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NORMS ON $F(X)$

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It is well known that if $\|\cdot\cdot\|$ is a norm on the field $F(X)$ of rational functions over a field F for which F is bounded, then $\|\cdot\cdot\|$ is equivalent to the supremum of a finite family of absolute values on $F(X)$, each of which is improper on F. Moreover, $\|\cdot\cdot\|$ is equivalent to an absolute value if and only if the completion of $F(X)$ for $\|\cdot\cdot\|$ is a field. We show that the analogous characterization of norms on $F(X)$ for which F is discrete is impossible by constructing for each infinite field F, a norm $\|\cdot\cdot\|$ on $F(X)$ such that F is discrete, $\|X\| < 1$, the completion of $F(X)$ for $\|\cdot\cdot\|$ is a field, but $\|\cdot\cdot\|$ is not equivalent to the supremum of finitely many absolute values.

1. Introduction and basic definitions. Let R be a ring and let $\mathfrak{T}$ be a ring topology on R, that is, $\mathfrak{T}$ is a topology on R making $(x, y) \to x - y$ and $(x, y) \to xy$ continuous from $R \times R$ to R. A subset A of R is *bounded* for $\mathfrak{T}$ if given any neighborhood U of zero, there exists a neighborhood V of zero such that $AV \subseteq U$ and $VA \subseteq U$. $\mathfrak{T}$ is a *locally bounded topology* on R if there exists a fundamental system of neighborhoods of zero for $\mathfrak{T}$ consisting of bounded sets.

We recall that a *norm* $\|\cdot\cdot\|$ on a ring R is a function from R to the nonnegative reals satisfying $\|x\| = 0$ if and only if $x = 0$, $\|x - y\| \leq \|x\| + \|y\|$ and $\|xy\| \leq \|x\|\,\|y\|$ for all x and y in R. If $\|\cdot\cdot\|$ is a norm on R, for each $\varepsilon > 0$ define B_ε by, $B_\varepsilon = \{r \in R\colon \|r\| < \varepsilon\}$. Then $\{B_\varepsilon\colon \varepsilon > 0\}$ is a fundamental system of neighborhoods of zero for a Hausdorff locally bounded topology $\mathfrak{T}_{\|\cdot\cdot\|}$ on R. Two norms on R are *equivalent* if they define the same topology. We note further that if $\|\cdot\cdot\|$ is a nontrivial norm on a field K (that is, $\mathfrak{T}_{\|\cdot\cdot\|}$ is nondiscrete), then a subset A of K is bounded for the topology defined by $\|\cdot\cdot\|$ if and only if A is bounded in norm.

It is classic that, to within equivalence, the only valuations on the field $F(X)$ of rational functions over a field F that are improper on F are the valuations v_p, where p is a prime polynomial of $F[X]$, and the valuation v_∞ defined by the prime polynomial X^{-1} of $F[X^{-1}]$ ([**1**, Corollary 2, p. 94]). For each valuation v, the function $|\cdot\cdot|_v$ defined by $|y|_v = 2^{-v(y)}$ for all y in $F(X)$ is an absolute value on $F(X)$ for which F is discrete. In [**2**, Theorem 2] we showed that if $\|\cdot\cdot\|$ is a nontrivial norm on $F(X)$ for which F is bounded, then $\|\cdot\cdot\|$ is equivalent to the supremum of finitely many of these absolute values. (This result was also obtained by Kiyek [**5**, Satz 2.11].) The analogous question of characterizing those norms $\|\cdot\cdot\|$ on $F(X)$ for which F is discrete has been considered in several papers. (See for example [**4**, Theorem 4] and [**10**, Lemma 3]. We note that in each case the author has actually assumed that F is bounded.) In this paper we modify a technique of Mutylin [**6**] to show that such a characterization is impossible by constructing for each infinite field F, a norm $\|\cdot\cdot\|$ on $F(X)$ for which F is discrete, $\|X\| < 1$, the completion of $F(X)$ is a field but $\|\cdot\cdot\|$ is not equivalent to the supremum of any finite family of absolute values on $F(X)$. In the process, we also obtain a norm $\|\cdot\cdot\|$ on the polynomial ring $F[X]$ such that F is discrete and $\|X\| < 1$ but $\|\cdot\cdot\|$ is not equivalent to the supremum of finitely many absolute values on $F[X]$. (For a characterization of all norms on $F[X]$ for which F is a bounded set, see [**3**, Theorem 2].)

2. Norms on $F(X)$.

LEMMA 1. *Let F be an infinite field and let E be its prime subfield.*

(1) *If F is finitely generated over E, then there exists a nested sequence $F_0, F_1, F_2, \ldots$ of subrings of F such that F_n is properly contained in F_{n+1} for all $n \geq 0$, $1 \in F_0$ and $F = \bigcup_{n=0}^{\infty} F_n$.*

(2) *If F is not finitely generated over E, then there exists a nested sequence $F_0, F_1, F_2, \ldots$ of subfields of F such that F_n is properly contained in F_{n+1} for all $n \geq 0$ and $F = \bigcup_{n=0}^{\infty} F_n$.*

Proof. (1) F is either a finite algebraic extension of Q or there exists a subfield K of F and an element z in F which is transcendental over K such that F is a finite algebraic extension of $K(z)$. If F is a finite algebraic extension of Q, let $p_0, p_1, \ldots$ be a sequence of distinct positive primes in Z and for each n, let $\hat{v}_n$ be an extension of the p_n-adic valuation from Q to F. Define F_n by,

$$F_n = O(\{\hat{v}_{n+1}, \hat{v}_{n+2}, \ldots\}) = \{a \in F\colon v_i(a) \geq 0 \text{ for } i \geq n+1\}.$$

Then $1 \in F_0$, each F_n is clearly a subring of F and $F_n \subseteq F_{n+1}$ for all $n \geq 0$. As $p_{n+2}/p_{n+1} \in F_{n+1} \backslash F_n$, F_n is properly contained in F_{n+1} for all $n \geq 0$. Finally, if $a \in F \backslash \{0\}$, then $\hat{v}_p(a) = 0$ for all but finitely many primes p. Hence $F = \bigcup_{n=0}^{\infty} F_n$.

If F is a finite algebraic extension of $K(z)$, let $p_0, p_1, \ldots$ be a sequence of distinct prime polynomials in $K[z]$ and proceed as before.

(2) Suppose $F \setminus E$ is a countably infinite set $\{s_0, s_1, \ldots\}$. By induction on n, we define integers $k_0, k_1, \ldots$ and subfields $F_0, F_1, \ldots$ of F satisfying:

(i) $k_0 < k_1 < \cdots$;

(ii) $F_n = E(s_0, s_1, \ldots, s_{k_n})$;

(iii) F_n is properly contained in F_{n+1}.

Let $k_0 = 0$ and let $F_0 = E(s_0)$. Assume $k_0, k_1, \ldots, k_n$ and $F_0, F_1, \ldots, F_n$ have been defined satisfying (i)–(iii). As F is not finitely generated over E, there exists an integer t such that $s_t \notin F_n$. Let k_{n+1} be the smallest integer t satisfying this property and let $F_{n+1} = E(s_0, s_1, \ldots, s_{k_{n+1}})$. Properties (i)–(iii) obviously hold for k_{n+1} and F_{n+1} thus defined. By (i) and (ii), $F = \bigcup_{n=0}^{\infty} F_n$ and hence $F_0, F_1, \ldots$ is the desired sequence of subfields of F.

Suppose $F \setminus E$ is uncountable. Then the transcendence degree of F over E is infinite. Hence there exists a subfield E_0 of F and distinct elements $x_0, x_1, \ldots$ of F such that $\{x_i : i \geq 0\}$ is a transcendence base for F over E_0. For each $n \geq 0$, let $F_n = \{a \in F: a \text{ is algebraic over } E_0(x_0, x_1, \ldots, x_n)\}$. $F_0, F_1, \ldots$ is then a sequence of subfields of F satisfying the desired properties.

(The author is grateful to the referee for suggesting the above lemma as well as its proof.)

Henceforth, let F be an infinite field and let $F_0, F_1, F_2, \ldots$ be a nested sequence of subrings of F such that F_n is properly contained in F_{n+1} for all $n \geq 0$, $1 \in F_0$ and $F = \bigcup_{n=0}^{\infty} F_n$. For each $a \in F$, let $\phi(a)$ denote the smallest nonnegative integer n such that $a \in F_n$. Clearly:

$$\phi(a \pm b) \leq \max\{\phi(a), \phi(b)\} \quad \text{for all } a, b \text{ in } F. \tag{1}$$

$$\phi(ab) \leq \max\{\phi(a), \phi(b)\} \quad \text{for all } a, b \text{ in } F. \tag{2}$$

Define $|\cdot\cdot|$ from F to $N \cup \{0\}$ by,

$$|a| = \begin{cases} 2^{\phi(a)} & \text{if } a \in F \setminus \{0\}, \\ 0 & \text{if } a = 0. \end{cases}$$

Obviously, $|a| = 0$ if and only if $a = 0$. Furthermore from (1) and (2) we obtain

$$|a \pm b| \leq \max\{|a|, |b|\} \quad \text{and} \quad |ab| \leq \max\{|a|, |b|\}$$

for all a and b in F. As $|a| \geq 1$ for all $a \in F \setminus \{0\}$, $|ab| \leq |a||b|$ for all a and b in F. Thus $|\cdot\cdot|$ is a norm on F.

Let x be any transcendental element over F in some field extension, let $F(x)$ be the field of rational functions over F and let $F((x))$ denote the field of formal power series over F, that is, $F((x)) = \{\sum_{i=m}^{\infty} a_i x^i\colon m \in Z, a_i \in F \text{ for all } i \geq m\}$. As $F((x))$ is the completion of $F(x)$ for the x-adic valuation v_x defined on $F(x)$ [**8**, p. 243], we may identify $F(x)$ with a subfield of $F((x))$.

Define N from $F((x))$ to $[0, \infty]$ by,

$$N(y) = \sup_i |a_i| 2^{-i} \quad \text{for } y = \sum a_i x^i \in F((x)).$$

LEMMA 2. (1) $N(y) = 0$ *if and only if* $y = 0$.
(2) $N(y_1 \pm y_2) \leq \max\{N(y_1), N(y_2)\}$ *for all* y_1, y_2 *in* $F((x))$.
(3) $N(y_1 y_2) \leq N(y_1)N(y_2)$ *for all* y_1, y_2 *in* $F((x))$.

Proof. As (1) and (2) follow easily from the corresponding properties of $|\cdot\cdot|$, it suffices to prove (3). Let $y_1 = \sum a_i x^i$ and $y_2 = \sum b_i x^i$ be elements of $F((x))$. Then $y_1 y_2 = \sum c_i x^i$ where $c_n = \sum_{i+j=n} a_i b_j$ for all $n \in Z$. Hence

$$\begin{aligned} N(c_n x^n) &= N\Big(\sum_{i+j=n} a_i x^i b_j x^j\Big) \leq \max_{i+j=n} N\big(a_i x^i b_j x^j\big) \\ &\leq \max_{i+j=n} N\big(a_i x^i\big) N\big(b_j x^j\big) \leq N(y_1)N(y_2). \end{aligned}$$

Therefore

$$N(y_1 y_2) = \sup_n N(c_n x^n) \leq N(y_1)N(y_2) \quad \text{for } y_1, y_2 \text{ in } F((x)).$$

By the above lemma, the set R defined by, $R = \{y \in F((x))\colon N(y) < \infty\}$, is a subring of $F((x))$ and N is a norm on R. Let D be the subset of R defined by,

$$D = \left\{\sum_{i=m}^{\infty} a_i x^i\colon m \in Z, a_i \in F \text{ for all } i \geq m \text{ and } \lim_{i \to \infty} |a_i| 2^{-i} = 0\right\}.$$

LEMMA 3. *D is a subfield of R, D is complete with respect to the N-topology and $F(x)$ is a dense subfield of D.*

Proof. Clearly, for any $a \in F$ and any $m \in Z$, $aD \subseteq D$ and $x^m D \subseteq D$. We first show that for any $y \in D \backslash \{0\}$, $y^{-1} \in D$. By the preceding observation, we may assume that $y = \sum_{i=0}^{\infty} a_i x^i$ where $a_0 = 1$. Then $y^{-1} = \sum_{i=0}^{\infty} b_i x^i$ where $b_0 = 1$ and for all $n \geq 1$, $b_n = -\sum_{i+j=n; 0 \leq j < n} a_i b_j$. For each $n \geq 0$, let $\gamma_n = \max\{|a_i| : 0 \leq i \leq n\}$. An inductive argument establishes that $|b_n| \leq \gamma_n$ for all $n \geq 0$. As $|a_n| 2^{-n} \to 0$, it follows that $\gamma_n 2^{-n} \to 0$ and so $|b_n| 2^{-n} \to 0$, that is, $y^{-1} \in D$.

To complete the proof of the lemma we shall make use of the following alternate construction of R, D and N. Let Z be given the discrete topology and let $v\colon Z \to (0, \infty)$ be defined by, $v(n) = 2^{-n}$ for all $n \in Z$. Denote the set of all continuous functions f from Z to F for which $\|f\|_v = \sup_{i \in Z} v(i)|f(i)| < \infty$ by $C^v(Z, F)$, the set of all f in $C^v(Z, F)$ such that f vanishes at ∞ (that is, for each $\varepsilon > 0$, there exists a compact subset K of Z such that $\|f \cdot \chi_{Z \setminus K}\|_v < \varepsilon$) by $C^v_\infty(Z, F)$, and the set of all f in $C^v_\infty(Z, F)$ with compact support by $C^v_0(Z, F)$. Then Z is a locally compact space, v is continuous, $C^v_\infty(Z, F)$ is a closed subgroup (under $+$) of the complete, normable group $C^v(Z, F)$ and $C^v_0(Z, F)$ is a dense subset of $C^v_\infty(Z, F)$. (The proof of this assertion is similar to the proof in the classical case where F is **R** or **C**. For a discussion of this case see, for example, [7].) For each $y = \sum a_i x^i \in F((x))$, we may identify y with the function f defined from Z to F by, $f(i) = a_i$ for all $i \in Z$. With this identification,

$$R = C^v(Z, F), \quad D = C^v_\infty(Z, F), \quad F[x] \subseteq C^v_0(Z, F) \subseteq F(x),$$
$$C^v_0(Z, F) \subseteq D \quad \text{and} \quad N(y) = \|y\|_v \quad \text{for all } y \text{ in } R.$$

Moreover, $C^v(Z, F)$ and $C^v_0(Z, F)$ are topological rings under the multiplication $(f \cdot g)(i) = \sum_{m+n=i} f(m)g(n)$. As $(C^v_\infty(Z, F), \|\cdot\|_v)$ is complete, (D, N) is complete as well. Further, as $\overline{C^v_0(Z, F)} = C^v_\infty(Z, F)$, D is a subring of R and hence a subfield of R by the previous observation. Thus $F(x) \subseteq D$ and so $D = \overline{C^v_0(Z, F)} \subseteq \overline{F(x)} \subseteq D$, that is, $F(x)$ is a dense subfield of D.

THEOREM 1. *Let F be an infinite field, let $F_0, F_1, F_2, \ldots$ be a nested sequence of subrings of F such that F_n is properly contained in F_{n+1} for all $n \geq 0$, $1 \in F_0$ and $F = \bigcup_{n=0}^{\infty} F_n$, and let x be any transcendental element over F in some field extension. Then there exists a norm $\|\cdot\|$ on $F(x)$ such that F is discrete, $\|x\| < 1$, the completion of $F(x)$ for $\|\cdot\|$ is a field but $\|\cdot\|$ is not equivalent to the supremum of a finite family of absolute values on $F(x)$. Moreover for each $n \geq 0$, the topology induced on $F_n(x)$ by $\|\cdot\|$ is the same as that induced on $F_n(x)$ by the x-adic valuation v_x defined on $F(x)$.*

Proof. Let $\|\cdot\|$ denote the restriction of N to $F(x)$. By Lemmas 2 and 3, $\|\cdot\|$ is a norm on $F(x)$ and the completion of $F(x)$ for $\|\cdot\|$ is a field. By definition, $\|x\| = 2^{-1} < 1$ and for each nonzero a in F, $\|a\| = |a| \geq 1$. Hence F is discrete for $\|\cdot\cdot\|$.

Suppose $\|\cdot\cdot\|$ is equivalent to the supremum of a finite family $\{|\cdot\cdot|_i: 1 \leq i \leq n\}$ of absolute values on $F(x)$. As the completion of $F(x)$ for $\|\cdot\cdot\|$ is a field, $n = 1$ by the Approximation Theorem for Absolute Values [**1**, Theorem 2, p. 136]. As F is discrete for $\|\cdot\cdot\|$, F is discrete for $|\cdot\cdot|_1$ as well, that is, $|a|_1 = 1$ for all a in $F\backslash\{0\}$. Thus F is a bounded set for the topology induced on $F(x)$ by $|\cdot\cdot|_1$. However, if n is any positive integer and a_n is any element of $F_n \backslash F_{n-1}$, then $\|a_n\| = |a_n| = 2^n$. Therefore F is not bounded for the topology defined on $F(x)$ by $\|\cdot\cdot\|$, a contradiction.

To prove the last assertion of the theorem, we note that for any $n \geq 0$ and for any y in $F_n(x)$,

$$2^{-v_x(y)} \leq \|y\| \leq 2^n 2^{-v_x(y)}.$$

In [**9**] Weber showed that if F is a field and x is any transcendental element over F, then F is finite if and only if for each Hausdorff, nondiscrete locally bounded topology $\mathfrak{T}$ on $F(x)$, there exists a nonempty proper subset S of $\mathfrak{P}' = \{p: p \text{ is a prime polynomial of } F[x]\} \cup \{\infty\}$ such that the set $O(S)$ defined by, $O(S) = \{y \in F(x): v_p(y) \geq 0 \text{ for all } p \in S\}$, is a bounded neighborhood of zero for $\mathfrak{T}$ (Satz 3.3). The following is a generalization of this result.

COROLLARY. *Let F be a field and let x be any transcendental element over F. The following are equivalent.*

(1) *F is a finite field.*

(2) *If $\mathfrak{T}$ is a Hausdorff, nondiscrete locally bounded topology on $F(x)$, then there exists a nonempty, proper subset S of $\mathfrak{P}'$ such that $O(S)$ is a bounded neighborhood of zero for $\mathfrak{T}$.*

(3) *If $\|\cdot\cdot\|$ is a nontrivial norm on $F(x)$ such that F is discrete and the completion of $F(x)$ for $\|\cdot\cdot\|$ is a field, then $\|\cdot\cdot\|$ is equivalent to an absolute value which is improper on F.*

Proof. By the above remarks, (1) and (2) are equivalent. By Theorem 1, (3) implies (1). So it suffices to show that (1) implies (3). Suppose F is a finite field and $\|\cdot\cdot\|$ is a nontrivial norm on $F(x)$ such that the completion of $F(x)$ for $\|\cdot\cdot\|$ is a field. Then F is bounded in norm and so by the corollary to Theorem 2 of [**2**], $\|\cdot\cdot\|$ is equivalent to an absolute value on $F(x)$ which is improper on F.

In [**3**] we characterized all norms on the polynomial ring $F[x]$ for which F is bounded (Theorem 2). We conclude this paper by showing that if F is any infinite field, the analogous characterization of the norms on $F[x]$ for which F is discrete is impossible.

THEOREM 2. *Let F be an infinite field and let x be any transcendental element over F in some field extension. Then there exists a norm $\|\cdot\cdot\|$ on $F[x]$ such that F is discrete and $\|x\| < 1$ but $\|\cdot\cdot\|$ is not equivalent to the supremum of a finite family of absolute values on $F[x]$.*

Proof. Let $\|\cdot\cdot\|$ be the norm on $F(x)$ constructed in the proof of Theorem 1 and let $\|\cdot\cdot\|'$ denote its restriction to $F[x]$. Clearly, F is discrete for $\|\cdot\cdot\|'$ and $\|x\|' < 1$. Suppose $\|\cdot\cdot\|'$ is equivalent to the supremum of a finite family $\{|\cdot\cdot|_i\colon 1 \le i \le n\}$ of absolute values on $F[x]$. Then each $|\cdot\cdot|_i$ is improper on F. Indeed, suppose there exist i, $1 \le i \le n$, and $a \in F$ such that $|a|_i > 1$. Let m be such that $|a^m x|_i > 1$. The sequence $\langle (a^m x)^r \rangle_{r=1}^{\infty}$ converges to 0 for $\|\cdot\cdot\|'$ but not for $|\cdot\cdot|_i$, a contradiction. Hence each $|\cdot\cdot|_i$ is improper on F. It then follows that F is bounded for the supremum topology but not for the topology defined on $F[x]$ by $\|\cdot\cdot\|'$, a contradiction.

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