\bar{h}_1,\dots,\bar{h}_r$ be the residue classes of $h_1,\dots,h_r$ in the $\mu(\mathfrak{A})$-dimensional vector space $\mathfrak{A}/\mathfrak{m} \cdot \mathfrak{A}$ over

R_0. We select a vector basis, say $\{\bar{h},\ldots,\bar{h}_{\mu(\mathfrak{A})}\}$. Then, by [**B**, Cor. 2, p. 248], $\{h_1,\ldots,h_{\mu(\mathfrak{A})}\}$ form a minimal set of generators of $\mathfrak{A}$. Therefore $\mu(\mathfrak{A}) = r$.

LEMMA 2. *Let $A = \bigoplus_{i\geq 0} A_i$ be a noetherian graded domain and let $I \subset A$ be a graded ideal. Let p be a graded prime ideal of A contained in $\bigoplus_{i>0} A_i$ such that there is $r_1 \in A_1 - p$. Then* (1) $\mu(I_{(p)}) = \mu(I_p)$; (2) depth $A_{(p)} =$ depth A_p; *and* (3) dim $A_p =$ dim $A_{(p)}$;

Proof. It follows from [**K1**, Theorem 2, p.456] and [**DE**, Theorem 1] that $A_p = A_{(p)}[r_q]_{\mathfrak{m}[r_1]}$ and r_1 is transcendental over $A_{(p)}$, where $\mathfrak{m}$ is the maximal ideal of $A_{(p)}$, and a similar argument yields $I_p = I_{(p)}[r_1]_{\mathfrak{m}[r_1]}$.

$$\mu(I_p) = \dim_{A_p/pA_p}(I_p/I_p \cdot pA_p),$$
$$\mu(I_{(p)}) = \dim_{A_{(p)}/\mathfrak{m}A_{(p)}}(I_{(p)}/I_{(p)} \cdot \mathfrak{m}A_{(p)}).$$

Since $A_p/pA_p = k(r_1)$, a transcendental field extension over $k = A_{(p)}/\mathfrak{m}A_{(p)}$, and

$$\begin{aligned} I_p/I_p \cdot pA_p &\cong I_p \otimes_{A_p} A_p/pA_p = \left(I_{(p)} \otimes_{A_{(p)}} A_{(p)}[r_i]_{\mathfrak{m}[r_1]}\right) \otimes_{A_p} A_p/pA_p \\ &\cong I_{(p)} \otimes_{A_{(p)}} \left(A_p \otimes_{A_p} A_p/pA_p\right) \cong I_{(p)} \otimes_{A_{(p)}} k(r_1) \\ &\cong \left(I_{(p)} \otimes_{A_{(p)}} k\right) \otimes_k k(r_1) \cong I_{(p)}/I_{(p)}\mathfrak{m}A_{(p)} \otimes_k k(r_1). \end{aligned}$$

Therefore (1) $\mu(I_p) = \mu(I_{(p)})$. For (2),

$$\begin{aligned} A_p \otimes_{A_{(p)}} k &\cong A_p \otimes_{A_{(p)}} A_{(p)}/\mathfrak{m}A_{(p)} \\ &= A_{(p)}[r_1]_{m[r_1]}/\mathfrak{m}A_{(p)}[r_1]_{\mathfrak{m}[r_1]} \cong k(r_1), \end{aligned}$$

which is of depth 0. $A_{(p)} \to A_p$ is a local flat homomorphism. It follows from [**M**, Cor. 1, p. 154] that depth $A_{(p)} =$ depth A_p. (3) [**DE**, Corollary to Theorem 1].

PROPOSITION. *Let $A = \bigoplus_{i\geq 0} A_i$ be a noetherian graded domain with A_0 a regular ring and let p be a graded prime ideal of A such that there is $r_1 \in A_1 - p$.*

(1) *$A_{(p)}$ is a complete intersection if and only if A_p is a complete intersection.*

(2) *$A_{(p)}$ is an almost complete intersection if and only if A_p is an almost complete intersection.*

Proof. Since A is Noetherian, A is finitely generated A_0-algebra. Let S be a polynomial ring over A_0, say $S = A_0[x_0,\dots,x_n]$ and let $I \subset (x_0,\dots,x_n)$ be the homogeneous prime ideal of S such that $A = S/I$. Let $P \subset S$ be the inverse image of p. Then $A_P = (S/I)_{P/I} = S_p/I \cdot S_P$. Let L_1 be an element of S of homogeneous degree 1 such that r_1 is the image of L_1 in A_1. Then

$$A_{(p)} = \left(S_{(L_1)}\right)_{P_{(L_1)}} / \left(I_{(L_1)}\right) \cdot \left(S_{(L_1)}\right) = S_{(P)}/I_{(P)} \cdot S_{(P)},$$

where $S_{(L_1)}$, $P_{(L_1)}$, $I_{(L_1)}$ are dehomogenized S, P and I with respect to L_1, respectively. That A_0 is regular implies $S = A_0[x_1,\dots,x_n]$ is regular. Thus S_p is regular and so is $S_{(P)}$ by [**K1**, Theorem 2e, p. 457]. A_p and $A_{(p)}$ are thus quotient rings of regular local rings. $\mathrm{ht}(I_{(P)}) = \mathrm{ht}(I_P)$ because $\dim S_P = \dim S_{(P)}$ and $\dim A_p = \dim A_{(p)}$. (1) If A_p is a complete intersection then, by Lemma 2, $\mu(I_P) = \mathrm{ht}(I_P) = \mathrm{ht}(I_{(P)}) = \mu(I_{(P)})$. Therefore $A_{(p)}$ is a complete intersection, and conversely.

(2) If A_p is an almost complete intersection then $\mu(I_P) = \mathrm{ht}(I_P) + 1 = \mathrm{ht}(I_{(P)}) + 1 = \mu(I_{(P)})$. Therefore $A_{(p)}$ is also an almost complete intersection, and conversely.

Note. (1) Also follows from a more general result [**A**, Theorem 2, p. 1413]: Let $f\colon (B, \mathfrak{A}) \to (A, \mathfrak{m})$ be a flat local homomorphism of noetherian local rings. Then A is a complete intersection if and only if the same is true of B and of $A \otimes_B B/\mathfrak{m}$. But our proofs is simple and direct for the case involved.

THEOREM. *Let $B = k[x_0, x_1,\dots,x_n]$ be a polynomial ring over a field k in the indeterminates $x_0,\dots,x_n$. Let $P \subset B$ be a homogeneous prime ideal such that P is an almost complete intersection. Let $A = B/P$. Then the following are equivalent*:

(1) *P^2 is P-primary.*

(2) *For all $p \in$* proj(A) *with* $\mathrm{ht}(p) = 1$, *the local ring $A_{(p)}$ is a complete intersection.*

(3) *P^i is P primary for all integers $i > 0$.*

Proof. For (1) $\leftrightarrow$ (2). Let $\mathrm{ht}(P) = r$. Let $M = (x_0,\dots,x_n) \cdot B$. Then by Lemma 1, $\mu(P) = \mu(PB_M) = r + 1$. Since for each $Q \in \mathrm{Spec}(B)$ and $Q \supset P$, $\mu(PB_Q) \le r + 1$ and $\mathrm{ht}(PB_Q) = \mathrm{ht}(P) = r$. Thus in the regular local ring B_M, $P \cdot B_M$ is locally a complete intersection or an almost complete intersection.

P^2 is P-primary if and only if $P^2 \cdot B_M$ is $P \cdot B_M$-primary. By [**K2**, Theorem 1, p. 15] $P^2 B_M$ is primary if and only if for each $q \in \mathrm{Spec}(A_{\mathfrak{m}})$ with $\mathrm{ht}(q) = 1$, the local ring A_q is a complete intersection, where $\mathfrak{m} = M/P$. Since A is a noetherian graded domain and $\mathfrak{m}$ is the irrelevant maximal ideal, and suppose q above is homogeneous, then by the above proposition $A_{(q)}$ is a complete intersection. Conversely, if $A_{(q)}$ is a complete intersection for all homogeneous ideals q of height 1 in $\mathrm{proj}(A)$, then A_q is a complete intersection by the proposition. Let $q' \in \mathrm{Spec}(A)$ and $q' \notin \mathrm{proj}(A)$ and $\mathrm{ht}(q') = 1$. Then by [**S**, Lemma 1, p. 618] A_q is a normal local domain. Since $\dim A_{q'} = 1$, then $A_{q'}$ is a regular local ring, in particular a complete intersection. Thus A_q is a complete intersection for all $q \in \mathrm{Spec}\, A$ and $\mathrm{ht}(q) = 1$. Since Lemma 1(2) implies P_M is an almost complete intersection and [**K2**, Theorem 1; p. 15] implies P_M^2 is P-primary, P^2 is P-primary.

For (1) $\leftrightarrow$ (3). As B is a polynomial ring, B is Cohen-Macauley and B_P is regular. It follows from [**H**, Theorem 3.1] as quoted in the introduction, that $P^2 = P^{(2)}$ if and only if $P^i = P^{(i)}$ for all integers $i > 0$. Since $P^{(i)} = P^i$ if and only if P^i is primary, (1) is equivalent to (3).

COROLLARY 1. *Let P, S be the same as in the theorem. Let V be the projective variety in the projective n space P_k^n defined by P. If V is free of singularities of codimension* 1, *in particular V is locally normal*, *then P^n is P-primary for $n > 0$.*

Proof. V is locally normal implies V is free of singularities of codimension 1. Let A be the homogeneous coordinate ring of V. Then $A_{(p)}$ is a regular local ring for each homogeneous prime ideal p of height 1. Thus by the theorem, P^n is P-primary for all integers $n > 0$.

COROLLARY 2. *Let $P \subset k[x_0, x_1, \ldots, x_n]$ be the homogeneous prime ideal of a nonsingular irreducible projective curve which is also an almost complete intersection. Then P^n is primary for all integers $n > 0$.*

Proof. This follows from Corollary 1.

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