ON THE SINGULARITIES OF ALMOST-SIMPLE PLANE CURVES

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Let Γ be a differentiable curve in a real projective plane P^2 intersected by every line at a finite number of points. A point of Γ is ordinary if Γ is locally convex at that point; otherwise, the point is singular. Let the singular points of Γ consist of n_1 inflections, n_2 cusps (cusps of the first kind) and n_3 beaks (cusps of the second kind). Then Γ is singular if $n(\Gamma) = n_1 + n_2 + n_3 > 0$; otherwise, Γ is non-singular. The following questions arise naturally: Under what conditions is Γ singular? What is then the minimum value of $n(\Gamma)$ and is it dependent or independent of the type of singularities that Γ may possess? Presently, we determine a class of curves Γ for which $n(\Gamma) \geq 2$ but $n_1 + 2n_2 + n_3 \geq 4$.

1. Introduction. Since Γ is intersected by every line at a finite number of points, all lines of P^2 which contain no singular points and are not a tangent at any point of Γ have the following property: either all meet Γ at an odd number of points or all meet Γ at an even number of points. In the case of the former [latter], we say that Γ is of odd [even] order. Then with regard to the questions posed, curves of odd order are singular, there exists a curve of odd order containing exactly one singular point (cf. pp. 1–7, [1]), and a theorem of Möbius [2] states that a simple (without self-intersections) curve of odd order with no cusps or beaks contains at least three inflections.

Henceforth we assume that our curves are of even order. Since those conics not met by some line of P^2 are trivially non-singular, we also assume that the curves of even order consideration are met by every line in P^2. The last restriction is associated with the number $m(\Gamma)$ of multiple (self-intersection) points that Γ may possess. In Figure 1, we present a non-singular Γ of even order which is met by every line in P^2 and contains three double points. Thus we may certainly restrict our attention to curves Γ for which $m(\Gamma)$ is finite. We also observe in this example that each double point decomposes the curve into two (not necessarily differentiable) closed curves, each of which is not met by some line. Thus we are faced with two possible restrictions: assume that $m(\Gamma) \leq 2$ or that no multiple point of Γ has the mentioned property of the double points in Figure 1. In subsequent papers we will consider the case $m(\Gamma) \leq 2$.

Presently we assume that Γ is almost-simple; that is, each multiple point of Γ decomposes Γ into curves, each of which is met by every line in P^2. Under these hypotheses, our main theorem states that $n(\Gamma) \geq 2$ and $n_1 + 2n_2 + n_3 \geq 4$. As examples of the minimal cases, we refer to Figure 3 ($n_2 = 2, n_1 = n_3 = 0$), Figure 4 ($n_1 = n_2 = n_3 = 1$) and Figure 5 ($n_1 = n_3 = 2, n_2 = 0$).
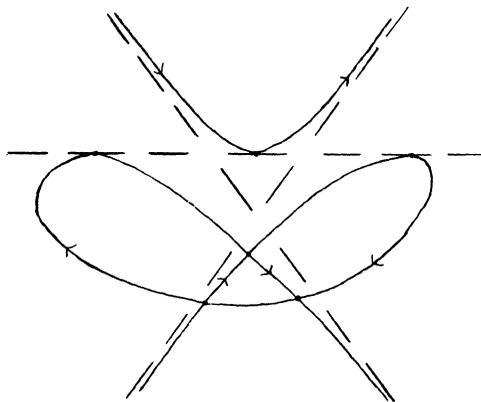

FIGURE 1

2. **Differentiable curves.** In this section we introduce our notation, present formal definitions and list results required for the proof of the main theorem. As these results are known or readily evident, they are presented without proofs. For a rigorous and complete treatment, we refer the reader to [3].

We assume that P^2 has the usual topology. Let $p, q, \ldots$ and $L, M, \ldots$ denoted the points and lines of P^2 respectively. The flat of P^2 spanned by $p, L, \ldots$ is denoted by $\langle p, L, \ldots \rangle$.

Let $T \subset P^2$ be an oriented line. For $t_0 \neq t_1$ in T, let $[t_0, t_1]$ denote the oriented closed segment of T with initial point t_0 and terminal point t_1. We put $(t_0, t_1) = [t_0, t_1] \backslash \{t_0, t_1\}$, $[t_0, t_1) = [t_0, t_1] \backslash \{t_1\}$ and $(t_0, t_1] \backslash \{t_0\}$. Hence $T = [t_0, t_1] \cup (t_1, t_0) = [t_0, t_1) \cup [t_1, t_0)$. If $U(t) = (t_0, t_1)$ is a neighbourhood of t in T then $U^-(t) = (t_0, t)$, $U^+(t) = (t, t_1)$ and $U'(t) = U^-(t) \cup U^+(t)$.

A *curve* Γ is a continuous map from T into P^2. Γ is (*directly*) *differentiable* if the *tangent line* $\Gamma_1(t) = \lim \langle \Gamma(t), \Gamma(t') \rangle$, as $t' \neq t$ tends to t in T, exists for each $t \in T$ and any line of P^2 meets Γ at a finite number of points. Henceforth Γ is differentiable and we identify $\Gamma(T)$ with Γ.

Let $\mathfrak{M} \subseteq T$ be a segment. We call $\Gamma/\mathfrak{M}$ a *subarc* of Γ and identify $\Gamma/\mathfrak{M}$ with $\Gamma(\mathfrak{M})$. If $n = \sup_{L \subset P^2} |L \cap \Gamma(\mathfrak{M})|$ is finite, we say that $\mathfrak{M}$, or equivalently $\Gamma(\mathfrak{M})$, is of *order n*. The *order of a point* $t \in T$, ord(t), is

the minimum order which an $U(t)$ can possess. Clearly ord$(t) \geq 2$. Then t is *ordinary* if ord$(t) = 2$; otherwise, t is *singular*. We say that t is *elementary* if there exist $U^-(t)$ and $U^+(t)$ of order two. A subarc of Γ is *ordinary* [*elementary*] if each of its points is ordinary [elementary].

Let $t \in T$ and $L \subset P^2$. We say that L *supports* Γ at t if there is an $L' \neq L$ with $\Gamma(t) \notin L'$ and an $U'(t)$ such that $\Gamma(U'(t))$ is contained in one of the open half-planes of P^2 determined by L and L'. When L does not support Γ at t, we say that L *cuts* Γ at t. Let $S(t) = \{L \subset P^2 \,|\, \Gamma(t) \in L \neq \Gamma_1(t)\}$. It is known that either all $L \in S(t)$ cut Γ at t or all $L \in S(t)$ support Γ at t. Thus there are four types of points in T with respect to Γ: t is *regular* if $L \in S(t)$ [$\Gamma_1(t)$] cuts [supports] Γ at t; t is an *inflection* if $L \in S(t)$ and $\Gamma_1(t)$ both cut Γ at t; t is a *beak* if $L \in S(t)$ and $\Gamma_1(t)$ both support Γ at t; t is a *cusp* if $L \in S(t)$ [$\Gamma_1(t)$] supports [cuts] Γ at t. In Figure 4, t_1 is a cusp, t_2 is a beak, t_3 is an inflection and all other points are regular. Trivially, an ordinary point is regular and hence inflections, cusps and beaks are singular.

Next, Γ is of *odd* [*even*] *order* if every line in P^2 cuts Γ at an odd [even] number of points. Since every line meets Γ at a finite number of points, Γ is clearly of odd or even order. Let $\mathfrak{M} \subseteq T$. The *index* of $\Gamma(\mathfrak{M})$, ind$(\Gamma(\mathfrak{M}))$, is the minimum number of points of $\Gamma(\mathfrak{M})$ which can lie on any line of P^2. Thus if Γ is of odd order, ind$(\Gamma) > 0$. Let $t \in \mathfrak{M}$. Then $\Gamma(t)$ is a *simple point* of $\Gamma(\mathfrak{M})$ if $\Gamma(t') \neq \Gamma(t)$ for $t' \in \mathfrak{M} \setminus \{t\}$; otherwise, $\Gamma(t)$ is a *multiple point* of $\Gamma(\mathfrak{M})$. We similarly define a *multiple tangent* of $\Gamma(\mathfrak{M})$. We say that $\Gamma(\mathfrak{M})$ is *simple* if each of its points is simple. Finally Γ is *almost-simple* if $\Gamma(t) = \Gamma(t')$ for $t \neq t'$ implies that ind$(\Gamma[t, t'])$ and ind$(\Gamma[t', t])$ are both positive.

As stated in the introduction, we are interested in almost-simple curves Γ with the property that Γ is of even order, $n(\Gamma) < \infty$ and $m(\Gamma) < \infty$. The conditions that $n(\Gamma)$ and $m(\Gamma)$ are finite however imply that Γ is elementary, that is, not only is Γ locally convex at all except a finite number of points but each point of Γ also has one-sided neighbourhoods of order two. Such curves have been extensively studied and we list some of their important properties.

1. A regular point is ordinary and hence the singular points of Γ consist of n_1 inflections, n_2 cusps and n_3 beaks.

2. $n_1 + 2n_2 + n_3$ is even if and only if Γ is of even order.

3. The tangents $\Gamma_1(t)$ of Γ depend continuously on $t \in T$.

4. Let $\Gamma(t)$ be simple and let $L = \lim \langle \Gamma(t'), \Gamma(t'') \rangle$ as $t' \neq t''$ tend to t. If t is an ordinary or an inflection point then $L = \Gamma_1(t)$.

5. Let $t \in T$ and $p \in P^2$. Then there is an $U'(t)$ such that $p \notin \Gamma_1(t')$ for $t' \in U'(t)$.

6. Let $\Gamma(t_1, t_2)$ be a regular (ordinary) simple subarc such that there is a line through $\Gamma(t_1)$ and $\Gamma(t_2)$ not meeting $\Gamma(t_1, t_2)$, $\Gamma(t_1) \neq \Gamma(t_2)$. If $\Gamma(t_1) \notin \Gamma_1(t)$ for $t \in (t_1, t_2)$ or $\Gamma(t_2) \notin \Gamma_1(t)$ for $t \in (t_1, t_2)$, then (t_1, t_2) is of order two.

Finally we note that our definitions and results were presented in terms of continuity and the manner in which lines of P^2 meet the curve. As the basic subarc of these curves is a subarc of order two, we also list some of the properties of such a subarc.

Let (t_0, t_1) be of order two and let N be an oriented line such that $N \cap \Gamma(t_0, t_1) = \varnothing$. For $t \in (t_0, t_1)$, $\Gamma_1(t)$ meets N at exactly one point which we denote by $\varphi(t)$.

7. $|\Gamma_1(t) \cap \Gamma[t_0, t_1]| = 1$ for $t \in (t_0, t_1)$, $\lim(\Gamma_1(t) \cap \Gamma_1(t_i)) = \{\Gamma(t_i)\}$ as t tends to t_i and $\lim(\langle \Gamma(t'), \Gamma(t'')\rangle \cap \Gamma_1(t_i)) = \{\Gamma(t_i)\}$ as $t' \neq t''$ tend to t_i in (t_0, t_1), $i = 0, 1$.

8. If φ is not onto then φ is a strictly monotone map.

3. The main theorem.

1. THEOREM. *Let Γ be almost-simple, elementary with positive index and of even order. Then*

(i) $n(\Gamma) \geq 2$

and

(ii) $n_1 + 2n_2 + n_3 \geq 4$.

Henceforth let $\Gamma: T \to P^2$ satisfy the hypothesis of 3.1. Let $t \in T$. Then there is an $U(t) = (t_0, t_1)$ such that both (t_0, t) and (t, t_1) are of order two. Let N be an oriented line such that $N \cap \Gamma(t_0, t_1) = \varnothing$, N meets $\Gamma_1(t')$ at the point $\varphi(t')$ for $t' \in \Gamma(t_0, t_1)$ and φ is not onto. If t is regular (equivalently ordinary), then we may assume that (t_0, t_1) is of order two and thus φ is strictly monotone by 1.8. In any case, $\varphi/(t_0, t]$ and $\varphi/[t, t_1)$ are both strictly monotone. As φ is not onto, $\varphi((t_0, t)) \cap \varphi((t, t_1)) \neq \varnothing$ implies that either $\varphi((t_0, t)) \subseteq \varphi((t, t_1))$ or $\varphi((t, t_1)) \subseteq \varphi((t_0, t))$. It is easy to check that.

2. LEMMA $\varphi((t_0, t_1)) \cap \varphi((t, t_1)) = \varnothing$ *if and only if t is a regular or a cusp point.*

We wish to relate this monotone intersection property of the tangents of $U(t)$ with N to the subarcs of Γ met by these tangents. To this end, we introduce the following definitions.

Let $\mathfrak{M}$ and $\mathfrak{M}'$ be segments of T such that $\Gamma(\mathfrak{M}) \cap \Gamma(\mathfrak{M}') = \varnothing$ and $\Gamma_1(t) \cap \Gamma(\mathfrak{M}') \neq \varnothing$ for each $t \in \mathfrak{M}$. If $|\Gamma_1(t) \cap \Gamma(\mathfrak{M}')| = 1$ for each $t \in \mathfrak{M}$, we write $\mathfrak{M} \to \mathfrak{M}'$. (Since Γ is elementary, $\mathfrak{M} \to \mathfrak{M}'$ and 1.3 readily imply that $\Gamma_1(t)$ cuts $\mathfrak{M}'$ at the point of intersection.) If for each $t' \in \mathfrak{M}'$ there is exactly one $t \in \mathfrak{M}$ such that $\Gamma(t') \in \Gamma_1(t)$ and moreover $\Gamma_1(t)$ cuts $\mathfrak{M}'$ at t', we write $\mathfrak{M} \leftarrow \mathfrak{M}'$. If both $\mathfrak{M} \to \mathfrak{M}'$ and $\mathfrak{M} \leftarrow \mathfrak{M}'$, then clearly $\Gamma_1(t), t \in \mathfrak{M}$, meets $\Gamma(\mathfrak{M}')$ in a strictly monotone manner and we write $\mathfrak{M} \leftrightarrow \mathfrak{M}'$.

3. **LEMMA.** *Let $\Gamma_1(s_0)$ cut Γ at $\Gamma(t_0) \neq \Gamma(s_0)$. Then there exist $U(s_0)$ and $U(t_0)$ such that $U(s_0) \to U(t_0)$ and $\Gamma(t_0) \notin \Gamma_1(s)$ for any $s \in U'(s_0)$.*

Proof. Clearly there exist open neighbourhoods $V(s_0)$ and $V(t_0)$ in T such that $\Gamma(V(s_0)) \cap \Gamma(V(t_0)) = \varnothing$, $\Gamma_1(s_0) \cap \Gamma(V(t_0)) = \{\Gamma(t_0)\}$ and by 1.5, $\Gamma(t_0) \notin \Gamma_1(s)$ for $s \in V^-(s_0)$. As Γ is elementary, we may assume that $V^-(s_0)$, $V^+(s_0)$, $V^-(t_0)$ and $V^+(t_0)$ are all of order two. Finally as $\Gamma_1(s_0)$ cuts Γ at t_0, we may assume that $\Gamma_1(s) \cap \Gamma(V(t_0)) \neq \varnothing$ for $s \in V(s_0)$.

Suppose there exist s_λ tending to s_0 in say $V^-(s_0)$ such that $\Gamma_1(s_\lambda)$ meets $V(t_0)$ at $t_\lambda \neq t_\lambda'$ for each s_λ. By 1.3, both t_λ and t_λ' tend to t_0 as s_λ tends to s_0. Since $\Gamma_1(s_0)$ cuts Γ at t_0, t_0 is not a beak. If t_0 is a cusp, then $\Gamma_1(s_0) = \Gamma_1(t_0)$. If t_0 is not a cusp, then 1.4 implies that

$$\Gamma_1(s_0) = \lim \Gamma_1(s_\lambda) = \lim \langle \Gamma(t_\lambda), \Gamma(t_\lambda') \rangle = \Gamma_1(t)$$

as s_λ tends to s_0 and thus t_0 is an inflection point. In any case, $\Gamma_1(s_0) = \Gamma_1(t_0)$ and $\operatorname{ord}(t_0) = 3$. From this, it readily follows that for $\Gamma_1(s_\lambda) = \langle \Gamma(t_\lambda), \Gamma(t_\lambda') \rangle$ close to $\Gamma_1(t_0)$, either $\Gamma_1(s_\lambda)$ supports and cuts Γ in $V'(t_0)$ or $\Gamma_1(s_\lambda)$ cuts Γ at three points of $V(t_0)$. Since $V^-(t_0)$ and $V^+(t_0)$ are of order two and $\Gamma(t_0) \notin \Gamma_1(s_\lambda)$, such $\Gamma_1(s_\lambda)$ is either a tangent of $\Gamma(V^-(t_0))$ or cuts Γ at two points of $V^-(t_0)$ say. In either case, 1.7 implies that

$$\{\Gamma(s_0)\} = \lim_{s_\lambda \to s_0} (\Gamma_1(s_\lambda) \cap \Gamma_1(s_0))$$

$$= \lim_{s_\lambda \to s_0} (\Gamma_1(s_\lambda) \cap \Gamma_1(t_0)) = \{\Gamma(t_0)\},$$

a contradiction. Hence $|\Gamma_1(s_\lambda) \cap \Gamma(V(t_0))| = 1$ for s_λ sufficiently close to s_0 in $V(s_0)$.

4. **LEMMA.** *Let $\Gamma_1(s_0)$ cut Γ at $\Gamma(t_0) \neq \Gamma(s_0)$. Then there exist $U(s_0)$ and $U(t_0)$ such that either $U^+(s_0) \leftrightarrow U^-(t_0)$ or $U^+(s_0) \leftrightarrow U^+(t_0)$ and either $U^-(s_0) \leftrightarrow U^-(t_0)$ or $U^-(s_0) \leftrightarrow U^+(t_0)$.*

Proof. By 3.3, there exist $V(s_0)$ and $V(t_0)$ such that $V^+(s_0)$ and $V^-(s_0)$ are both of order two, $V(s_0) \rightarrow V(t_0)$ and $\Gamma(t_0) \notin \Gamma_1(s)$ for $s \in V'(s_0)$. By 1.3, $\Gamma_1(s) \cap \Gamma(V^+(t_0)) = \varnothing$ for $s \in V^+(s_0)$ say and hence $V^+(s_0) \rightarrow V^-(t_0)$. Set $V^-(t_0) = (t_1, t_0)$ and let $U^+(s_0) \subseteq V^+(s_0)$ such that some $\bar{t} \in (t_1, t_0)$ does not lie on any tangent of $U^+(s_0)$.

Suppose there is a $t^* \in (t_1, t_0)$ such that $\Gamma(t^*) = \Gamma_1(s_1) \cap \Gamma_1(s_2)$, $s_1 < s_2$ in $U^+(s_0)$. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the closed half-planes determined by $\Gamma_1(s_1)$ and $\Gamma_1(s_2)$. Since $U^+(s_0)$ is of order two, $\Gamma(U^+(s_0)) \subset \mathcal{P}_1$ say by 1.7. Let $\mathcal{P}_{11}$ and $\mathcal{P}_{12}$ be the open disjoint regions of $\mathrm{int}(\mathcal{P}_1)$ determined by $\Gamma(s_1, s_2)$, $\Gamma(s_0, s_1) \subset \mathcal{P}_{11}$ say. Then each point of $\mathcal{P}_2 \cup \mathcal{P}_{12}$ lies on at least one tangent of $U^+(s_0)$. Hence $\Gamma_1(s) \cap \Gamma(V^+(t_0) \cup \{t_0\}) = \varnothing$ for $s \in U^+(s_0)$ implies that $\Gamma(V^+(t_0) \cup \{t_0\}) \subset \mathcal{P}_{11}$. As $U^+(s_0) \rightarrow V^-(t_0)$, $\Gamma(t^*) = \Gamma_1(s_1) \cap \Gamma_1(s_2)$ and $\Gamma(t_0) \in \mathcal{P}_{11}$ yield that $\Gamma(t^*, t_0) \subset \mathcal{P}_{11}$. Clearly each point of (t^*, t_0) lies on a tangent of $U^+(s_0)$ by 1.3 and thus $\bar{t} \in (t_1, t^*)$. But as both $\Gamma_1(s_1)$ and $\Gamma_1(s_2)$ cut Γ at t^*, $\Gamma(t^*, t_0) \subset \mathcal{P}_{11}$ and $\Gamma(U^+(s_0)) \cap \Gamma(V^-(t_0)) = \varnothing$ imply that $\Gamma(t_1, \bar{t}] \subset \Gamma(t_1, t^*) \subset \mathcal{P}_{12}$, a contradiction.

Since each point of $V^-(t_0)$ lies on at most one tangent of $U^+(s_0)$, it is clear that there is an $U^-(t_0) \subset V^-(t_0)$ such that $U^+(s_0) \leftarrow U^-(t_0)$ and thus $U^+(s_0) \leftrightarrow U^-(t_0)$.

The preceding is symmetric in $U^+(s_0)$ and $U^-(s_0)$.

COROLLARY. *Under the hypothesis of* 3.4, *there exist* $U(s_0)$ *and* $U(t_0)$ *such that* $U(s_0) \leftrightarrow U(t_0)$ *if and only if* s_0 *is a regular or a cusp point.*

Proof. Let $U(s_0) \rightarrow U(t_0)$. Then clearly $U(s_0) \leftrightarrow U(t_0)$ if and only if either $U^+(s_0) \leftrightarrow U^+(t_0)$ and $U^-(s_0) \leftrightarrow U^-(t_0)$ or $U^+(s_0) \leftrightarrow U^-(t_0)$ and $U^-(s_0) \leftrightarrow U^+(t_0)$. We now apply 3.2.

Let $\Gamma_1(s)$ meet Γ at $\Gamma(t) \neq \Gamma(s)$. We say that t is *s-negative* [*s-positive*] if there exist $U^+(s)$ of order two and $U^-(t)[U^+(t)]$ such that $U^+(s) \leftrightarrow U^-(t)\,[U^+(s) \leftrightarrow U^+(t)]$.

By 3.4, if $\Gamma_1(s)$ cuts Γ at $\Gamma(t) \neq \Gamma(s)$ then t is either s-positive or s-negative. We note that t may be both s-positive and s-negative or neither s-positive nor s-negative. In either case, $\Gamma_1(s)$ supports Γ at t if $\Gamma(s) \neq \Gamma(t)$.

Finally we observe that the following two lemmas are a direct consequence of 3.4 and the definitions.

5. LEMMA. *Let* t_0 *be* s_0-*negative* [s_0-*positive*]. *Then there exist* $U^+(s_0)$ *and* $U^-(t_0)\,[U^+(t_0)]$ *such that* $U^+(s_0) \leftrightarrow U^-(u_0)\,[U^+(s_0) \leftrightarrow U^+(t_0)]$ *and each* $t \in U^-(t_0)\,[U^+(t_0)]$ *is* s-*negative* [s-*positive*] *where* $\Gamma(t) \in \Gamma_1(s)$, $s \in U^+(s_0)$.

6. **Lemma.** *Let t_0 be s_0-negative, $U^+(s_0) \leftrightarrow U^-(t_0)$ and each $t \in U^-(t_0)$ be s-negative where $\Gamma(t) \in \Gamma_1(s)$, $s \in U^+(s_0)$. Let $\Gamma_1(s_i)$ cut Γ at $t_i \in U^-(t_0)$, $i = 1, 2$. Then $s_1 < s_2$ (s_1 precedes s_2) in $U^+(s_0)$ implies that $t_2 < t_1$ in $U^-(t_0)$ and $\Gamma[s_2, t_2] \subset \Gamma[s_1, t_1]$.*

We now present another characterization of singular points and give sufficient conditions for their existence. With that purpose, let $\mathfrak{M}$ and $\mathfrak{M}'$ be open segments in T such that $\Gamma(\mathfrak{M}) \cap \Gamma(\mathfrak{M}') = \varnothing$ and $\Gamma_1(s) \cap \Gamma(\mathfrak{M}') \neq \varnothing$ for each $s \in \mathfrak{M}$. If each point $t \in \mathfrak{M}'$ lies on a tangent of $\mathfrak{M}$ and $\Gamma(t) \in \Gamma_1(s) \cap \Gamma(\mathfrak{M}')$, $s \in \mathfrak{M}$, implies that $\Gamma_1(s)$ cuts $\mathfrak{M}'$ at t and t is s-negative, we write $\mathfrak{M} \sim \mathfrak{M}'$.

7. **Theorem.** *Let $t_0 \in T$. Then t_0 is singular if and only if there exists an $U(t_0)$ of minimum order such that both $U^-(t_0)$ and $U^+(t_0)$ are of order two and (with T possibly reoriented) $U^+(t_0) \sim U^-(t_0)$.*

Proof. Since t_0 is elementary, there exists a $V(t_0)$ of minimum order such that $V^-(t_0)$ and $V^+(t_0)$ are both of order two.

If t_0 is regular then $V(t_0)$ is of order two and $|\Gamma_1(t) \cap \Gamma(V(t_0))| = 1$ for $t \in V(t_0)$ by 1.7. Hence the sufficient condition follows.

If t_0 is an inflection or a cusp then $V(t_0)$ is of order three, $|\Gamma_1(t_0) \cap \Gamma(V(t_0))| = 1$ and $\Gamma_1(t_0)$ meets, and cuts, Γ at exactly t_0 in $V(t_0)$. Let $s \in V^+(t_0)$ tend to t_0. Then $\lim \Gamma_1(s) = \Gamma_1(t_0)$ by 1.3 and this readily implies that $\Gamma_1(s)$ meets, and cuts, $V^-(t_0)$ at exactly one point t for sufficiently close to s_0. Hence there exist $W^+(t_0) \subseteq V^+(t_0)$ and $W^-(t_0) \subset V^-(t_0)$ such that $W^+(t_0) \to W^-(t_0)$. By arguing as in the proof of 3.4, there exist $U^+(t_0) \subseteq W^+(t_0)$ and $U^-(t_0) \subseteq W^-(t_0)$ such that $U^+(t_0) \leftrightarrow U^-(t_0)$. Since $t \in U^-(t_0)$ tends to t_0 as $s \in U^+(t_0)$ tends to t_0, it is immediate that each t is s-negative.

If t_0 is a beak then $V(t_0)$ is of order two and $\Gamma_1(t_0)$ meets, and supports, Γ at exactly t_0 in $V(t_0)$. It is easy to check that for all $s \in V^+(t_0)$ close to t_0, either $\Gamma_1(s) \cap \Gamma(V^-(t_0)) = \varnothing$ or $\Gamma_1(s)$ meets, and cuts, $V^-(t_0)$ at exactly two points. By reorienting T if necessary, we may assume that $|\Gamma_1(s) \cap \Gamma(V^-(t_0))| = 2$ for $s \in V^+(t_0)$ close to t_0. Then by arguing as in the preceding, we obtain that there exist $U^+(t_0) \subseteq V^+(t_0)$ and $U^-(t_0) \subseteq V^-(t_0)$ such that each point of $U^-(t_0)$ lies on a tangent of $U^+(t_0)$ and every tangent of $U^+(t_0)$ cuts $U^-(t_0)$ at each point of contact. Then $\lim(\Gamma_1(s) \cap \Gamma(U^-(t_0))) = \{\Gamma(t_0)\}$ as $s \in U^+(t_0)$ tends to t_0 implies that $U^+(t_0) \sim U^-(t_0)$.

8. LEMMA. *Let $\Gamma(s_0, t_0)$ be simple. Let $U^+(s_0) = (s_0, s_1)$, $s_0 \neq s_1$, be the maximal (open right-neighbourhood of s_0) in (s_0, t_0) such that there exists an $U^-(t_0) = (t_1, t_0)$ with $U^+(s_0) \sim U^-(t_0)$. Then $s_1 = t_1$ or $\Gamma_1(s_1)$ supports Γ at a point $t \in [t_1, t_\theta)$ or s_1 is a beak or an inflection point.*

Proof. Let $(s_0, s_1) \sim (t_1, t_0)$, $s_1 \neq t_1$. Then for any $s_2 \in (s_1, t_1)$ there is an $s \in (s_0, s_2)$ such that (1) $\Gamma_1(s) \cap \Gamma(t_1, t_0) = \varnothing$ or (2) $\Gamma_1(s)$ supports Γ at a point $t \in (t_1, t_0)$ or (3) $\Gamma_1(s)$ cuts Γ at a point $t \in (t_1, t_0)$ and t is s-positive.

If (1) for any $s_2 \in (s_1, t_1)$, then 1.3, 3.3 and the hypotheses yield that $\Gamma_1(s_1) \cap \Gamma(t_1, t_0) = \varnothing$ and $\Gamma_1(s_1)$ supports Γ at t_1. If (2) for any $s_2 \in (s_1, t_1)$, then $s \in [s_1, s_2]$ by the hypotheses and $\Gamma_1(s_1)$ supports Γ at a point $t \in [t_1, t_0)$ by 1.3. If (3) for any $s_2 \in (s_1, t_1)$, then again $s \in [s_1, s_2)$ and thus $\Gamma_1(s_1)$ cuts Γ at a point $t \in [t_1, t_0)$ such that t is s_1-positive. By the Corollary of 3.4, s_1 is clearly a beak or an inflection.

REMARK. Since $(s_0, s_1) \sim (t_1, t_0)$ implies that each tangent of (s_0, s_1) meets (t_1, t_0), the choice of (t_1, t_0) in 3.8 is not arbitrary. In fact if s_1 is not singular, then (t_1, t_0) is the maximal $U^-(t_0)$ in (s_0, t_0) such that $(s_0, s_1) \sim (t_1, t_0)$. We also note that $\Gamma_1(s_1)$ supporting Γ at a point $t \in [t_1, t_0)$ does not exclude $\Gamma_1(s_1)$ meeting Γ at t_0 in some fashion.

9. LEMMA. *Let $\Gamma(s_0, t_0)$ be a simple subarc such that t_0 is s_0-negative and no tangent supports Γ at more than one point. Then (s_0, t_0) is not regular (contains a singular point).*

Proof. By 3.5, there is a mximal $U^+(s_0) = (s_0, s_1)$ in (s_0, t_0) such that $U^+(s_0) \sim U^-(t_0)$ for some $U^-(t_0) = (t_1, t_0) \subset (s_1, t_0)$. By 3.8, $s_1 = t_1$ or $\Gamma_1(s_1)$ supports Γ at some $t \in [t_1, t_0)$ or s_1 is singular.

If $s_1 = t_1$, then $(s_0, t_0) = (s_0, s_1) \cup \{s_1\} \cup (s_1, t_0)$. Then $(s_0, s_1) \sim (s_1, t_0)$ and 3.6 (with T reoriented) imply that s_1 is singular. If $s_1 \neq t_1$ and $\Gamma_1(s_1)$ supports Γ at $t \in [t_1, t_0)$, then $\Gamma_1(s_1) \neq \Gamma_1(t)$ or $\Gamma_1(s_1)$ cuts Γ at s_1 by the hypotheses. If $\Gamma_1(s_1) \neq \Gamma_1(t)$, then t is a cusp or a beak. If $\Gamma_1(s_1)$ cuts Γ at s_1, then s_1 is an inflection or a cusp.

10. LEMMA. *Let $\Gamma(s_0, t_0)$ be a subarc such that t_0 is s_0-negative and $\Gamma_1(s_0) \cap \Gamma(s_0, t_0) = \varnothing$. Then (s_0, t_0) is not regular.*

Proof. Since Γ is almost-simple, $\Gamma(s_0, t_0)$ is simple and we may assume that 3.9 does not apply to $\Gamma(s_0, t_0)$.

Since Γ is elementary, 3.5 implies there exist $V^+(s_0)$ and $V^-(t_0)$, both of order two, in (s_0, t_0) such that $V^+(s_0) \sim V^-(t_0)$. Let $s \in V^+(s_0)$. Then $\Gamma_1(s) \cap \Gamma(s, t_0) \neq \varnothing$ and there is a $t \in (s, t_0)$ such that $\Gamma_1(s) \cap \Gamma(s, t] = \{\Gamma(t)\}$. By 1.3 and 1.7, $\Gamma_1(s_0) \cap \Gamma(s_0, t_0) = \varnothing$ clearly implies that t tends to t_0 as s tends to s_0. Hence we may assume that $t \in V^-(t_0)$ (hence t is s-negative) for $s \in V^+(s_0)$ arbitrarily close to s_0. Let $s' > s$ be arbitrarily close to s_0 in $V^+(s_0)$. Then $\Gamma_1(s')$ cuts Γ at a point $t' \in V^-(t_0)$ such that $\Gamma_1(s') \cap \Gamma(s', t') = \varnothing$ and by 3.6, $(s', t') \subset (s, t) \subset (s_0, t_0)$.

It is now immediate that there is a maximal $U^+(s_0) = (s_0, s_1)$ in (s_0, t_0) such that (s_0, s_1) is regular and for each $s \in (s_0, s_1)$, $\Gamma_1(s)$ cuts Γ at a point $t \in (s, t_0)$ such that t is s-negative and $\Gamma_1(s) \cap \Gamma(s, t) = \varnothing$. Let t_1' be an accumulation point of t as $s \in (s_0, s_1)$ tends to s_1. By 1.3 and 3.6, $\Gamma(t_1') \subset \Gamma_1(s_1)$ and $t \in (t_1', t_0)$ for $s \in (s_0, s_1)$. Hence t' is the limit point of t. (At this point, one should note that we do not claim that $(s_0, s_1) \sim (t_1', t_0)$. We are only concerned with how $\Gamma_1(s_1)$ meets $\Gamma(s_1, t_1']$.) From the maximality of (s_0, s_1), it readily follows that either s_1 is singular or $s_1 \neq t_1'$ and $\Gamma_1(s_1)$ supports Γ at t_1' or $\Gamma_1(s_1) \cap \Gamma(s_1, t_1') \neq \varnothing$. Assuming that s_1 is regular, $s_1 \neq t_1'$ and $\Gamma_1(s_1)$ supporting Γ at t_1' also yield that $\Gamma_1(s_1) \cap \Gamma(s_1, t_1') \neq \varnothing$. Clearly $\Gamma_1(s_1)$ supports $\Gamma(s_1, t_1')$ at each point of contact and there is a $t_1 \in (s_1, t_1')$ such that $\Gamma_1(s_1) \cap \Gamma(s_1, t_1] = \{\Gamma(t_1)\}$. Then s_1 regular and (s_0, s_1) maximal imply that t_1 is both s_1-positive and s_1-negative. We note that if $\Gamma_1(s_1) \neq \Gamma_1(t_1)$, then t_1 is a beak or a cusp.

In summary, (s_0, t_0) is not regular or there exist $s_1 < t_1$ in (s_0, t_0) such that t_1 is s_1-negative, $\Gamma_1(s_1) \cap \Gamma(s_1, t_1) = \varnothing$ and $\Gamma_1(s_1)$ is a multiple tangent supporting $\Gamma(s_0, t_0)$ at least twice.

By reiteration of the preceding arguments and 1.1, there exist $\bar{s} < \bar{t}$ in $(s_1, t_1) \subset (s_0, t_0)$ such that $\bar{t}$ is $\bar{s}$-negative, $\Gamma_1(\bar{s}) \cap \Gamma(\bar{s}, \bar{t}) = \varnothing$ and $\Gamma(\bar{s}, \bar{t})$ possesses no multiple tangents. Then either the lemma follows by 3.9 or there exist $s' < t'$ in $(\bar{s}, \bar{t})$ such that $\Gamma_1(s')$ supports Γ at t'. If s' is regular, then $\Gamma_1(s') \neq \Gamma_1(t')$ and t' is singular.

Let $\Gamma(t_1, t_2)$ be a subarc of order two. By 1.7, $|\Gamma_1(t) \cap \Gamma[t_1, t_2]| = 1$ for $t \in \Gamma(t_1, t_2)$ and thus there is an $L \subset P^2$ such that $L \cap \Gamma[t_1, t_2] = 0$ and $\Gamma[t_1, t_2]$ is simple. Lety $H(\Gamma[t_1, t_2])$ be the convex hull of $\Gamma[t_1, t_2]$ (in $P^2 \backslash L$). Then $\mathcal{R} = H(\Gamma[t_1, t_2])$ is the closed region in P^2 bounded by $\Gamma[t_1, t_2]$ and $\langle \Gamma(t_1), \Gamma(t_2) \rangle$ such that $|\Gamma_1(t) \cap \mathcal{R}| = 1$ for $t \in (t_1, t_2)$. We note that any line through any point of int($\mathcal{R}$) meets $\Gamma(t_1, t_2)$ at least once.

The following result should be well-known.

11. LEMMA. *Let $\Gamma(t_1, t_2)$ be a regular subarc such that $\Gamma_1(t_1) \cap \Gamma(t_1, t_2]$* $= \{\Gamma(t_2)\}$. *Then $\Gamma(t_1, t_2)$ is of order two.*

Proof. $\Gamma[t_1, t_2]$ is clearly simple. As (t_1, t_2) is regular, $\Gamma_1(t_1) \cap \Gamma(t_1, t_2) = \varnothing$ and 3.10 imply that t_2 is not t_1-negative. Then t_1 elementary, 1.7, 3.5 and 1.3 readily yield that there exists a maximal $U^+(t_1) = (t_1, s_1)$ in (t_1, t_2) such that $|\Gamma_1(s) \cap \Gamma[t_1, t_2]| = 1$ for $s \in (t_1, s_1)$. If $s_1 = t_2$, the lemma follows by 1.6.

Suppose $s_1 \neq t_2$. Then $\Gamma(t) \in \Gamma_1(s_1)$ for some $t \neq s_1$ in $[t_1, t_2]$. If $t \in (t_1, t_2)$, then clearly $\Gamma_1(s_1)$ supports Γ at t and t is both s_1-positive and s_1-negative. By the maximality of (t_1, s_1), $t \notin (t_1, s_1)$ and hence we may assume that $\Gamma_1(s_1) \cap \Gamma(s_1, t) = \varnothing$. As t is s_1-negative, 3.10 implies that $(s_1, t) \subset (t_1, t_2)$ is not regular, a contradiction. Hence $t \in \{t_1, t_2\}$ and $\Gamma_1(s_1) \cap \Gamma(t_1, t_2) = \{\Gamma(s_1)\}$.

Suppose $\Gamma(t_2) \in \Gamma_1(s_1)$ and choose $U^-(t_2) \subset (s_1, t_1)$. Then $\Gamma_1(s) \cap \Gamma(U^-(t_2)) = \varnothing$ for $s \in (t_1, s_1)$ yields that t_2 is s_1-negative [s_1-negative and s_1-positive] if $\Gamma_1(s_1)$ cuts [supports] Γ at s_2. Hence (s_1, t_2) is not regular by 3.10, a contradiction.

Suppose $\Gamma_1(s_1) \cap \Gamma[t_1, t_2] = \{\Gamma(t_1), \Gamma(s_1)\}$. Then $\Gamma(t_1, s_1)$ is of order to by 1.6. Let $\mathcal{R} = H(\Gamma[t_1, s_1])$ (in $P^2 \backslash L$, $L \cap \Gamma[t_1, s_1] = \varnothing$). Since $\Gamma_1(t_1) \cap \Gamma(t_1, s_1) = \varnothing$, $\Gamma(t_2) \in \Gamma_1(t_1)$ yields that $\Gamma(t_2) \notin \mathcal{R}$. But s_1 ordinary implies there is an $U(s_1) \subset (t_1, t_2)$ of order two. Then $\Gamma(U^-(s_1)) \subset \mathrm{bd}(\mathcal{R})$ and 1.7 imply that $\Gamma_1(s) \cap \Gamma(U^+(s_1)) = \varnothing$ for $s \in U^-(s_1)$ and $\Gamma(U^+(s_1)) \subset \mathrm{int}(\mathcal{R})$. Thus $\mathrm{bd}(\mathcal{R}) \cap \Gamma(s_1, t_2) \neq \varnothing$. Then $\mathrm{bd}(\mathcal{R}) \subset \Gamma[t_1, s_1] \cup \Gamma_1(s_1)$ and $\Gamma[t_1, t_2]$ simple imply that $\Gamma_1(s_1) \cap \Gamma(s_1, t_2) \neq \varnothing$, a contradiction.

12. **Lemma.** *Let* $\Gamma(t_1, t_2)$ *be simple. Let* $s_1 \in (t_1, t_2)$ *such that* $\Gamma_1(s_1) \cap \Gamma[t_1, t_2] = \{\Gamma(t_1), \Gamma(s_1), \Gamma(t_2)\}$. *Then* (t_1, t_2) *is not regular.*

Proof. We assume that $(t_1, s_1]$ is regular. Then $\Gamma_1(s_1) \cap \Gamma(t_1, s_1) = \varnothing$ yields that $\Gamma[t_1, s_1]$ is simple and by 3.11, $\Gamma(t_1, s_1)$ is of order two. Let $\mathcal{R} = H(\Gamma[t_1, s_1])$. Arguing as in the proof of 3.11, we observe that $\Gamma_1(s_1) \cap \Gamma(s_1, t_2) = \{\Gamma(t_2)\}$ implies that (cf. Figure 2) $\Gamma[s_1, t_2] \subset \mathcal{R}$, $\Gamma[s_1, t_2]$ is simple and t_2 is s_1-negative. Hence by 3.10, $(s_1, t_2) \subset (t_1, t_2)$ is not regular.
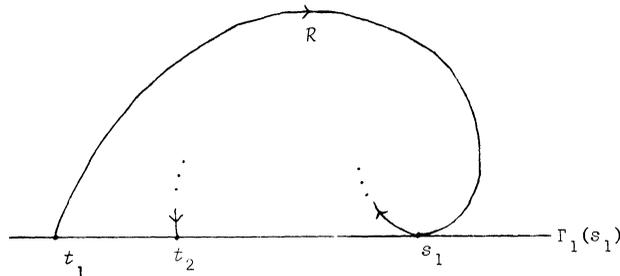

FIGURE 2

Before proceeding with the proof of Theorem 3.1, we still need to investigate the regular simple subarcs of Γ.

13. LEMMA. *Let* $\Gamma(t_1, t_2)$ *be a regular, simple subarc such that* $\langle \Gamma(t_1), \Gamma(t_2) \rangle \cap \Gamma(t_1, t_2) = \varnothing$. *Then* $\Gamma(t_1, t_2)$ *is of order two.*

Proof. Suppose there is an $s \in (t_1, t_2)$ such that $\Gamma(t_2) \in \Gamma_1(s)$. Then there is an $s' \in (s, t_2]$ such that $\Gamma_1(s) \cap \Gamma(s, s') = \{\Gamma(s')\}$. Then $\Gamma(s, s')$ is of order two by 3.11 and $\Gamma_1(s) \cap \Gamma[t_1, s] = \varnothing$ by 3.12. Hence $\Gamma(t_1) \in \text{int}(H(\Gamma[s, s']))$ from the proof of 3.11. But then $\langle \Gamma(t_1), \Gamma(t_2) \rangle \cap \Gamma(s, s') \neq \varnothing$, a contradiction. The lemma now follows by 1.6.

14. LEMMA. *Let* $\Gamma(t_1, t_2)$ *be regular and simple. Then*
(1) *there is a* $t \in (t_1, t_2)$ *such that* $|\Gamma_1(t) \cap \Gamma[t_1, t_2]| = 1$ *and*
(2) *there is an* L *such that* $L \cap \Gamma[t_1, t_2] = \varnothing$.

Proof. Clearly (1) implies (2).

If $\Gamma_1(t_1) \cap \Gamma(t_1, t_2] = \varnothing$ then t_1 elementary, 1.7, 1.3, and 1.5 imply that there is a $t \in (t_1, t_2)$, arbitrarily close to t_1, such that $|\Gamma_1(t) \cap \Gamma[t_1, t_2]| = 1$.

Let $\Gamma_1(t_1) \cap \Gamma(t_1, t_2] \neq \varnothing$. Then there is an $s_1 \in (t_1, t_2]$ such that $\Gamma_1(t_1) \cap \Gamma(t_1, s_1] = \{\Gamma(s_1)\}$. By 3.11, $\Gamma(t_1, s_1)$ is of order two, $\Gamma[t_1, s_1]$ is simple and (1) follows if $s_1 = t_2$. Let $s_1 \neq t_2$. If $\Gamma_1(t_1)$ supports Γ at s_1 then $\Gamma_1(t_1) = \Gamma_1(s_1)$ and by 3.12, $\Gamma_1(s_1) \cap \Gamma(s_1, t_2] = \varnothing$. Hence (cf. the proofs of 3.11 and 3.12) $\Gamma(s_1, t_2] \subset H(\Gamma[t_1, s_1])$ and any $t \in (t_1, s_1)$ satisfies (1).

Let $\Gamma_1(t_1)$ cut Γ at s_1. Then s_1 is t_1-positive by 3.10. Since $\Gamma_1(t_1) \cap \Gamma(t_1, s_1) = \varnothing$, 3.5 and 3.12 imply that there exist

(a) $t \in (t_1, t_2)$, arbitrarily close to t_1, such that $\Gamma_1(t)$ cuts Γ at a point $s \in (s_1, t_2) \subset (t_1, t_2)$ where $\Gamma_1(t) \cap \Gamma(t, s) = \varnothing$, s is t-positive and $\Gamma_1(t) \cap \Gamma[t_1, t) = \varnothing$.

The preceding argument is symmetric in t_1 and t_2 with the orientation of T reversed; that is, we may assume that there is an $s_2 \in (t_1, t_2)$ such that $\Gamma_1(t_2)$ cuts Γ at s_2, $\Gamma_1(t_2) \cap \Gamma(s_2, t_2) = \varnothing$ and t_2 is s_2-negative. Then there eixst

(b) $t \in (t_1, t_2)$, arbitrarily close to t_2, such that $\Gamma_1(t)$ cuts Γ at a point $s \in (t_1, s_2) \subset (t_1, t_2)$, $\Gamma_1(t) \cap \Gamma(s, t) = \varnothing$, s is t-negative and $\Gamma_1(t) \cap \Gamma(t, t_2] = \varnothing$.

Since either $\Gamma_1(t) \cap \Gamma[t_1, t) = \varnothing$ or $\Gamma_1(t) \cap \Gamma(t, t_2] = \varnothing$ for any $t \in (t_1, t_2)$ by 3.12, (1) readily follows from (a), (b) and 1.3.

15. **Lemma.** *Let $\Gamma(t_1, t_2)$ be regular and simple. Then there exist $s_1 < s_2$ in $[t_1, t_2]$ such that $\Gamma(s_1, s_2)$ is of order two and $\Gamma[t_1, t_2] \subset H(\Gamma[s_1, s_2])$.*

Proof. By 3.14, there is a $\bar{t} \in (t_1, t_2)$ such that $|\Gamma_1(\bar{t}) \cap \Gamma[t_1, t_2]| = 1$. Since $\bar{t}$ is ordinary, 1.7 and 1.3 imply that there is a maximal $U(\bar{t}) = (s_1, s_2)$ of order two in (t_1, t_2) such that $|\Gamma_1(t) \cap \Gamma[t_1, t_2]| = 1$ for $t \in (s_1, s_2)$.

Let $\Re = H(\Gamma[s_1, s_2])$. If $s_2 \neq t_2$, then $\Gamma_1(s_2)$ meets either $\Gamma(s_2, t_2]$ or $\Gamma[t_1, s_1]$ by 3.12. From the proof of 3.11, we obtain that $\Gamma_1(s_2) \cap \Gamma(s_2, t_2] = \varnothing$ and hence there is a $t \in [t_1, s_1]$ such that $\Gamma_1(s_2) \cap \Gamma[t, s_2) = \{\Gamma(t)\}$. If $t = t_1$, then clearly $t_1 = s_1$ and $\Gamma[t_1, t_2] \subset H(\Gamma[t_1, s_2]) = \Re$. If $t \neq t_1$ then $\Gamma_1(s_2) = \Gamma_1(t)$ and by 3.12, $\Gamma_1(s_2) \cap \Gamma[t_1, t) = \varnothing$. Since $\Gamma_1(s_2)$ supports Γ at t, we again obtain from the proof of 3.11 that $t = s_1$ and $\Gamma[t_1, t_2] \subset \Re$.

The preceding argument is symmetric in s_1 and s_2.

We call $\Gamma[s_1, s_2]$ in 3.15, the *convex cover* of $\Gamma[t_1, t_2]$. We note that Lemmas 3.13–3.15 are known in some form or another. The proofs have been included for the sake of completeness and because the approach is new.

Since Γ is of even order with positive index, we observe:

16. **Lemma.** *If L supports Γ at a simple point, then L meets Γ at three distinct points of T.*

17. **Lemma.** *Let $\Gamma(t_1, t_2)$ be simple and regular. Let $L \cap \Gamma[t_2, t_1] = \varnothing$. Then L meets, and cuts, Γ at exactly two points, at least one of which lies in the convex cover of $\Gamma[t_1, t_2]$.*

Proof. Let $\Gamma[s_1, s_2]$ be the convex cover of $\Gamma[t_1, t_2]$, $\Re = H(\Gamma[s_1, s_2])$. Then $\operatorname{ind}(\Gamma) > 0$ and $L \cap \Gamma[t_2, t_1] = \varnothing$ imply that $L \cap \Re \neq \varnothing$. Since $\Gamma[t_1, t_2] \subset \Re$ is simple, $|L \cap \Gamma(t_1, t_2)| \geq 2$ by 3.16 and L separates $\Re$ into two closed regions $\Re_1$ and $\Re_2$ such that $\Re_1 \cup \Re_2 = \Re$ and $\Re_1 \cap \Re_2 = L \cap \Re$. Since Γ is of even order, L cuts $\Gamma(t_1, t_2)$ at an even number of points and thus $\{\Gamma(t_1), \Gamma(t_2)\} \subset \Re_1$ say.

We first observe that if L meets $\Gamma(t_1, t_2)$ in less than three points, then L cuts $\Gamma(t_1, t_2)$ exactly twice by 3.16. If $|L \cap \Gamma(t_1, t_2)| \geq 3$, then L supports $\Gamma(t_1, t_2)$ at most once by 3.12.

Suppose that $|L \cap \Gamma(t_1, t_2)| \geq 3$. Then $\Re_2$ convex implies that $\operatorname{int}(\Re_2) \cap \Gamma(t_1, t_2) \neq \varnothing$. Let $\Gamma(s) \in (\Re_2 \cap \Gamma(s_1, s_2)) \setminus L$. Then $\Gamma_1(s) \cap \operatorname{int}(\Re) = \varnothing$ and $\Gamma_1(s)$ meets $L \setminus \Re$ at a point p. We note that $\Re_1$ and $\Re_2$ are contained in the different closed half-planes bounded by $\Gamma_1(s)$ and L.

Hence $\Gamma_1(s) \cap \text{int}(\mathfrak{R}_2) = \varnothing$ and $\Gamma(t_1, t_2) \cap \text{int}(\mathfrak{R}_2) \neq \varnothing$ imply that there is a line N such that $p \in N$, $N \cap \mathfrak{R}_1 = \varnothing$ and either (a) N supports Γ at some $t \in (t_1, t_2) \setminus (s_1, s_2)$ or (b) $\Gamma(t_1)$ or $\Gamma(t_2)$ lie on L. Suppose (a) and $t \in (t_1, s_1)$ say. Then $\Gamma(t) \in \text{int}(\mathfrak{R})$ and $(s_1, s_2) \subset (t_1, s_2)$ imply there is a $t' \in (t, s_2)$ such that $\Gamma(t') \in N = \Gamma_1(t)$ and $\Gamma_1(t) \cap \Gamma(t, t') = \varnothing$. By 3.11, $\Gamma(t, t')$ is of order two.

Let $\mathfrak{R}' = H(\Gamma[t, t'])$. We note that $\Gamma_1(t) \cap \Gamma[t_1, t) = \varnothing$ by 3.12 and thus $\Gamma(t_1) \in \text{int}(\mathfrak{R}')$ (cf. the proof of 3.11). Since L separates $\Gamma(t)$ and $\Gamma(t_1)$ in $\mathfrak{R}$, L also separates $\Gamma(t)$ and $\Gamma(t_1)$ in $\mathfrak{R}'$. Let $\mathfrak{R}'_1$ and $\mathfrak{R}'_2$ be the closed regions of $\mathfrak{R}'$ determined by L, $\Gamma(t_1) \in \mathfrak{R}'_1$ and $\Gamma(t) \in \mathfrak{R}'_2$ say. The $\mathfrak{R}' \cap \mathfrak{R}_i = \mathfrak{R}'_i$, $i = 1, 2$. Since $L \cap N = \{p\} \not\subset \mathfrak{R}$, $L \cap N \cap \mathfrak{R}' = \varnothing$ and $N = \langle \Gamma(t), \Gamma(t') \rangle$ yield that there exist $r < r'$ in (t, t') such that $\mathfrak{R}'_1$ is bounded by $\Gamma[r, r']$ and L. Finally we observe that 3.14 and $\mathfrak{R} = H(\Gamma[s_1, s_2])$ imply that there is an L' such that $L' \cap \mathfrak{R} = \varnothing$.

Then $\text{ind}(\Gamma) > 0$ and $\mathfrak{R}'_1 \subset \mathfrak{R}' \subset \mathfrak{R}$ imply that $\Gamma[t_2, t_1] \not\subset \mathfrak{R}'_1$. Hence $\Gamma(t_1) \in \mathfrak{R}'_1$ and $L \cap \Gamma[t_2, t_1] = \varnothing$ yield that there exist $r^* \in (r, r') \subset (t_1, t_2)$ and $t^* \in (t_2, t_1)$ such that $\Gamma(r^*) = \Gamma(t^*)$ and $\Gamma[t^*, t_1] \subset \mathfrak{R}'_1$. Then $\Gamma[t^*, r^*] = \Gamma[t^*, t_1] \cup \Gamma[t_1, r^*] \subset \mathfrak{R}$ and $L' \cap \mathfrak{R} = \varnothing$ imply that γ is not almost-simple. Similarly $N \cap ((t_1, s_1) \cup (s_2, t_2)) = \varnothing$, (b) and $\text{ind}(\Gamma) > 0$ imply that Γ is not almost-simple, a contradiction.

COROLLARY. *Let* $\Gamma(t_1, t_2]$ *be simple and regular. Let* $L = \lim L_\lambda$ *where* $L_\lambda \cap \Gamma[t_2, t_1] = \varnothing$ *for each* L_λ. *Then* L *cuts* Γ *in at most two points.*

Proof of Theorem 3.1. (i). If Γ is simple, then $\text{ind}(\Gamma) > 0$ implies that $\Gamma(T \setminus \{t\})$ contains a singular point for any $t \in T$ by 3.14. Hence $n(\Gamma) \geq 2$.

If there exist $r_1 \neq r_2$ in T such that $\Gamma(r_1) = \Gamma(r_2)$, then 1.3 implies that there exist $t_1 < t_2$ in $[r_1, r_2]$ and $s_2 < s_1$ in $[r_2, r_1]$ such that $\Gamma(t_1) = \Gamma(t_2)$, $\Gamma(s_2) = \Gamma(s_1)$ and both $\Gamma(t_1, t_2)$ and $\Gamma(s_2, s_1)$ are simple. As Γ is almost-simple, every line in P^2 meets both $\Gamma[t_1, t_2]$ and $\Gamma[s_2, s_1]$. Hence neither (t_1, t_2) nor (s_2, s_1) are regular by 3.14 and $n(\Gamma) \geq 2$.

Proof of Theorem 3.1 (ii). By 3.1 (i), Γ contains at least two singular points t_1 and t_2. If t_1 and t_2 are both cusps, then $n_1 + 2n_2 + n_3 \geq 2n_2 \geq 4$. For the curve represented in Figure 3, $n_1 = n_3 = 0$ and $n_2 = 2$.

If say only t_1 is a cusp, then Γ contains a beak or an inflection point $t_3 \neq t_2$ by 1.2. Therefore $n_1 + 2n_2 + n_3 \geq 2 + n_1 + n_3 \geq 4$. For the curve represented in Figure 4, $n_1 = n_2 = n_3 = 1$.
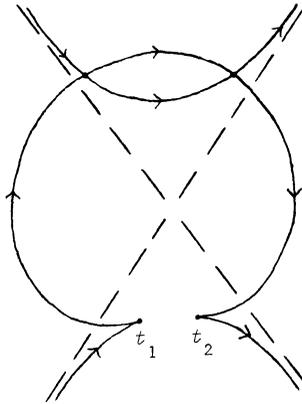
FIGURE 3

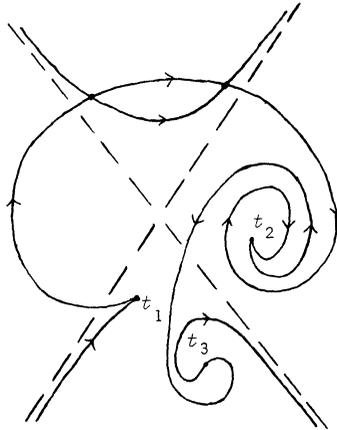

FIGURE 4

If $n_2 = 0$ and $n(\Gamma) = n_1 + n_3 \neq 2$, then $n(\Gamma) \geq 4$ by 1.2. Hence we show that $n(\Gamma) = 2$ implies that $n_2 > 0$. Let (t_1, t_2) and (t_2, t_1) be regular. Then $\Gamma[t_1, t_2]$ and $\Gamma[t_1, t_2]$ are simple from the proof of 3.1 (i). By 3.15, there exist points $t_1 \leq s_1 < s_2 \leq t_2 \leq u_2 < u_1 \leq t_1$ such that $\Gamma[t_1, t_2] \subset \mathfrak{R}_1$ $= H(\Gamma[s_1, s_2])$ and $\Gamma[t_2, t_1] \subset \mathfrak{R}_2 = H(\Gamma[u_2, u_1])$. Let $L_1 = \langle \Gamma(s_1), \Gamma(s_2) \rangle$ and $L_2 = \langle \Gamma(u_2), \Gamma(u_1) \rangle$. By 3.17 Corollary, L_2 and L_2 each cut Γ at most twice.

We observe that $\Gamma[t_1, t_2] \subset \mathfrak{R}_1$ and $\mathrm{ind}(\Gamma) > 0$ imply that $\Gamma[t_2, t_1] \not\subset \mathfrak{R}_1$. Hence there exist $v_2 < v_1$ in $[t_2, t_1]$ such that $\Gamma[v_1, v_2]$ is the largest subarc of Γ contained in $\mathfrak{R}_1$. As Γ is almost-simple, $\Gamma[v_1, v_2]$ is simple.

Case 1. $s_1 = t_1$ and L_1 supports Γ at s_1.

Then t_1 is a cusp or a beak and we may assume that t_1 is a beak.

Suppose that L_1 supports Γ at $s_2 \neq t_2$. Then clearly $\Gamma(t_2) \in \mathrm{int}(\mathfrak{R}_1)$, $v_2 \neq t_2$, L_1 cuts Γ at v_2 ($\Gamma[v_1, v_2]$ is simple) and $L_1 \neq \Gamma_1(v_2)$. Since L_1

does not cut Γ at any point of $[t_1, v_2)$ and Γ is of even order, L_1 cuts Γ at a point $t' \in (v_2, t_1)$. We note that L_1 cuts Γ at exactly v_2 and t'. Hence $(v_2, t_1) \subset (t_2, t_1)$ and 3.12 imply that L_1 meets Γ at only the points t_1, s_2, v_2 and t'. Since t_1 is a beak, $L_1 \cap \Gamma[t', s_2] = \{\Gamma(t'), \Gamma(t_1), \Gamma(s_2)\}$ readily yields that either $\Gamma[t_1, s_2] \subset H(\Gamma[t', t_2]) \subset \mathfrak{R}_2$ or $\Gamma[t', t_1] \subset H(\Gamma[t_1, s_2]) = \mathfrak{R}_1$. The former yields that $\mathfrak{R}_1 \subset \mathfrak{R}_2$ and $\mathrm{ind}(\Gamma) = 0$ and therefore $\Gamma[t', t_1] \subset \mathfrak{R}_1$ and in particular $\Gamma[t', v_2] \subset \mathfrak{R}_1$. Since L_1 cuts Γ at t', $t' = v_1$ and $L_1 \neq \Gamma_1(v_1)$. Let $\mathfrak{R}_{11}$ and $\mathfrak{R}_{12}$ be the closed regions of $\mathfrak{R}_1$ determined by $\Gamma_1(v_1)$; that is, $\mathfrak{R}_{11} \cup \mathfrak{R}_{12} = \mathfrak{R}_1$ and $\mathfrak{R}_{11} \cap \mathfrak{R}_{12} = \Gamma_1(v_1)$ $\cap \mathfrak{R}_1$. Let $\Gamma(t_1) \in \mathfrak{R}_{11}$. Then $\Gamma[v_1, v_2]$ simple implies that $\{\Gamma(s_2), \Gamma(v_2)\}$ $\subset \mathfrak{R}_{12}$. Since $\Gamma(t_2, t_1)$ is regular and simple, 3.13 and $L_1 \cap \Gamma(v_2, v_1) = \varnothing = L_1 \cap \Gamma(v_1, t_1)$ imply that both $\Gamma(v_2, v_1)$ and $\Gamma(v_1, t_1)$ are of order two. Hence $\Gamma_1(v_1) \neq L_1$ yields that $\Gamma_1(v_1)$ does not cut Γ at any point of $[v_2, t_1]$. But $\Gamma(t_1) \in \mathfrak{R}_{11}$, $\Gamma(v_2) \in \mathfrak{R}_{12}$ and $\Gamma(t_1, v_2) \subset \mathfrak{R}_1$ simple imply that $\Gamma_1(v_1)$ cuts $\Gamma(t_1, v_2)$, and hence Γ, at an odd number of points, a contradiction.

Suppose that L_1 supports Γ at $s_2 = t_2$. We may again assume that t_2 is a beak. We observe that $L_1 \cap \Gamma(t_1, t_2) = \varnothing$ and by 3.12, L_1 does not support Γ at any point of (t_2, t_1). By 3.16 and 3.17 Corollary, L_1 cuts Γ at exactly two points, say $t'' < t'$, in (t_2, t_1). Since t_1 and t_2 are beaks, we argue as in the preceding and obtain that $t' = v_1$, $t'' = v_2$, $\Gamma(v_2, v_1)$ and $\Gamma(v_1, t_1)$ are of order two, $\Gamma_1(v_1)$ does not cut Γ at any point of $[v_2, v_1]$ and $\Gamma_1(v_1)$ separates $\Gamma(t_1)$ and $\Gamma(v_2)$ in $\mathfrak{R}_1$. Hence $\Gamma_1(v_1)$ cuts Γ at an odd number of points, a contradiction.

Suppose L_1 cuts Γ at s_2. Then $s_2 = t_2 = v_2$ and L_1 cuts Γ at exactly one other point $t' \in (v_2, t_1)$. As in the preceding, $v_1 = t'$ and $\Gamma_1(v_1)$ cuts Γ at an odd number of points, a contradiction. Thus t_1 must be a cusp.

Case 2. $s_1 = t_1$ and L_1 cuts Γ at s_1.

Then $t_1 = v_1$ and L_1 cuts Γ at exactly one other point $t'' \in [s_2, t_1)$. We show that $L_1 = \Gamma_1(t_1)$.

If $t'' = s_2$ then $s_2 = t_2 = v_2$, $\Gamma(t_1, t_2)$ is of order two and $\Gamma_1(t_1) \cap \Gamma(t_1, t_2) = \varnothing$. By 3.12, L_1 does not support Γ at any point of (t_2, t_1) and hence $L_1 \cap \Gamma(t_2, t_1) = \varnothing$. Then $\Gamma(t_2, t_1)$ is of order two by 3.13 and thus $\Gamma_1(t_1) \cap \Gamma(t_2, t_1) = \varnothing$. If $\Gamma(t_2) \notin \Gamma_1(t_1)$, then $\Gamma_1(t_1)$ meets Γ at exactly the point $\Gamma(t_1)$, a contradiction. Hence $\Gamma_1(t_1) = \langle \Gamma(t_1), \Gamma(t_2) \rangle = L_1$.

If $t'' \neq s_2$, then L_1 supports Γ at s_2 and either $s_2 = t_2$ or $s_2 \neq t_2$ and $t'' = v_2$. As in (1), $s_2 = t_2$ and L_1 supports Γ at s_2 leads to a contradiction and thus $s_2 \neq t_2$ and $t'' = v_2$. Then L_1 does not support Γ at any point of (v_2, v_1) by 3.12 and hence $L_1 \cap \Gamma(v_2, t_1) = \varnothing$ and $\Gamma(v_2, t_1)$ is of order

two. If $\Gamma(s_2) \notin \Gamma_1(t_1)$, then $|\Gamma_1(t_1) \cap \mathcal{R}_1| = 1$ and in particular $|\Gamma_1(t_1) \cap (\mathcal{R}_1 \cup \Gamma(v_2, t_1))| = 1$. This is impossible since $\Gamma = \Gamma[t_1, v_2] \cup \Gamma(v_2, t_1)$ $\subset \mathcal{R}_1 \cup \Gamma(v_2, t_1)$ and therefore $L_1 = \langle \Gamma(t_1), \Gamma(s_2) \rangle = \Gamma_1(t_1)$.

Since $\Gamma_1(t_1)$ cuts Γ at t_1, t_1 is a cusp or an inflection point. Since $\Gamma_1(t_1)$ cuts Γ at $t'' = v_2$, there exist $U(t_1)$ and $U(v_2)$ such that $U(t_1) \to U(v_2)$ by 3.3. Since $\Gamma[t_1, v_2] \subset \mathcal{R}_1$, $\Gamma(U^+(t_1)) \subset \mathrm{bd}(\mathcal{R}_1)$ and $\Gamma(U^-(v_2)) \subset \mathcal{R}_1$. Then no tangent of $U^+(t_1)$ meets $\Gamma(U^-(v_2))$ and thus $U^+(t_1) \leftrightarrow U^+(v_2)$ by 3.4. Since $\Gamma(v_2, t_1)$ is of order two, $\Gamma(U^-(t_1)) \cup \Gamma(U^+(v_2)) \subset \Gamma(v_2, t_1)$ implies that no tangent of $U^-(t_1)$ meets $\Gamma(U^+(v_2))$ and hence $U^-(t_1) \leftrightarrow U^-(v_2)$. Therefore $U(t_1) \leftrightarrow U(v_2)$ and by 3.4 Corollary, t_1 is a cusp.

Case 3. $s_1 \neq t_1$.

We note that (1) and (2) are symmetric in t_1 and t_2, hence we may assume that $s_2 \neq t_2$. Then $L_1 = \Gamma_1(s_i)$, $s_i \neq v_i$, $\Gamma(t_i) \in \mathrm{int}(\mathcal{R}_1)$, L_1 supports [cuts] Γ at $s_i[v_i]$ and $L_1 \neq \Gamma_1(v_i)$, $i = 1, 2$. Since L_1 cuts Γ at only v_1 and v_2, it readily follows from 3.10 and 3.12 that L_1 meets Γ at exactly s_1, s_2, v_1 and v_2. By 3.13, $\Gamma(v_2, v_1)$ is of order two and thus $\{\Gamma(v_1), \Gamma(v_2)\} \subset \mathcal{R}_1$ implies that either both or neither of $\Gamma(s_1)$ and $\Gamma(s_2)$ are contained in $\mathcal{R}_2' = H(\Gamma[v_2, v_1])$.

As $[\Gamma(v_1), \Gamma(v_2)] \subset \mathcal{R}_2$, $\mathcal{R}_2$ convex implies that $\Gamma(v_1)$ or $\Gamma(v_2)$, say $\Gamma(v_2)$, lies in the boundary of $\mathcal{R}_2$. Let $\mathcal{R}_{11}$ and $\mathcal{R}_{12}$ be the closed regions of $\mathcal{R}_1$ determined by $\Gamma_1(v_2)$; $\Gamma(s_1) \in \mathcal{R}_{11}$. Let L_1 be oriented so that $s_2 < s_1$ in $L_1 \cap \mathcal{R}_1$. Then $\Gamma[v_1, v_2] \subset \mathcal{R}_1$ simple implies that $s_2 < v_2 < v_1 < s_1$ in $L_1 \cap \mathcal{R}_1$ and in particular, $\Gamma(s_2) \in \mathcal{R}_{12}$ and $\Gamma(v_1) \in \mathcal{R}_{11}$. Since $\Gamma(v_2) \in \mathrm{bd}(\mathcal{R}_2)$, $\Gamma_1(v_2) = \lim L_\lambda$ where $L_\lambda \cap \mathcal{R}_2 = \varnothing$ for each L_λ. In the proof of 3.17, we showed that $L_\lambda \cap \Gamma[t_2, t_1] = \varnothing$ and $\Gamma[t_1, t_2] \subset \mathcal{R}_1$ imply that L_λ does not separate $\Gamma(t_1)$ and $\Gamma(t_2)$ in $\mathcal{R}_1$. Similarly $L_\lambda \cap \Gamma[v_2, v_1] = \varnothing$ and $\Gamma[v_1, v_2] \subset \mathcal{R}_1$ imply that L_λ does not separate $\Gamma(v_1)$ and $\Gamma(v_2)$, and hence separates $\Gamma(v_2)$ and $\Gamma(s_2)$, in $\mathcal{R}_1$. Then $\Gamma(v_1) \subset \mathcal{R}_{11}$ implies that $l_1 \cap \mathcal{R}_2' \subset \mathcal{R}_{11}$ and $\{\Gamma(s_1), \Gamma(s_2)\} \cap \mathcal{R}_1' = \varnothing$. It is immediate that $\Gamma(t_2) \in \mathcal{R}_{12}$.

Let $r \in (s_2, v_2)$. Since $\Gamma(s_2, v_2) \subset \mathrm{int}(\mathcal{R}_1)$, $1 \leq |\Gamma_1(r) \cap \Gamma(s_1, s_2)| \leq 2$ and $\Gamma_1(r)$ cuts $\Gamma(s_1, s_2)$ at each point of contact. Let $\Gamma_1(r)$ cut Γ at r' and r'' in (s_1, s_2). Since (s_2, t_2) [resp. (t_2, s_2)] is regular, it readily follows from 3.4 Corollary that either r' and r'' are r-positive for all $r \in (s_2, t_2)[(t_2, s_2)]$ or r' and r'' are r-negative for all $r \in (s_2, t_2)[(t_2, s_2)]$. Since $\Gamma_1(s_2)$ supports Γ at s_1, it is immediate that s_1 is both s_2-positive and s_2-negative and hence r' and r'' are r-positive for all $r \in (s_2, t_2)$ by 3.5.

We note that $\langle \Gamma(v_1), \Gamma(r) \rangle$ tends to L_1 and $\Gamma_1(r)$ tends to $\Gamma_1(v_2)$ as $r \in (t_2, v_2)$ tends to v_2. Let $r \in (t_2, v_2)$ be arbitrarily close to v_2. Since L_1 cuts Γ at v_1 and v_2 and $\Gamma_1(v_2)$ cuts Γ at exactly one point of (s_1, s_2), we

may assume by 3.3 that $\langle \Gamma(v_1), \Gamma(r) \rangle \cap \Gamma(v_1, r) = \varnothing$ and $|\Gamma_1(r) \cap \Gamma(s_1, s_2)| = 1$. Then $r' = r''$ and by 3.13, $\Gamma(r, v_2)$ is of order two with $\mathfrak{R}_2' \subset H(\Gamma[r, v_1])$. Let $\Gamma_1(v_2)$ cut Γ at the point $r_2' \in (s_1, s_2)$ and $\Gamma_1(r)$ meet L_1 at the point p. Since $L_1 \cap \mathfrak{R}_2' \subset \mathfrak{R}_{11}$ and $\Gamma(s_2) \notin \mathfrak{R}_{11}$, 1.3 and $\mathfrak{R}_2' \subset H(\Gamma[r, v_1])$ clearly imply that $p \notin \mathfrak{R}_{11}$; that is, $s_2 < p < v_2$ in $L_1 \cap \mathfrak{R}_1$. Since $\Gamma[s_1, s_2] \cup (L_1 \cap \mathfrak{R}_1)$ is convex, $s_2 < p < v_2$ in $L_1 \cap \mathfrak{R}_1$ readily yields that $s_1 < r' < r_2'$ in (s_1, s_2). Since r' tends to r_2' in (s_1, s_2) as r tends to v_2 in (t_2, v_2), it follows that r' and r'' are r-positive for all $r \in (t_2, v_2)$.

Let $\Gamma_1(t_2)$ cut Γ at $s \in (s_1, s_2)$. By 3.3, there exist $U(t_2) \subset (s_2, v_2)$ and $U(s) \subset (s_1, s_2)$ such that $U(t_2) \to U(s_2)$. As $U^-(t_2) \subset (s_2, t_2)$, $U^+(t_2) \subset (t_2, v_2)$, 1.3 and the preceding imply that $U^-(t_2) \leftrightarrow U^-(s_2)$ and $U^+(t_2) \leftrightarrow U^+(s_2)$. Hence t_2 is a cusp by 3.4 Corollary.
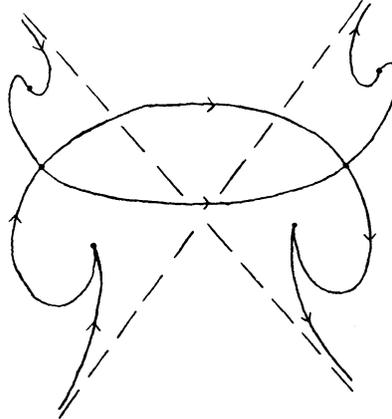

FIGURE 5

Thus $n_2 = 0$ implies that $n(\Gamma) \geq 4$. In Figure 5, we represent a curve with $n_1 = n_3 = 2$ and $n_2 = 0$.

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