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A DESCRIPTION OF THE TOPOLOGY ON THE DUAL SPACE OF A NILPOTENT LIE GROUP

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The study of convergence of sequences of elements of the dual space $\hat{G}$, for a nilpotent Lie group G, is done by reducing the study to convergence of sequences of subgroup representation pairs, whose subgroup component has dimension less than the dimension of G. The main results are then applied to give a new proof to the fact that the Kirillov correspondence is a homeomorphism for nilpotent Lie groups.

1. Introduction. Let G be a real, connected, simply-connected nilpotent Lie group. By the dual space $\hat{G}$ of G, we mean the set of all equivalence classes of irreducible unitary representations equipped with the hull-Kernel topology. Kirillov [4] has shown that the elements of $\hat{G}$ are in one-to-one correspondence with orbits of real linear functionals on $\mathfrak{g}$ (the Lie algebra of G). The fact that this correspondence is actually a homeomorphism was conjectured by Kirillov and first proved by Brown [1].

In this paper we study the convergence of sequences $\{W_n\}$ in $\hat{G}$ by studying sequences of subgroup-representation pairs $\{(H_n, S_n)\}$, where H_n is a subgroup of G, $S_n \in \hat{H}_n$ and $\dim H_n < \dim G$. We develop necessary and sufficient conditions for the convergence of sequences in $\hat{G}$, in terms of convergence of associated sequences of subgroup-representation pairs in the subgroup-representation topology of Fell [3]. Then using the theorems of Fell, we give a new proof of the Kirillov conjecture that avoids the use of the free nilpotent Lie algebras of Brown.

2. Preliminaries. Let G be a simply connected nilpotent Lie group. We will be primarily interested in the case when G is nonabelian. For if G were abelian, then $G \cong \mathbf{R}^n$ for some n, and the irreducible representations of G are characters. In fact, $\hat{\mathbf{R}}^n \cong \mathbf{R}^n$. Therefore, the case when G is abelian is essentially trivial and, unless otherwise noted, we will assume our groups are nonabelian.

We introduce the following conventions in notation:

(1) $\mathcal{K}(G)$ will denote the set of all closed subgroups of G, equipped with the compact-open topology

(2) $\mathcal{Q}(G)$ will denote the set of all subgroup-representation pairs of G, equipped with the subgroup-representation pair topology of Fell [3]

(3) If $\mathfrak{g}$ is the Lie algebra of G, there is a one-to-one correspondence between closed, connected subgroups of G and Lie subalgebras of $\mathfrak{g}$. If H is a closed, connected subgroup of G we will denote its corresponding subalgebra by $\mathfrak{h}$, and conversely.

(4) We will denote by $\operatorname{ind}_H^G S$ the representation of G, induced by the representation S of H, and by $W|_H$ the restriction of the representation W to the subgroup H.

(5) By Kirillov [**4**], the irreducible representations of G are in one-to-one correspondence with certain orbits in $\mathfrak{g}^*$ (the conjugate space of $\mathfrak{g}$). If $W \in \hat{G}$, we will denote its correspondence orbit by Ω_W.

3. The topology of $\hat{G}$. Let G be a simply connected, nilpotent Lie group.

3.1 Definition. Let $(H, S) \in \mathcal{Q}(G)$ where $S \in \hat{H}$ and S is not a character. A pair $(H', S') \in \mathcal{Q}(G)$ is called inducing for (H, S) if H' is of codimension one in H and $S = \operatorname{ind}_{H'}^H S'$.

3.2 Remark. The existence of inducing pairs, when S is not a character, follows from [**4**].

3.3 Lemma. *Let H be a subgroup of G of codimension one, and let $W \in \hat{G}$. Let $f \in \Omega_W$ and T be chosen so that $f|_{\mathfrak{h}} \in \Omega_T$. Then $W \subseteq_{\mathrm{w}} \operatorname{ind}_H^G T$ and $T \in \operatorname{sp}(W|_h)$.*

Proof. By [**4**] the orbit Ω_W can be classified as one of two types. The Lemma then follows directly from Lemma 6.2 of [**4**] and Theorem 4.3 of [**3**], or by Lemma 6.3 of [**4**] and Theorem 4.5 of [**3**]. □

We now develop necessary and sufficient conditions for a sequence of subgroup-representation pairs to converge in $\mathcal{Q}(G)$.

3.4 Theorem. *Let $\{(H_n, S_n)\}$ be a sequence in $\mathcal{Q}(G)$ such that, for each n, $S_n \in \hat{H}_n$ and S_n is not a character. Then $(H_n, S_n) \to (H, S)$ in $\mathcal{Q}(G)$ if and only if, for each subsequence of $\{(H_n, S_n)\}$, there exists a subsequence $\{(H_i', S_i')\}$ such that:*

(1) *there exist inducing pairs (H_n'', T_n) for each (H_n', S_n') such that*

$$(H_n'', T_n) \to (H'', T) \quad \text{in } \mathcal{Q}(G)$$

for some $T \in \hat{H}''$.

(2) $S \subseteq_{\mathrm{w}} \operatorname{ind}_{H''}^H T$.

Proof. ($\Rightarrow$) Let $(H_n, S_n) \to (H, S)$ in $\mathcal{Q}(G)$, and let $\{(H_i', S_i')\}$ be a subsequence of $\{(H_n, S_n)\}$. For each i, let (H_i'', S_i'') be an inducing pair for (H_i', S_i'). By restricting to subsequences, it is sufficient to assume that all the H_i'' have the same dimension. Therefore since $\mathcal{K}(G)$ is compact, by further restricting to a subsequence, we can assume there exists an $H'' \in \mathcal{K}(G)$ of codimension one in H such that $H_i'' \to H''$ in $\mathcal{K}(G)$.

By continuity of restriction (Theorem 3.2 of [**2**]) we have

$$\left(H_i'', S_i'|_{H_i''}\right) \to \left(H'', S|_{H''}\right) \quad \text{in } \mathcal{Q}(G).$$

Pick $f \in \Omega_s$ and T such that $f|_{\mathfrak{h}''} \in \Omega_T$. By Lemma 3.3, $T \in \operatorname{sp}(S|_{H''})$ and $S \subseteq_{\mathrm{w}} \operatorname{ind}_{H''}^{H} T$. Now T is in the closure of the set

$$E = \bigcup_i \left\{(H_i'', T''): T'' \in \operatorname{sp}\left(S_i'|_{H_i''}\right)\right\}$$

(Lemma 3.3 of [**5**]), and the closure of E is the same as the closure of

$$\bigcup_i \{(H_i'', T''): T'' \text{ is in the orbit of } H_i'' \text{ associated with } S_i'\}$$

(Theorem 4.5 of [**4**]). Therefore, for each i, there exists T_i contained in the orbit of $\hat{H}_i''$ associated with S_i' such that $(H_i'', T_i) \to (H, T)$ in $\mathcal{Q}(G)$. Since S_i'' and T_i must be in the same orbit of $\hat{H}_i''$ associated with S_i', we have that (H_i'', T_i) is inducing for (H_i', S_i').

The sequence $\{(H_i', S_i')\}$ and the pair (H, T) satisfy the conditions of the Theorem.

($\Leftarrow$) Suppose that for each subsequence of $\{(H_n, S_n)\}$ there exists a subsequence $\{(H_i', S_i')\}$ such that the condition holds. Then by continuity of inducing (Theorem 1.2 of [**3**]),

$$\left(H_i', \operatorname{ind}_{H_i''}^{H_i'}(T_i)\right) \to \left(H, \operatorname{ind}_{H''}^{H}(T)\right) \quad \text{in } \mathcal{Q}(G).$$

Since (H_i'', T_i) is inducing for (H_i', S_i'), we have

$$\operatorname{ind}_{H_i''}^{H_i'}(T_i) \cong S_i'$$

for each i. Thus

$$(H_i', S_i') \to \left(H, \operatorname{ind}_{H''}^{H}(T)\right) \quad \text{in } \mathcal{Q}(G).$$

But $S \subseteq_{\mathrm{w}} \operatorname{ind}_{H''}^{H}(T)$, which implies

$$(H_i', S_i') \to (H, S) \quad \text{in } \mathcal{Q}(G).$$

Therefore,

$$(H_n, S_n) \to (H, S) \quad \text{in } \mathcal{Q}(G). \qquad \square$$

3.5 COROLLARY. *Let* $\{(H_n, S_n)\}$ *be a sequence in* $\mathcal{Q}(G)$ *such that* $S_n \in \hat{H}_n$ *and* S_n *is not a character. Then* $(H_n, S_n) \to (H, S)$ *in* $\mathcal{Q}(G)$ *if and only if, for every subsequence of* $\{(H_n, S_n)\}$, *there exists a subsequence* $\{(H_i', S_i')\}$ *such that*:

(1) *There exist pairs* $(K_i, \chi_i) \in \mathcal{Q}(G)$, *where* χ_i *is a character of* K_i *and* $S_i' \cong \operatorname{ind}_{K_i}^{H_i'} \chi_i$, *such that*

$$(K_i, \chi_i) \to (K, \chi) \quad \text{in } \mathcal{Q}(G)$$

where χ *is a character of* K.

(2) $S \subseteq_{\mathrm{w}} \operatorname{ind}_K^H(\chi)$

Proof. Let $(H_n, S_n) \to (H, S)$ in $\mathcal{Q}(G)$. By successively applying Theorem 3.4, and reducing to appropriate sequences, the sequence $\{(K_i, \chi_i)\}$ and the pair (K, χ) can be produced that satisfy the corollary. The opposite direction is shown in the same manner as in the proof of Theorem 3.4. $\square$

We can now use Theorem 3.4 to prove a lemma which clearly suffices to prove the Kirillov conjecture.

LEMMA. *Let* $(H_n, S_n) \to (H, S)$ *in* $\mathcal{Q}(G)$. *If* $f \in \mathfrak{g}^*$ *such that* $f|_{\mathfrak{h}} \in \Omega_S$, *then for every subsequence of* $\{(H_n, S_n)\}$, *there is a subsequence* $\{(H_i', S_i')\}$ *such that for each* i, *there exists* $f_i \in \mathfrak{g}^*$ *such that* $f_i|_{\mathfrak{h}_i'} \in \Omega_{S_i'}$ *and* $f_i \to f \in \mathfrak{g}^*$.

Proof. Let $\{(H_i', S_i')\}$ be a subsequence of $\{(H_n, S_n)\}$ and restrict to a subsequence so that we may assume the dimensions of all H_i' are constant.

Let this constant be k. Then $\dim H = k$.

If each S_i' is a character of H_i', then S is a character of H. In this case, it is easily seen that Ω_S and the $\Omega_{S_i'}$ contain only one element.

Let $f \in \mathfrak{g}^*$. By passage to another subsequence, we may assume there exist bases $h_1^{(i)}, \dots, h_n^{(i)}$ of $\mathfrak{g}$ such that $h_1^{(i)}, \dots, h_k^{(i)}$ is a basis of $\mathfrak{h}_i$, $h_j^{(i)} \to h_j$ in $\mathfrak{g}$, $1 \le j \le n$, and $h_1, \dots, h_n$ is a basis of H.

Define $f_i \in \mathfrak{g}^*$ by

$$f_i\left(h_j^{(i)}\right) = \begin{cases} f_i'\left(h_j^{(i)}\right), & j \le k, \\ f\left(h_j\right), & j > k, \end{cases}$$

where f_i' is the singleton element of $\Omega_{S_i'}$.

Clearly $f_i \to f$ in $\mathfrak{g}^*$ and the result follows.

The lemma proceeds by induction on k. If $k = 1$, then each S_i' is a character and we are done. Assume, then, that the lemma has been shown for all sequences $\{(H_i', S_i')\}$ whose first elements all have dimension less than k.

By the above and by passage to a subsequence, we may assume each S_i' is not a character of H_i'. By Theorem 3.4 there exist inducing pairs (H_n'', T_n) for each (H_n', S_n') such that

$$(H_i'', T_n) \to (H'', T) \quad \text{in } \mathscr{C}(G)$$

for some $T \in \hat{H}''$.

Now using the inductive hypothesis there exist $f_i \in \mathfrak{g}^*$ such that $f_i|_{\mathfrak{h}_i''} \in \Omega_{T_i}$ for each i, and $f_i \to f$ in $\mathfrak{g}^*$. But since (H_i'', T_i) is inducing for (H_i', S_i'), $f_i|_{\mathfrak{h}_i} \in \Omega_{S_i'}$ and the desired sequence is constructed. □

Given this lemma, the fact that the Kirillov correspondence is a homeomorphism is straightforward.

3.7 Theorem (Kirillov-Brown). *$W_n \to W$ in $\hat{G}$ if and only if $\Omega_{W_n} \to \Omega_W$ in the space of orbits equipped with the quotient topology.*

Proof. If $W_n \to W$ the theorem follows directly from Lemma 3.8. The opposite direction has been shown by Kirillov [**4**].

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