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ON "TAUBERIAN THEOREMS VIA BLOCK DOMINATED MATRICES"

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Tauberian theorems for gap sequences are given in which the Tauberian condition is determined by the blocks of consecutive terms that dominate the rows of a regular summability matrix.

1. Introduction. Let A be a summability matrix and denote by $(Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k$ the nth term of the sequence Ax. J. A. Fridy [3] has defined the notion of A being $\{B_n\}$-dominated. For each $n \geq 0$ let $B_n = \{k\colon \mu(n) < k \leq \nu(n)\}$, where μ and ν are integer sequences with $\mu(n) \geq -1$ and $\mu(n+1) < \nu(n)$. (We depart here from Fridy's requirement that μ and ν be nonnegative integer sequences in order to allow $0 \in B_n$.) Let $L_n = \nu(n) - \mu(n)$. The complex valued matrix A is said to be $\{B_n\}$-dominated if

$$\liminf_n \left\{ \left| \sum_{k \in B_n} a_{nk} \right| - \sum_{k \notin B_n} |a_{nk}| \right\} > 0. \tag{1}$$

It is easy to see that if A is regular, then it admits a block sequence $\{B_n\}$ that satisfies (1).

Let x be a complex valued sequence and $\Delta x = x_n - x_{n+1}$. We say x satisfies the gap condition determined by the increasing sequence κ of nonnegative integers if $(\Delta x)_k = 0$ when $k \neq \kappa(m)$, $m = 0, 1, \ldots$. Gap sequences are also called stretchings since a gap sequence x may be thought of as the result of the finite repetition of each term of a sequence t. Thus $x_i = t_0$ if $i \leq \kappa(0)$ and $x_i = t_j$ if $\kappa(j-1) < i \leq \kappa(j)$ and $j > 0$. Each increasing sequence of nonnegative integers κ has an associated regular row finite stretching matrix S such that $St = x$. The matrix S is defined by $s_{ij} = 1$ if $i \leq \kappa(0)$ and $j = 0$ or if $\kappa(j-1) < i \leq \kappa(j)$ and $j > 0$, and $s_{ij} = 0$ otherwise.

The main purpose of this paper is to correct the statements of two results in [3] and to show how the corrected versions relate to some of the literature on gap sequences. In §2, we demonstrate a counterexample to a proposition in [3] involving bounded gap sequences and provide a corrected statement and proof of the proposition. The case for unbounded gap sequences is dealt with in §3. Several results about gap sequences, some of which are known, are discussed in §4, and their relationship to the corrected propositions in §2 and §3 are investigated. Some brief final comments may be found in §5.

2. Bounded gap sequences. In [**3**] and [**4**] J. A. Fridy establishes the following theorem as a main result.

THEOREM 1F. (*Fridy*) *Suppose A is a regular matrix that is $\{B_n\}$-dominated. If x is a bounded sequence such that Ax is convergent and*

$$\max_{k\in B_n} |(\Delta x)_n| = o\left(L_n^{-1}\right),$$

then x is convergent.

He then points out [**3**, p. 83] that a proof of the following proposition may be obtained by altering the argument he used to establish Theorem 1F.

COROLLARY 1F. (*Fridy*) *Let A be a regular matrix that is $\{B_n\}$-dominated and let x be a bounded sequence satisfying the gap condition $(\Delta x)_k = 0$ if $k \neq \kappa(m)$, $m = 0, 1, 2, \ldots$. If $\{B_n\}$ and κ satisfy $\kappa(m) \leq \mu(n) < \nu(n) \leq \kappa(m+1)$ for infinitely many n, then Ax and x either both converge or both diverge.*

Fridy's proof of Theorem 1F is valid and his applications of Theorem 1F to the Tauberian theorems in §3 and §4 of [**3**] remain true, but the argument used to prove Theorem 1F cannot be altered as he suggests to obtain a proof of Corollary 1F. The following example illustrates this fact.

Let $a_{2n,3n} = 1$, $a_{2n+1,3n+2} = a_{2n+1,3n+1} = \frac{1}{2}$, for $n = 0, 1, 2, \ldots$ and $a_{pq} = 0$ otherwise. Let $\mu(2t) = \mu(2t+1) = 3t - 1$ for $t = 0, 1, 2, \ldots$, $\nu(0) = 1$, and $\nu(2t) = \nu(2t-1) = 3t$ for $t = 1, 2, 3, \ldots$. The matrix A is regular and $\{B_n\}$-dominated by the blocks determined by μ and ν. Let $x_{3t+1} = -1$, $x_{3t+2} = 1$, and $x_{3t} = 0$ for $t = 0, 1, 2, \ldots$, and consider x as a gap sequence determined by $\kappa(m) = m$ for $m = 0, 1, 2, \ldots$. It follows that $\kappa(m) \leq \mu(n) < \nu(n) \leq \kappa(m+1)$ for $m = 3t - 1$ and $n = 2t$ where $t = 1, 2, 3, \ldots$. Thus the hypothesis of the corollary is satisfied, yet x is divergent and Ax is a constant sequence of zeros.

There does exist a valid corollary to the proof of Theorem 1F having a statement similar to Corollary 1F. In essence, we must strengthen the hypothesis that "$\{B_n\}$ and κ satisfy $\kappa(m) \leq \mu(n) < \nu(n) \leq \kappa(m+1)$ for infinitely many n" by assuming that the inequalities hold "for all sufficiently large n." Although the proof of this restated result follows the basic argument outlined by Fridy for his proof of Theorem 1F, we include it here for completeness.

THEOREM 1. *Let A be a regular matrix that is $\{B_n\}$-dominated and let x be a bounded gap sequence such that $(\Delta x)_k = 0$ if $k \neq \kappa(m)$, $m = 0, 1, \ldots$. If there exists M such that each $m \geq M$ satisfies $\kappa(m) \leq \nu(n) < \nu(n) \leq \kappa(m+1)$ for some n, then Ax and x either both converge or both diverge.*

Proof. Since A is regular, it is clear that if x converges then so does Ax. Choose x to be a bounded divergent sequence and select r as a candidate for $\lim_n (Ax)_n$. Let $R = \lim\sup_k |x_k - r|$ and $0 < \varepsilon < R$. Choose K such that $k \geq K$ implies $|x_k - r| < R + \varepsilon$. Since A is regular, then

$$
\begin{aligned}
|(Ax)_n - r| &= o(1) + \left| \sum_{k=0}^{\infty} a_{nk}(x_k - r) \right| \\
&\geq o(1) + \left| \sum_{k \in B_n} a_{nk}(x_k - r) \right| - \sum_{k \notin B_n} |a_{nk}|\, |x_k - r| \\
&\geq o(1) + \left| \sum_{k \in B_n} a_{nk}(x_k - r) \right| - \sum_{\substack{k \notin B_n \\ k \geq K}} |a_{nk}|\, |x_k - r| \\
&\geq o(1) + \left| \sum_{k \in B_n} a_{nk}(x_k - r) \right| - (R + \varepsilon) \sum_{k \notin B_n} |a_{nk}| \\
&= o(1) + |x_{\nu(n)} - r| \left| \sum_{k \in B_n} a_{nk} \right| - R \sum_{k \notin B_n} |a_{nk}| - \varepsilon \sum_{k \notin B_n} |a_{nk}|.
\end{aligned}
$$

Since x is divergent, there exists an infinite number of $m \geq M$ such that $|x_{\kappa(m+1)} - r| > R - \varepsilon$. Let $\|A\| = \sup_n \sum_{k=0}^{\infty} |a_{nk}|$ and n be chosen such that $\kappa(m) \leq \mu(n) \leq \nu(n) \leq \kappa(m+1)$. Then

$$
|(Ax)_n - r| > o(1) + R\left\{ \left| \sum_{k \in B_n} a_{nk} \right| - \sum_{k \in B_n} |a_{nk}| \right\} - 2\varepsilon \|A\|.
$$

But n is forced to become large as m increases and ε may be chosen as small as desired, therefore, by (1), $\lim\sup_n |(Ax)_n - r| > 0$ and Ax is divergent, which completes the proof.

3. Unbounded gap sequences. In [3, p. 83], Fridy provides the following example to show the assumption of x being bounded is necessary in the statement of Theorem 1F. Let A be the matrix given by

$$a_{nk} = \begin{cases} 3/2 & \text{if } k = n, \\ -1/2 & \text{if } k = n^3, \\ 0 & \text{otherwise.} \end{cases}$$

A is block dominated by $\{B_n\} = \{n\}$; thus $\mu(n) = n - 1$, $\nu(n) = n$, and $L_n = 1$. Let $x_n = \log n$ if $n > 0$ and $x_0 = 0$. Then $(Ax)_n = 0$ for each n, yet x diverges to $+\infty$. This example also serves to show the necessity of x being bounded in our Theorem 1 and to illustrate that the boundedness of x is still required even if A is chosen to be row finite.

Our next objective is to show that in Theorem 1 the boundedness of x may be dropped if $a_{nk} = 0$ for $k > \nu(n)$. This will provide a correct version of Corollary 2 of [3]. Again, the quantifying phrase "for infinitely many n" must be strengthened to read "for all sufficiently large n." The example given in §2 shows that Corollary 2 of [3] is not true with the weaker hypothesis.

THEOREM 2. *Let A be a regular matrix that is $\{B_n\}$-dominated so that $a_{nk} = 0$ whenever $k > \nu(n)$; and let x be a gap sequence such that $(\Delta x)_k = 0$ if $k \neq \kappa(m)$, $m = 0, 1, \ldots$. If there exists M such that each $m \geq M$ satisfies $\kappa(m) \leq \mu(n) < \nu(n) \leq \kappa(m+1)$ for some n, then Ax and x converge or diverge together.*

Proof. We choose x to be divergent and by virtue of Theorem 1 only consider the case where x is unbounded. Let H be an arbitrary positive number. Let $m \geq M$ such that $|x_{\kappa(m)}| \geq \max_{k \leq \kappa(m)}\{|x_k|, H\}$. If $\kappa(m-1) \leq \mu(n) < \nu(n) \leq \kappa(m)$, then

$$|(Ax)_n| \geq |x_{\kappa(m)}| \left| \sum_{k \in B_n} a_{nk} \right| - \sum_{k \leq \mu(n)} |a_{nk} x_k|$$

$$\geq H \left\{ \left| \sum_{k \in B_n} a_{nk} \right| - \sum_{k \leq \mu(n)} |a_{nk}| \right\}.$$

Since H is arbitrary, it follows from (1) that $\limsup_n |(Ax)_n| = +\infty$. Hence, Ax is divergent, and the proof is complete.

4. Applications to stretchings. We first consider applications for stretchings of bounded sequences. In [2], D. F. Dawson establishes the following result.

THEOREM 1D. (*Dawson*) *If A is regular, x is bounded, and Ay is convergent for every stretching y of x, then x is convergent.*

An alternate proof of this theorem may be obtained by applying the following corollary of our Theorem 1.

COROLLARY 1. *If A is regular, then there exists a stretching matrix S such that if x is bounded, then $A(Sx)$ and x converge or diverge together.*

Proof. Since A is regular, it admits a block domination sequence $\{B_n\}$ such that $\lim_n \mu(n) = +\infty$. It is thus possible to define a stretching matrix S having the property that $\kappa(m) \leq \mu(n) < \nu(n) \leq \kappa(m+1)$ for each m. Hence, by Theorem 1, Sx and $A(Sx)$ converge or diverge together, and the proof is complete.

Let ε be a positive term null sequence. We say y contains an ε-copy of x if there exists a subsequence $y_{p(i)}$ of y such that $|y_{p(i)} - x_i| < \varepsilon_i$ for $i = 1, 2, 3, \ldots$. D. F. Dawson **[1, 2]** and this author **[5, 6]** have obtained results connecting the concepts of ε-copies and stretchings of sequences. The following theorem is a form of a result in **[6]** which is based on Theorem 1 of **[2]**. The theorem is not a consequence of our Theorem 1, and the original proof presented in **[6]** does not formally utilize the concept of a block dominated matrix, but the argument below illustrates one way a broader application of the concept of block dominated matrices can be made to the summability of gap sequences.

THEOREM 3. *Let A be a regular matrix and x be a bounded sequence. If ε is a positive term null sequence, then there exists a stretching matrix S such that $A(Sx)$ contains an ε-copy of x.*

Proof. Since A is regular it admits a block sequence $\{B_n\}$ with $\lim_n \mu(n) = +\infty$, $\lim_n |\sum_{k \in B_n} a_{nk}| = 1$, and $\lim_n \sum_{k \notin B_n} |a_{nk}| = 0$. Let $T > 0$ such that $|x_k| < T$ for each k. Suppose that $\kappa(0), \ldots, \kappa(m-1)$ and $N(0), \ldots, N(m-1)$ have been chosen. Let $N(m) > N(m-1)$ such that $\mu(N(m)) \geq \kappa(m-1)$, $T \sum_{k \in B_{N(m)}} |a_{N(m),k}| < \varepsilon_m/2$, and $|x_m| |\sum_{k \in B_{N(m)}} a_{N(m),k} - 1| < \varepsilon_m/2$. Also let $\kappa(m) = \nu(N(m))$. Then regardless of the remaining choices for terms of κ, $|(A(Sx))_{N(m)} - x_m| < \varepsilon_m$, and the proof is complete.

With appropriate modifications, the three applications in this section also apply to unbounded gap sequences. Typical of these is the following corollary to our Theorem 2.

COROLLARY 2. *If A is regular and row finite and x is an arbitrary sequence, then there exists a stretching matrix S such that $A(Sx)$ and x converge or diverge together.*

5. Concluding remarks. As a final note, emphasis should be given to the fact that the major results obtained in [3] stand. Propositions 1F and 2F are peripheral to the main thrust of [3], and the present paper in no way detracts from the significance of that work. This author also wishes to thank Professor Fridy for his cooperation and several helpful suggestions in the preparation of this paper.

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