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COMPACT ELEMENTS OF WEIGHTED GROUP ALGEBRAS

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For a locally compact group G let $L^1(G, \omega\lambda)$ be the weighted group algebra. We characterize elements $g \in L^1(G, \omega\lambda)$ for which the operator $T_g(f) = f * g$ $(f \in L^1(G, \omega\lambda))$ is compact. We conclude a result due to S. Sakai that if G is a locally compact non-compact group, then 0 is the only compact element of $L^1(G, \lambda)$, and a result due to C. Akemann that if G is a compact group, then every element of $L^1(G, \lambda)$ is compact.

In a recent paper ([**2**], Theorem 2.2) W. G. Bade and H. G. Dales, among other things, characterize compact elements of $L^1(R^+, \omega)$. Niels Grønbæk, in his Ph.D. thesis ([**6**] Proposition 2.4), for a large class of semigroups (including cancellation semigroups) characterizes compact elements of the discrete weighted semigroup algebras.

In this paper, we characterize the compact elements of the weighted group algebras of locally compact groups. S. Sakai has proved that if G is a locally compact non-compact group, then 0 is the only compact element of $L^1(G, \lambda)$, (see [**10**], Theorem 1), and C. Akemann has proved that if G is a compact group, then every element of $L^1(G, \lambda)$ is compact ([**1**], Theorem 4). These two results will immediately follow from our characterization of the compact elements of the weighted group algebras. Also, a technique somewhat similar to ours provides other proofs for the Bade-Dales theorem (for bounded ω) and the theorem of Grønbæk.

By a weight function on a locally compact group G we mean a positive and continuous function ω on G such that $\omega(st) \leq \omega(s)\omega(t)$ $(s, t \in G)$. If λ is a left Haar measure on G and ω is a weight function on G, we set

$$L^1(G, \omega\lambda) = \left\{ f\colon \|f\| = \int_G |f(t)|\omega(t)\, d\lambda(t) < \infty \right\}.$$

Then, $L^1(G, \omega\lambda)$ is a Banach space: as usual, we equate functions equal λ almost everywhere. Under the convolution product defined by the equation

$$(f * g)(x) = \int_G f(xy^{-1})g(y)\, d\lambda(y) \qquad (f, g \in L^1(G, \omega\lambda))$$

$L^1(G, \omega\lambda)$ becomes a Banach algebra. We call an element $g \in L^1(G, \omega\lambda)$ compact if the operator $T_g(f) = f * g$ $(f \in L^1(G, \omega\lambda))$ is a compact operator.

Now we state the main result of this paper.

THEOREM 1. *An element $g \in L^1(G, \omega\lambda)$ is compact if and only if the function F_g defined on G by*

$$F_g(s) = \int_G \frac{\omega(st)}{\omega(s)} |g(t)| d\lambda(t) \qquad (s \in G), \tag{1.1}$$

vanishes at infinity.

Before we proceed to prove Theorem 1, we let $C_0(G, \omega)$ be the Banach space of all complex functions f on G such that $f/\omega \in C_0(G)$, and where the norm is taken to be

$$\|f\| = \sup_{x \in G} \left| \frac{f(x)}{\omega(x)} \right|.$$

Let $M(G, \omega)$ be the Banach space of all complex regular Borel measures μ on G such that

$$\|\mu\| = \int_G \omega(t)\, d|\mu|(t) < \infty;$$

then by the pairing $\langle \mu, \psi \rangle = \int_G \psi(x)\, d\mu(x)$ $(\mu \in M(G, \omega), \psi \in C_0(G, \omega))$ we have $(C_0(G, \omega))^* = M(G, \omega)$ and we can define the product of μ, $\nu \in M(G, \omega)$ by

$$\int_G \psi(x)\, d(\mu * \nu)(x) = \int_G \int_G \psi(xy)\, d\mu(x)\, d\nu(y) \qquad (\psi \in C_0(G, \omega))$$

to make $M(G, \omega)$ a Banach algebra. The map $f \to \mu_f$, where $d\mu_f(x) = f(x)\, d\lambda(x)$ defines an isometric isomorphism from $L^1(G, \omega\lambda)$ into $M(G, \omega)$, and $L^1(G, \omega\lambda)$ can be identified with a closed ideal of $M(G, \omega)$. We define the topology (so) on $M(G, \omega)$ as follows: for a net $(\mu_\alpha) \subset M(G, \omega)$ we let $\mu_\alpha \xrightarrow{(so)} \mu$ if and only if $\mu_\alpha * f \xrightarrow{\|\cdot\|} \mu * f$, for every $f \in L^1(G, \omega\lambda)$, (see [4] and [5]). The algebra $L^1(G, \omega\lambda)$ possesses a bounded approximate left identity (see [8], p. 84).

The proof of the next lemma is formally the same as the proof of Theorem 20.4 of [7] and is therefore omitted.

LEMMA 1. *The map* $x \to 1/\omega(x)\delta_x$ *from* G *into* $M(G, \omega)$ *is* (so) *continuous.*

LEMMA 2. *An element* $g \in L^1(G, \omega\lambda)$ *is compact if and only if it is a compact element of* $M(G, \omega)$.

Proof. Suppose that the operator $T_g(f) = f * g$ $(f \in L^1(G, \omega\lambda))$ is compact. It is to be shown that the operator $\overline{T}_g(\mu) = \mu * g$ $(\mu \in M(G, \omega))$ is compact.

Let $\{f_\alpha : \alpha \in A\}$ be a bounded approximate left identity for $L^1(G, \omega\lambda)$. If $\mu \in M(G, \omega)$, then

$$\overline{T}_g(\mu) = \mu * g = \lim \mu * f_\alpha * g = \lim T_g(\mu * f_\alpha).$$

Hence the set $\{\overline{T}_g(\mu) : \|\mu\| \leq 1\}$ is contained in the norm closure of the set $\{T_g(\mu * f_\alpha) : \|\mu\| \leq 1,\ \alpha \in A\}$, which is compact by compactness of T_g. Thus, $\overline{T}_g$ is compact.

The converse is obvious, since $L^1(G, \omega\lambda)$ is a closed subspace of $M(G, \omega)$ and is invariant under $\overline{T}_g$ the restriction of which to $L^1(G, \omega\lambda)$ is T_g.

Proof of Theorem 1. Suppose g is a compact element of $L^1(G, \omega\lambda)$, then by lemma 2 it is a compact element of $M(G, \omega)$. If the function F_g defined by (1.1) does not vanish at infinity, then there exists an $\alpha > 0$ such that for every compact set $K \subset G$, there exists $s \notin K$, with

$$\left\|\overline{T}_g\left(\frac{1}{\omega(s)}\delta_s\right)\right\| = \int_G \frac{\omega(st)}{\omega(s)}|g(t)|\,d\lambda(t) \geq \alpha. \tag{1}$$

The set $\mathcal{K}$ of all compact subsets of G is a directed set under set inclusion. For each $K \in \mathcal{K}$, choose $s(K) \notin K$ such that

$$\left\|\overline{T}_g\left(\frac{1}{\omega(s(K))}\delta_{s(K)}\right)\right\| = \int_G \frac{\omega(s(K)t)}{\omega(s(K))}|g(t)|\,d\lambda(t) \geq \alpha. \tag{2}$$

Thus, we obtain a net $\{s(K) : K \in \mathcal{K}\}$ the terms of which satisfy (2). By the boundedness of the net

$$\left\{\frac{1}{\omega(s(K))}\delta_{s(K)} : K \in \mathcal{K}\right\}$$

and compactness of $\overline{T}_g$, there exists a subnet

$$\left\{\frac{1}{\omega(s(K_i))}\delta_{s(K_i)} : i \in I, \succ\right\}$$

and a measure μ such that

$$\overline{T}_g\left(\frac{1}{\omega(s(K_i))}\delta_{s(K_i)}\right) \xrightarrow{\|\cdot\|} \mu. \tag{3}$$

From (2) and (3) it follows that $\|\mu\| \geq \alpha$, whence there exists $\psi \in C_0(G, \omega)$ with $\|\psi\| = 1$ and with

$$|\langle \mu, \psi \rangle| > \alpha/2. \tag{4}$$

By (3)

$$\left|\left\langle \overline{T}_g\left(\frac{1}{\omega(s(K_i))}\delta_{s(K_i)}\right), \psi\right\rangle\right| \to |\langle \mu, \psi \rangle|. \tag{5}$$

From (4) and (5) it follows that there exist $i_0 \in I$, such that for $i \succ i_0$, we have

$$\left|\int_G \frac{\psi(s(K_i)y)}{\omega(s(K_i))} g(y)\, d\lambda(y)\right| = \left|\left\langle \overline{T}_g\left(\frac{1}{\omega(s(K_i))}\delta_{s(K_i)}\right), \psi\right\rangle\right| > \frac{\alpha}{2}. \tag{6}$$

Choose $h \in L^1(G, \omega\lambda)$ with compact support K_h and such that $\|g - h\| < \alpha/4$. Then

$$\begin{aligned}
&\left|\left\langle \overline{T}_g\left(\frac{1}{\omega(s(K_i))}\delta_{s(K_i)}\right), \psi\right\rangle - \left\langle \overline{T}_h\left(\frac{1}{\omega(s(K_i))}\delta_{s(K_i)}\right), \psi\right\rangle\right| \\
&\qquad \leq \int_G \left|\frac{\psi(s(K_i)y)}{\omega(s(K_i)y)}\right| \frac{\omega(s(K_i)y)}{\omega(s(K_i))} |g(y) - h(y)| d\lambda(y) \\
&\qquad \leq \|\psi\| \int_G \frac{\omega(s(K_i))\omega(y)}{\omega(s(K_i))} |g(y) - h(y)| d\lambda(y) \\
&\qquad = \|g - h\| < \frac{\alpha}{4}.
\end{aligned} \tag{7}$$

Hence, for $i \succ i_0$, by (6) and (7), we have

$$\begin{aligned}
\left|\left\langle \overline{T}_h\left(\frac{1}{\omega(s(K_i))}\delta_{s(K_i)}\right), \psi\right\rangle\right| &> \left|\left\langle \overline{T}_g\left(\frac{1}{\omega(s(K_i))}\delta_{s(K_i)}\right), \psi\right\rangle\right| - \frac{\alpha}{4} \\
&> \frac{\alpha}{2} - \frac{\alpha}{4} = \frac{\alpha}{4}.
\end{aligned} \tag{8}$$

Now, let $K_\psi \in \mathcal{K}$, be such that

$$\left|\frac{\psi(z)}{\omega(z)}\right| < \frac{\alpha}{4(1 + \|h\|)},$$

for $z \notin K_\psi$. Choose $i \succ i_0$ such that $K_i \supset K_\psi K_h^{-1}$. Then $s(K_i)$ satisfies (8). But, if $y \in K_h$, then $s(K_i)y \notin K_\psi$, whence

$$\left|\frac{\psi(s(K_i)y)}{\omega(s(K_i)y)}\right| < \frac{\alpha}{4(1+\|h\|)}.$$

Thus,

$$(9) \qquad \left|\left\langle \bar{T}_h\left(\frac{1}{\omega(s(K_i))}\delta_{s(K_i)}\right), \psi\right\rangle\right|$$

$$= \left|\int_{K_h} \frac{\psi(s(K_i)y)}{\omega(s(K_i)y)} \frac{\omega(s(K_i)y)}{\omega(s(K_i))} h(y)\, d\lambda(y)\right|$$

$$\leq \int_{K_h} \left|\frac{\psi(s(K_i)y)}{\omega(s(K_i)y)}\right| \frac{\omega(s(K_i))\omega(y)}{\omega(s(K_i))} |h(y)| d\lambda(y)$$

$$\leq \frac{\alpha}{4(1+\|h\|)}\|h\| \leq \frac{\alpha}{4}$$

which contradicts (8). Thus, F_g vanishes at infinity.

Conversely, suppose that the function F_g defined by (1.1) vanishes at infinity. If $g = 0$, then it is obviously a compact element. If $g \neq 0$, then the vanishing of F_g at infinity implies that G is σ-compact. In fact, if for each positive integer n we let K_n be a compact subset of G such that $|F_g(x)| < 1/n$ for $x \notin K_n$, then if $x \in G$, we have $F_g(x) \neq 0$, whence for some positive integer n we have $1/n < F_g(x)$. Hence $x \in K_n$, which implies $G = \bigcup_{n=1}^\infty K_n$. By Lemma 2 it suffices we prove that the operator $\bar{T}_g(\mu) = \mu * g$ ($\mu \in M(G, \omega)$) is compact. The operator $\bar{T}_g$ is the adjoint of the operator R_g defined on $C_0(G, \omega)$ by

$$(R_g f)(x) = \int_G f(xy)g(y)\, d\lambda(y) \qquad (f \in C_0(G, \omega), x \in G).$$

The map τ: $C_0(G, \omega) \to C_0(G)$ defined by $(\tau f)(x) = f(x)/\omega(x)$ ($x \in G$) is a (linear) isometry. Therefore, it suffices to show that the operator $\tilde{R}_g = \tau R_g \tau^{-1}$, defined on $C_0(G)$ by

$$\tilde{R}_g(f)(x) = \int_G \frac{\omega(xs)}{\omega(x)} f(xs) g(s)\, d\lambda(s) \qquad (f \in C_0(G), x \in G),$$

is compact.

Let (f_n) be a bounded sequence in $C_0(G)$, and $(K_n)_{n=1}^\infty$ be as defined earlier. We note that $K_n \subset K_{n+1}$, $n = 1, 2, \ldots$.

First we show that the sequence $(\tilde{R}_g f_n)$ has a subsequence the restriction of whose terms to K_1 converges uniformly to a function $h_1 \in C(K_1)$. Let $s \in K_1$ and $(s_\alpha) \subset K_1$ be a net converging to s. Then,

$$\begin{aligned}
&\left|\tilde{R}_g f_n(s_\alpha) - \tilde{R}_g f_n(s)\right| \\
&= \left|\int_G f_n(y)\left(\left(\frac{1}{\omega(s)}\delta_s * g\right)(y) - \left(\frac{1}{\omega(s_\alpha)}\delta_{s_\alpha} * g\right)(y)\right)\omega(y)\,d\lambda(y)\right| \\
&\leq \sup_n \|f_n\| \int_G \omega(y)\left|\left(\frac{1}{\omega(s)}\delta_s * g\right)(y) - \left(\frac{1}{\omega(s_\alpha)}\delta_{s_\alpha} * g\right)(y)\right| d\lambda(y) \\
&= \sup_s \|f_n\| \left\|\left(\frac{1}{\omega(s_\alpha)}\delta_{s_\alpha} - \frac{1}{\omega(s)}\delta_s\right) * g\right\| \to 0,
\end{aligned}$$

as $s_\alpha \to s$ uniformly in (f_n), (by Lemma 1). Also the sequence $(\tilde{R}_g f_n)$ is uniformly bounded on K_1. Thus, by the Ascoli-Arzela theorem ([**3**], Theorem 7, p. 266) the set of restrictions of the terms of $(\tilde{R}_g f_n)$ to K_1 is a conditionally compact subset of $C(K_1)$, whence there exists a function $h_1 \in C(K_1)$ and a subsequence $(\tilde{R}_g f_{1,k})$ such that $\tilde{R}_g f_{1,k} \to h_1$, as $k \to \infty$, uniformly on K_1.

Let us now consider sequences $S_1, S_2, S_3, \ldots$, which we represent by the array

$$\begin{array}{llll}
S_1\colon \tilde{R}_g f_{1,1} & \tilde{R}_g f_{1,2} & \tilde{R}_g f_{1,3} \cdots. & \\
S_2\colon \tilde{R}_g f_{2,1} & \tilde{R}_g f_{2,2} & \tilde{R}_g f_{2,3} \cdots. & \\
S_3\colon \tilde{R}_g f_{3,1} & \tilde{R}_g f_{3,2} & \tilde{R}_g f_{3,3} & \tilde{R}_g f_{3,4} \cdots. \\
\vdots & & &
\end{array}$$

and which have the following properties:

(a) S_n is a subsequence of S_{n-1}, for $n = 2, 3, 4, \ldots$.

(b) $(\tilde{R}_g f_{n,k})$, when restricted to K_n, converges uniformly to a function $h_n \in C(K_n)$, as $k \to \infty$.

(c) The order in which the functions appear is the same in each sequence.

Thus, h_{n+1} is an extension of h_n from K_n to K_{n+1}, and from the definition of $\tilde{R}_g(f_n)(x)$ we have

$$|h_{n+1}(x)| \leq \sup_m \|f_m\| \frac{1}{n}, \quad \text{for } x \in (K_{n+1} \backslash K_n),\ n = 1, 2, 3, \ldots.$$

Consider the diagonal array

$$S: \tilde{R}_g f_{1,1} \quad \tilde{R}_g f_{2,2} \quad \tilde{R}_g f_{3,3} \cdots .$$

By (c), the sequence S (except possibly its first $n-1$ terms) is a subsequence of S_n for $n = 1, 2, \ldots$. Hence (b) implies that $(\tilde{R}_g f_{n,n})$ converges uniformly on K_i to $h_i \in C(K_i)$, $i = 1, 2, \ldots$. Now, if h is a function on G, the restriction of which to K_i is equal to h_i, then $\tilde{R}_g f_{n,n} \to h$, uniformly on G, and the proof is complete.

Remark. Our method of finding the convergent subsequence $(\tilde{R}_g f_{n,n})$ is similar to the well-known process of finding a pointwise convergent subsequence of a pointwise bounded sequence of functions defined on a countable set, (see **[9]**, Theorem 7.23).

Corollary 1. *If the group G is a compact group, then every element of $L^1(G, \omega\lambda)$ is compact.*

For the special case $\omega(t) = 1$ $(t \in G)$ we obtain:

Corollary 2. (C. Akemann **[1]**, Theorem 4.) *If G is a compact group, then every element of $L^1(G, \lambda)$ is compact.*

Corollary 3. (S. Sakai **[10]**, Theorem 1.) *If G is a locally compact non-compact group, then 0 is the only compact element of $L^1(G, \lambda)$.*

Proof. If $g \neq 0$ is compact, then $\|g\| = |F_g(x)| < \frac{1}{2}\|g\|$ for every $x \notin K$, where K is a proper compact subset of G, a contradiction.

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