PACIFIC JOURNAL OF MATHEMATICS
Vol. 117, No 2, 1985

THE GENERALIZED SCHWARZ LEMMA FOR THE BERGMAN METRIC

MASAAKI SUZUKI

The function-theoretic criterion for the Bergman metric to be dominated by the Kobayashi metric on the domain in $\mathbf{C}^n$ is given. For this, we use the distinguished family of plurisubharmonic functions and P-metric of N. Sibony.

1. Introduction. Let D be a hyperbolic domain in $\mathbf{C}^n$ (cf. Kobayashi [7]). On D we can define some intrinsic metrics: the Carathéodory metric C_D, the Kobayashi metric K_D, and the Bergman metric B_D. It is known that $C_D \leq K_D$ and $C_D \leq B_D$. In this paper we investigate when the Bergman metric is dominated by the Kobayashi metric. Using N. Sibony's P-metric we give a function-theoretic criterion for the following condition (#) to hold:

(#) $B_D \leq cK_D$ on the tangent bundle of D, $c > 0$ constant.

Under (#) every holomorphic mapping $F\colon U \to D$ satisfies $F^*B_D \leq 2^{-1/2}cB_U$, where U is the unit disc in $\mathbf{C}$ with the Bergman metric B_U. This theorem is called the generalized Schwarz lemma for the Bergman metric. According to N. Sibony [10], we introduce the family of functions

$$S_p(D) = \{u\colon D \to [0,1);\ u(p) = 0,\ C^2\text{-class in a neighborhood of } p \text{ and } \log u \text{ is plurisubharmonic in } D\}.$$

Taking a Bergman kernel $k(z, \bar{w})$ of a domain D, we construct a function ϕ_w for a fixed point w in D as follows;

$$\phi_w(z) = \phi_{w,\alpha}(z) = 1 - \left(\frac{|k(z,\bar{w})|^2}{k(z,\bar{z})k(w,\bar{w})}\right)^\alpha \equiv 1 - v^\alpha,$$

where α is a positive constant chosen for D. It is clear that $0 \leq \phi_w \leq 1$ and $\phi_w(w) = 0$. Our main results are stated as follows.

(I) Let D be a Bergman domain. If there is a constant $\alpha > 0$ such that $\phi_w = \phi_{w,\alpha}$ belongs to $S_w(D)$ for each w in D, then $B_D \leq \alpha^{-1/2}K_D$; hence B_D satisfies the generalized Schwarz lemma.

(II) If the Bergman metric B_D of a domain D satisfies the following condition; there exists a positive constant α such that, for each w in D,

$$(*)\quad \phi_w(z)B_D^2(z,\xi) \geq \alpha\left|\partial_\xi \log v(z)\right|^2 \quad \text{for all } \xi \in \mathbf{C}^n \text{ and } z \in \{z \in D;\, 0 < v(z) < 1\},$$

then $B_D \leq \alpha^{-1/2}K_D$; therefore, B_D satisfies the generalized Schwarz lemma.

In §2 we give some properties of the family $S_p(D)$. In §3 we arrange the basic properties of the intrinsic metrics, especially of the P^*-metric. In §4 we prove the main results.

In §5, for the classical domains, we construct the function ϕ_w and directly verify that each ϕ_w belongs to $S_w(D)$. Hence, we have the generalized Schwarz lemma for the classical domains (cf. Kobayashi [**7**, **8**]).

The author is grateful to K. Azukawa for helpful discussions of this material.

2. The family $S_p(D)$. Though our argument is available on a complex manifold, we work mainly on a domain in $\mathbf{C}^n$ $(n \geq 1)$.

Let p be a fixed point of D and

$$S_p(D) = \{u\colon D \to [0,1);\ u(p) = 0,\ C^2\text{-class in a neighborhood of } p \text{ and } \log u \text{ is plurisubharmonic in } D\},$$

$$A_p(D) = \left\{u = |f|^2;\ f \in \operatorname{Hol}(D,U),\ f(p) = 0\right\},$$

where U is the unit disc in $\mathbf{C}$ and $\operatorname{Hol}(D,U)$ denotes the family of all holomorphic mappings $f\colon D \to U$. It is clear that $S_p(D) \supset A_p(D)$.

A nonnegative plurisubharmonic function u such that $\log u$ is also plurisubharmonic ($\log 0 = -\infty$) is called a logarithmically plurisubharmonic function. Hereafter we abbreviate them p.s.h. and log.p.s.h. $S_p(D)$ is a special family of log.p.s.h. functions on D. We give some lemmas about the family $S_p(D)$.

LEMMA 2.1. (1) *If u_1, u_2 belong to $S_p(D)$, then $ru_1 + (1-r)u_2 \in S_p(D)$ for any real number r with $0 \leq r \leq 1$.*

(2) *If f_j $(j = 1,\ldots,k)$ are holomorphic in D and vanish at p and $u(z) = \Sigma|f_j(z)|^2 < 1$ on D, then $u \in S_p(D)$.*

Proof. (1) It is sufficient to prove that if $\log u_j$ $(j = 1,2)$ is subharmonic in an open set G in C, then $\log(r_1u_1 + r_2u_2)$ is subharmonic in G for $r_j \geq 0$, with $r_1 + r_2 = 1$. As in the book of Hörmander [**4**], we take a disc $G^* \subset G$ and a polynomial $P(t)$ ($t \in \mathbf{C}$) such that $\log u \leq \operatorname{Re} P$ on ∂G^*, where $u = r_1 u_1 + r_2 u_2$. Then $u \leq \exp(\operatorname{Re} P)$ on ∂G^*. Since $\log u_j - \operatorname{Re} P$ is subharmonic in G^*, $r_j u_j |\exp(-P)|$ is also subharmonic in G^*. Hence $(r_1 u_1 + r_2 u_2)|\exp(-P)| = u|\exp(-P)|$ is subharmonic in G^*. From $u|\exp(-P)| \leq 1$ on ∂G^* and the maximum principle for the subharmonic function, we have $u|\exp(-P)| \leq 1$ on G^*, that is, $\log u \leq \operatorname{Re} P$ in G^*. Since G^* is an arbitrary disc in G, $\log u$ is subharmonic in G.

(2) follows directly from (1). □

The following lemma will be used in §4.

LEMMA 2.2. *Let $v(z)$ be a $\mathscr{C}^2$-function on a domain D with $0 < v < 1$.*

(1) *$1/v(z)$ is log.p.s.h. if and only if*

$$\left|\sum v_i \xi_i\right|^2 - v \sum v_{i\bar{j}} \xi_i \bar{\xi}_j \geq 0 \quad \textit{for all } \xi \in \mathbf{C}^n,$$

where $v_i = \partial v / \partial z_i$, $v_{\bar{j}} = \partial v / \partial \bar{z}_j$ and $v_{i\bar{j}} = \partial^2 v / \partial z_i \partial \bar{z}_j$.

(2) *The function $\phi(z) = 1 - v(z)^\beta$ (β is a positive constant) is log.p.s.h. if and only if*

$$\phi\left[\left|\sum v_i \xi_i\right|^2 - v \sum v_{i\bar{j}} \xi_i \bar{\xi}_j\right] \geq \beta \left|\sum v_i \xi_i\right|^2 \quad \textit{for all } \xi \in \mathbf{C}^n. \tag{2.3}$$

Proof. We show (2) only. It is clear that $(\log \phi)_{i\bar{j}} = (\phi_{i\bar{j}} \phi - \phi_i \phi_{\bar{j}}) \phi^{-2}$. We have

$$\phi_i = -\beta v^{\beta - 1} v_i \quad \text{and} \quad \phi_{i\bar{j}} = \beta v^{\beta-2}\big((1 - \beta) v_i v_{\bar{j}} - v v_{i\bar{j}}\big).$$

Therefore,

$$\phi_{i\bar{j}} \phi - \phi_i \phi_{\bar{j}} = \beta v^{\beta - 2}\left[(\phi - \beta) v_i v_{\bar{j}} - \phi v v_{i\bar{j}}\right].$$

Hence (2.3) is equivalent to the fact that the matrix $[(\log \phi)_{i\bar{j}}]$ is positive semidefinite. □

3. The intrinsic metrics. Let M be a complex manifold and TM its holomorphic tangent bundle. According to Kobayashi [**8**], we call a function $X = X_M(p, \xi)$ on TM a complex Finsler metric on M when it satisfies the following conditions;

(i) $X_M(p, \xi)$ is an upper semicontinuous positive function on TM,

(ii) $X_M(p, \lambda\xi) = |\lambda| X_M(p, \xi)$ for any $\lambda \in \mathbf{C}$.

The intrinsic metric is the biholomorphic invaraiant complex Finsler metric which is determined only by the complex analytic structure of the complex manifold. As examples of intrinsic metrics, there are the Carathéodory, the Kobayashi, and the Bergman metrics. In this section we give the basic properties of them and introduce a new intrinsic metric, that is, the P^*-metric (cf. [**10**] and [**14**]).

When M is a domain in $\mathbf{C}^n$, since $TD \cong D \times \mathbf{C}^n$, we may assume a tangent vector $\xi \in \mathbf{C}^n$. Let $(z_1, \ldots, z_n)$ be the canonical coordinates of $\mathbf{C}^n$, and let (p, ξ) denote a pair of TD.

The Carathéodory metric (C-metric) C_D of a domain D is given by

$$C_D(p, \xi) = \sup\left\{|\partial_\xi f(p)|; f \in \mathrm{Hol}(D, U), f(p) = 0\right\}$$

where $\partial_\xi f(p) = \sum_i \partial f/\partial z_i(p)\xi_i$.

The Kobayashi metric (K-metric) K_D of D is defined by

$$K_D(p, \xi) = \inf\{1/r;\, F \in \mathrm{Hol}(U, D),\, F(0) = p,\, F'(0) = r\xi,\, r > 0\}.$$

Though they are not always positive, we mainly work on hyperbolic domains. The following theorem is well known.

THEOREM 3.1 [**7**, **8**]. *Let D be a hyperbolic domain in $\mathbf{C}^n$.*

(1) *The C-metric is continuous on TD and the K-metric is upper semicontinuous on TD.*

(2) *The C and K-metrics are decreasing for any holomorphic mappings: for $F \in \mathrm{Hol}(D, E)$ (D, E are domains),*

$$C_E(F(p), F'(p)\xi) \le C_D(p, \xi),$$
$$K_E(F(p), F'(p)\xi) \le K_D(p, \xi).$$

N. Sibony [**10**] introduced the P-metric as follows:

$$P_D(p, \xi) = \sup\left\{L(u; p, \xi)^{1/2};\, u \in S_p(D)\right\},$$

where

$$L(u; p, \xi) = \sum \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}(p)\xi_i \bar{\xi}_j$$

is the Levi form of u. P_D is locally integrable, but its upper semicontinuity is unknown yet. We define

$$P_D^*(p, \xi) = \limsup P_D(q, \zeta) \quad \text{as } (q, \zeta) \to (p, \xi).$$

Then P_D^* is upper semicontinuous and $P_D \le P_D^*$ and $P_U^*(0, 1) = 1$ for the unit disc U in $\mathbf{C}$. P_D^* is called the P^*-metric of D.

LEMMA 3.2 (cf. [**10**], [**14**]). *The P^*-metric is an intrinsic metric having the decreasing property for holomorphic mappings, and $C_D \le P_D^* \le K_D$ on TD.*

Proof. The first half follows from the result of Sibony **[10]** (see also **[14]**). We give a simple proof for the last half. Noting that $S_p(D) \supset A_p(D)$ and $L(|f|^2; p, \xi) = |\partial_\xi f(p)|^2$, we have $C_D \leq P_D^*$. Taking a mapping F in $\mathrm{Hol}(U, D)$ with $F(0) = p$, $F'(0) = r\xi$ $(r > 0)$, from the decreasing property for F, we have

$$P_D^*(p, r\xi) = P_D^*(F(0), F'(0)) \leq P_U^*(0,1) = 1.$$

Therefore, $P_D^*(p, \xi) \leq 1/r$. Since the K-metric is the infimum of such $1/r$, we obtain the inequality $P_D^* \leq K_D$. □

Let $X_D(p, \xi)$ be one of the intrinsic metrics on a domain D. A point p in D is said to be an X-hyperbolic point if there is a neighborhood V of p and a constant $c > 0$ such that $X_D(q, \xi) \geq c\|\xi\|$ for all $\xi \in \mathbf{C}^n$ at every point q in V, where $\| \ \|$ is the Euclidean norm of $\mathbf{C}^n$. If each oint of D is X-hyperbolic point, then D is said to be an X-hyperbolic domain.

4. The generalized Schwarz lemma. Let D be a domain in $\mathbf{C}^n$. The reproducing kernel $k(z, \overline{w})$ of the Hilbert space $L^2H(D)$ of L^2-holomorphic functions on D is called the Bergman kernel of D. It is holomorphic in $D \times \overline{D}$ (where $\overline{D}$ is the complex conjugate of D.) Defining

$$B_D^2(z, \xi) = \sum \left(\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log k(z, \bar{z}) \right) \xi_i \bar{\xi}_j, \qquad \xi \in \mathbf{C}^n,$$

we have the Bergman metric B_D of D provided that the right side of the above is positive for all $\xi \neq 0$. The Bergman metric is a Kähler metric and an intrinsic metric, but does not always have the decreasing property for holomorphic mappings.

We call a domain D with $B_D \neq 0$ the Bergman domain.

For the upper semicontinuous Finsler metric, the holomorphic curvature is defined (cf. **[12]**) and coincides with the holomorphic sectional curvature if the metric is C^2-hermitian. Let U be the unit disc in $\mathbf{C}$ with the canonical metric $(1 - |t|^2)^{-2}|dt|^2$. The following lemma is a generalization of the Schwarz lemma (cf. [7, III, Theorem 2.1]) to the upper semicontinuous case.

LEMMA 4.1 (*cf.* **[14]**.) *Let $X_M(p, \xi)$ be a complex Finsler metric on a K-hyperbolic manifold M. If its holomorphic curvature is bounded from above by a negative constant $-c^2$ $(c > 0)$, then $X_M \leq 2c^{-1}K_M$, and for any holomorphic mapping F: $U \to M$, $X_M(F(t), F'(t)) \leq 2c^{-1}K_U(t, 1)$.*

Since the holomorphic curvature coincides with the holomorphic sectional curvature for the Bergman metric, we have the following lemma.

LEMMA 4.2. *Let D be a Bergman domain. If the holomorphic sectional curvature of the Bergman metric B_D of D is bounded from above by a negative constant $-k$, then $B_D \le 2k^{-1/2}K_D$, and for every holomorphic mapping $F\colon U \to D$,*

$$(*) \qquad B_D(F(t), F'(t)) \le (2/k)^{1/2} B_U(t, 1), \qquad t \in U.$$

Proof. The first half follows from Lemma 4.1. For any holomorphic mapping $F\colon U \to D$, we have

$$\begin{aligned} B_D(F(t), F'(t)) &\le 2k^{-1/2} K_D(F(t), F'(t)) \\ &\le 2k^{-1/2} K_U(t, 1) = (2/k)^{1/2} B_U(t, 1), \end{aligned}$$

since $\sqrt{2}\, K_U = B_U$. □

From the proof of this lemma, we see that if the Bergman metric is dominated by the Kobayashi metric, then any $F \in \mathrm{Hol}(U, D)$ is decreasing with respect to the Bergman metric (i.e. $(*)$ holds).

For a domain D in $\mathbf{C}^n$, we consider the following condition:

(#) $B_D \le cK_D$ on TD for some constant $c > 0$.

(A) If D is a bounded homogeneous domain, then C_D, K_D, P_D^* and B_D are all equivalent metrics because of their biholomorphic invariantness and the homogeneity of D. Especially, (#) is satisfied.

(B) If the Bergman metric of D has strictly negative holomorphic sectional curvature, then by Lemma 4.2, (#) is satisfied.

Here we shall give a function-theoretic criterion for (#). *Let D denote the Bergman domain in $\mathbf{C}^n$ with the Bergman kernel $k(z, \bar{w})$. For a fixed point w in D, we construct a function on D by*

$$\phi_w(z) = \phi_{w,\alpha}(z) = 1 - \left(\frac{|k(z, \bar{w})|^2}{k(z, \bar{z})k(w, \bar{w})} \right)^{\alpha},$$

where α is a positive constant properly chosen for D. When $\alpha = 1/2$, this is a square of the invariant distance $\rho_D(z, w)$ of Skwarczynski (cf. **[11]**).

LEMMA 4.3. *With the above notations, the following hold.*

(1) $0 \le \phi_w \le 1$ *and* $\phi_w(w) = 0$.

(2) *If* $k(z, \bar{w}) \ne 0$, *then* $L(\phi_w; w, \xi) = \alpha B_D^2(w, \xi)$.

(3) ϕ_w *is biholomorphically invariant, i.e. for any biholomorphic automorphism g of D,* $\phi_{g(w)}(g(z)) = \phi_w(z)$.

Proof. (1) is clear. (3) follows from the fact that

$$k\big(g(z), \overline{g(w)}\big) J_g(z) \overline{J_g(w)} = k(z, \bar{w}),$$

where J_g is the Jacobian determinant of g. It remains to show (2). By simple calculations we have

$$\begin{aligned}\partial^2\phi_w/\partial z_i\partial\bar{z}_j(w) &= \alpha k(w,\overline{w})^{-2}\{k_{i\bar{j}}(w,\overline{w})k(w,\overline{w}) - k_i(w,\overline{w})k_{\bar{j}}(w,\overline{w})\}\\ &= \alpha\big(\partial^2\log k/\partial z_i\partial\bar{z}_j\big)(w).\end{aligned}$$

Hence, $L(\phi_w; w, \xi) = \alpha B_D^2(w, \xi)$. □

A domain D in $\mathbf{C}^n$ is called a Lu Qi-keng domain if $k(z,\overline{w}) \neq 0$ for all z, w in D.

THEOREM 4.4. *Let D be a Bergman domain in $\mathbf{C}^n$. If there is positive constant α such that $\phi_w = \phi_{w,\alpha}$ belongs to $S_w(D)$ for each w in D, then D is a Lu Qi-keng domain and*

$$B_D(w,\xi) \leq \alpha^{-1/2}P_D^*(w,\xi) \leq \alpha^{-1/2}K_D(w,\xi) \quad \textit{on } TD.$$

Proof. When ϕ_w belongs to $S_w(D)$ for each w in D, ϕ_w is a p.s.h. function. Hence, by the maximum principle, $0 \leq \phi_w < 1$ in D, that is, $k(z,w) \neq 0$ for all z, w in D. So D is a Lu Qi-keng domain. From the definition of P^*-metric and Lemma 4.3(2), we have

$$\alpha^{1/2}B_D \leq P_D^* \leq K_D \quad \text{on } TD.$$ □

Using Lemma 2.2 we can rewrite the condition that ϕ_w belongs to $S_w(D)$ for each w in D. For a fixed point w in D, let $v(z) = |k(z,\overline{w})|^2/k(z,\bar{z})k(w,\overline{w})$. We remark that $k(z,\overline{w})$ is holomorphic in $z \in D$. We set

$$\begin{aligned}A_w &= \{z \in D; v(z) = 0\} = \{z \in D; k(z,\overline{w}) = 0\},\\ E_w &= \{z \in D; v(z) = 1\} = \{z \in D; \phi_w(z) = 0\},\\ D_w &= D\setminus(A_w \cup E_w).\end{aligned}$$

Then A_w is an analytic set in D, and E_w contains at least w.

PROPOSITION 4.5. *For fixed w in D, $\phi_w = 1 - v^\alpha$ is log.p.s.h. if and only if the following inequality holds*:

$$\phi_w(z)B_D^2(z,\xi) \geq \alpha\left|\partial_\xi \log v(z)\right|^2 \tag{4.6}$$

for all $\xi \in \mathbf{C}^n - \{0\}$ at all $z \in D_w$.

Proof. Since $0 < v < 1$ on D_w, we may verify that (2.3) in Lemma 2.2 is equivalent to (4.6) (we set $\beta = \alpha$). Assume that this was proved. Then

$\log \phi_w$ is p.s.h. in D_w. For z in E_w, we set $\log \phi_w(z) = -\infty$. Hence $\log \phi_w$ is p.s.h. in $D \setminus A_w$. Remarking that $\log \phi_w$ is negative in D and A_w is the analytic set in D, we can extend $\log \phi_w$ plurisubharmonically to D (by the p.s.h. extension theorem of Grauert and Remmert). Thus it remains to show that (2.3) is equivalent to (4.6) for each w in D. For a fixed w in D, we write $\phi = \phi_w$, $k = k(z, \bar{z})$, $h = k(z, \bar{w})$, $a = k(w, \bar{w})$. Then $v = h\bar{h}/ak$, $\phi = 1 - (h\bar{h}/ak)^\alpha$. By partial differentiation we have the following:

$$v_i = \bar{h}(h_i k - h k_i)/ak^2,$$

$$v_{i\bar{j}} = \left\{h_i \bar{h}_{\bar{j}} k^2 - \left(\bar{h} h_i k_{\bar{j}} + h\bar{h}_{\bar{j}} + h\bar{h} k_{i\bar{j}}\right)k + 2h\bar{h}k_i k_{\bar{j}}\right\}/ak^3.$$

Substituting these in

$$A \equiv \phi\left|\sum v_i \xi_i\right|^2 - \phi v \sum v_{i\bar{j}} \xi_i \bar{\xi}_j - \alpha\left|\sum v_i \xi_i\right|^2,$$

we get

$$\begin{aligned} A &= \phi v^2 \left\{k \sum k_{i\bar{j}} \xi_i \bar{\xi}_j - \left|\sum k_i \xi_i\right|^2\right\}/k^2 - \alpha v^2 \left|\sum \frac{v_i}{v} \xi_i\right|^2 \\ &= v^2\left\{\phi B_D^2(z, \xi) - \alpha\left|\partial_\xi \log v\right|^2\right\}. \end{aligned}$$

Therefore (2.3) holds if and only if (4.6) holds because of $0 < v < 1$ on D_w. □

THEOREM 4.7. *If the Bergman metric B_D of a Bergman domain D satisfies the following condition: for each w in D there is a positive constant α such that* (4.6) *holds, then $B_D \le \alpha^{-1/2} K_D$. Hence every holomorphic mapping $F\colon U \to D$ satisfies $F^* B_D \le (2\alpha)^{-1/2} B_U$.*

Proof. By Proposition 4.5 we have $\phi_w \in S_w(D)$ for each w in d. From Theorem 4.4 the conclusion is obtained.

Also, for any $F \in \mathrm{Hol}(U, D)$.

$$\begin{aligned} B_D\left(F(t), F'(t)\right) &\le \alpha^{-1/2} K_D\left(F(t), F'(t)\right) \\ &\le \alpha^{-1/2} K_U(t, 1) = (2\alpha)^{-1/2} B_U(t, 1), \end{aligned}$$

since $B_U = 2^{1/2} K_U$. □

REMARKS. (1) E_w is the polar set of $\log \phi_w$. If the coordinate functions $z_1, \ldots, z_n$ are in $L^2H(D)$ and the volume of D is finite, then $E_w = \{w\}$.

(2) There exist the bounded homogeneous domains in $\mathbf{C}^n$ $(n \ge 7)$ of which the Bergman metrics have positive holomorphic sectional curvatures (cf. D'Atri **[2]**).

(3) There is a bounded pseudoconvex domain in $\mathbf{C}^3$ with C^∞-boundary, which does not satisfy condition (#) (cf. Diederich-Fornaess [**3**]).

(4) The annulus $A = \{t \in \mathbf{C};\ r < |t| < 1\}$ is not the Lu Qi-keng domain. Therefore $\log \phi_w$ is not p.s.h. for some $w \in A$ (cf. Skwarczynski [**11**]).

5. The classical domains. In this section we construct the function ϕ_w for the classical domains (the bounded symmetric domains of four main type) R and directly verify that they belong to $S_w(R)$ for some properly chosen α.

We begin with the unit ball.

EXAMPLE 5.1. Let D be the unit ball $B_n = \{z \in \mathbf{C}^n;\ \|z\| < 1\}$. Then its Bergman kernel is $k(z, \bar{w}) = c_n(1 - z \cdot \bar{w})^{-n-1}$, where $c_n = n!\pi^{-n}$. Let 0 be the origin of $\mathbf{C}^n$. Taking $\alpha = 1/(n+1)$ we have $\phi_0(z) = \|z\|^2$ for $w = 0$. It is clear that $\phi_0 \in S_0(B_n)$. For any other point w in B_n, taking an automorphism g of B_n with $g(0) = w$, we have $\phi_w(z) = \phi_0(g(z)) = \|g(z)\|^2$, which belongs to $S_w(B_n)$, and

$$1 - \left(1 - \|z\|^2\right)\left(1 - \|w\|^2\right)/|1 - z \cdot \bar{w}^2| = \|g(z)\|^2.$$

The last formula is well known (cf. Rudin [**9**], p. 26).

The four classical domains R_{I}, R_{II}, R_{III}, and R_{IV}, are given as follows ($M(m, n)$ denotes the set of all $m \times n$ matrices):

$$R_{\mathrm{I}} = \{Z \in M(m, n);\ I_n - Z^*Z > 0\},$$

$$R_{\mathrm{II}} = \{Z \in M(n, n);\ Z' = Z, I_n - Z^*Z > 0\},$$

$$R_{\mathrm{III}} = \{Z \in M(n, n);\ Z' = -Z, I_n - Z^*Z > 0\},$$

$$R_{\mathrm{IV}} = \left\{z \in \mathbf{C}^n;\ \left|\sum z_i^2\right|^2 + 1 - 2\|z\|^2 > 0, \left|\sum z_i^2\right| < 1\right\},$$

where I_n is the $n \times n$ unit matrix and Z' is the transpose of Z and $Z^* = \bar{Z}'$.

Let R or R_j denote one of these domains, and 0 be the zero matrix or the origin of $\mathbf{C}^n$.

PROPOSITION 5.2. (1) *For R_{I}, choosing $\alpha = 1/(m+n)n$, we set*

$$\phi_0(Z) = 1 - \left(\det(I_n - Z^*Z)\right)^{1/n},$$

(2) *For R_{II}, choosing $\alpha = 1/(n+1)n$, we set*

$$\phi_0(Z) = 1 - \left(\det(I_n - Z^*Z)\right)^{1/n},$$

(3) *For R_{III}, choosing $\alpha = 1/(n-1)[n/2]$, we set*

$$\phi_0(Z) = 1 - \left(\det(I_n - Z^*Z)\right)^{[n/2]},$$

(4) *For R_{IV}, choosing $\alpha = 1/2n$, we set*

$$\phi_0(z) = 1 - \left(1 + \left|\sum z_i^2\right|^2 - 2\|z\|^2\right)^{1/2}.$$

Then each ϕ_0 is log.p.s.h., hence it belongs to $S_0(R)$; also $\phi_w(Z) = \phi_0(g(Z))$ belongs to $S_w(R)$ for any point w in R, where g is an automorphism of R with $g(w) = 0$.

We quote two lemmas to prove this proposition.

LEMMA 5.3 ([7], p. 34). *Let B_j be the Bergman metric of R_j ($j = $ I, II, III, IV). Then*

$$B_{\mathrm{I}}^2(Z, \xi) = (m+n)\mathrm{Trace}\left[(I_n - Z^*Z)^{-1}\xi(I_m - ZZ^*)^{-1}\xi^*\right],$$
$$B_{\mathrm{II}}^2(Z, \xi) = (n+1)\mathrm{Trace}\left[(I_n - Z^*Z)^{-1}\xi(I_n - ZZ^*)^{-1}\xi^*\right],$$
$$B_{\mathrm{III}}^2(Z, \xi) = (n-1)\mathrm{Trace}\left[(I_n - Z^*Z)^{-1}\xi(I_n - ZZ^*)^{-1}\xi^*\right],$$
$$B_{\mathrm{IV}}^2(z, \xi) = 2nA^{-2}\xi\left[A(I_n - 2z'\bar{z}) + 2(I_n - z'\bar{z})z^*z(I_n - z'\bar{z})\right]\xi^*,$$

where

$$A = \left|\sum z_i^2\right|^2 + 1 - 2\|z\|^2.$$

LEMMA 5.4 ([5]). (1) *For $Z \in R_{\mathrm{I}}$, there exists an element g of G_0 (the isotropy subgroup of automorphism group of R_{I}) such that*

$$gZ = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \\ & 0 & \end{pmatrix}, \qquad 1 > \lambda_1 \geq \cdots \geq \lambda_n \geq 0, m \geq n.$$

(2) *For $Z \in R_{\mathrm{II}}$, there is a $g \in G_0$ such that*

$$gZ = \mathrm{diagonal}[\lambda_1, \ldots, \lambda_n], \qquad 1 > \lambda_1 \geq \cdots \geq \lambda_n \geq 0.$$

(3) *For $Z \in R_{\mathrm{III}}$, there is a $g \in G_0$ such that*

$$gZ = \mathrm{diagonal}\left[\begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & \lambda_k \\ -\lambda_k & 0 \end{pmatrix}, (0)\right],$$

where $k = [n/2]$, and the last term is 0 if n is odd.

(4) *For* $z \in R_{\mathrm{IV}}$, *there is a* $g \in G_0$ *such that*

$$gz = (\lambda, i\mu, 0, \dots, 0), \qquad \lambda, \mu \text{ real numbers}.$$

Proof of Proposition 5.2. Since

$$\phi_0(Z) = 1 - \left(|k(Z,0)|^2 / \left(k(Z,\bar{Z})k(0,0)\right)\right)^{\alpha}$$

and $k(Z,0)$ is a constant, each ϕ_0 is as above. (For the Bergman kernel of the classical domains, see, for example, [7, p. 34].) We directly verify (4.6) for the domain R_{I} $(m \geq n)$. Note that R_{I} is a Lu Qi-keng domain; thus A_0 is the empty set and E_0 is point 0 only. From the biholomorphic invariantness of ϕ_0 and Lemma 5.4, it is sufficient to show (4.6) at the point

$$Z_0 = \begin{pmatrix} \lambda_1 & & & \\ & & 0 & \\ & \ddots & & \\ & 0 & & \\ & & & \lambda_n \\ & & 0 & \end{pmatrix} \qquad (1 > \lambda_1 \geq 0).$$

$$V(Z_0)^{\alpha} = \prod_1^n \left(1 - \lambda_j^2\right)^{1/n};$$

thus

$$\phi_0(Z_0) = 1 - \prod_1^n \left(1 - \lambda_j^2\right)^{1/n}.$$

From Lemma 5.3 we have

$$B_D^2(Z_0, \xi) = (m+n)\left[\sum_{i,j}^{n} \frac{|\xi_{ij}|^2}{\left(1-\lambda_i^2\right)\left(1-\lambda_j^2\right)} + \sum_{i=1}^{n} \sum_{k>n}^{m} \frac{|\xi_{ki}|^2}{1-\lambda_i^2}\right].$$

By partial differentiation

$$\frac{\partial}{\partial z_{\mu\nu}} \det(I_n - Z^*Z) = (-1)^{\mu+1} \delta_{\mu\nu} \lambda_{\mu} \prod_{i \neq \mu} \left(1 - \lambda_i^2\right) \quad \text{at } Z = Z_0.$$

Hence,

$$\begin{aligned} \alpha \left|\partial_{\xi} \log v(Z_0)\right|^2 &= \frac{m+n}{n} \left|\partial_{\xi} \log \det\left(I_n - Z_0^* Z_0\right)\right|^2 \\ &= \frac{m+n}{n} \left|\sum_{\mu,\nu} \frac{\partial}{\partial z_{\mu\nu}} \left(\det\left(I_n - Z_0^* Z_0\right) \xi_{\mu\nu}\right)\right|^2 \left|\det\left(I_n - Z_0^* Z_0\right)\right|^{-2} \\ &= \frac{m+n}{n} \left|\sum_{\mu} \frac{\lambda_{\mu}}{1 - \lambda_{\mu}^2} \xi_{\mu\mu}\right|^2. \end{aligned}$$

Let $T(Z_0)$ be the difference of the left and right sides of (4.6) at Z_0. Then

$$T(Z_0) = (m+n)\left(1 - \prod_j \left(1-\lambda_j^2\right)^{1/n}\right)$$

$$\cdot\left(\sum_{i,j} \frac{|\xi_{ij}|^2}{\left(1-\lambda_i^2\right)\left(1-\lambda_j^2\right)} + \sum_i \sum_{k>n} \frac{|\xi_{ki}|^2}{1-\lambda_i^2}\right)$$

$$- \frac{m+n}{n}\left|\sum_\mu \frac{\lambda_\mu}{1-\lambda_\mu^2}\xi_{\mu\mu}\right|^2 .$$

Noting that $\prod(1-\lambda_j^2)^{1/n} \le 1 - (1/n)\sum\lambda_j^2$, we have

$$T(Z_0) \ge \frac{m+n}{n}\left(\sum\lambda_j^2\right)\left(\sum|\xi_{ij}|^2\left(1-\lambda_i^2\right)^{-1}\left(1-\lambda_j^2\right)^{-1}\right)$$

$$- \frac{m+n}{n}\left|\sum\lambda_\mu\xi_{\mu\mu}\left(1-\lambda_\mu^2\right)^{-1}\right|^2 \ge 0$$

by Schwarz' inequality.

It is the same for (2) and (3) as (1). For R_{IV} we may verify (2.3) directly at $(\lambda, i\mu, 0, \dots, 0)$. The proof is reduced to the following inequality;

$$\begin{aligned}(\phi - 1/2)&\left[\lambda^2(\lambda^2-\mu^2-1)^2|\xi_1|^2 + \mu^2(\lambda^2-\mu^2+1)|\xi_2|^2\right.\\ &\left.+ i\lambda\mu(\lambda^2-\mu^2-1)(\lambda^2-\mu^2+1)\left(\bar{\xi}_1\xi_2 - \xi_1\bar{\xi}_2\right)\right]\\ &-2\phi v\left[(2\lambda^2-1)|\xi_1|^2 + (2\mu^2-1)|\xi_2|^2 + 2i\lambda\mu\left(\bar{\xi}_1\xi_2 - \xi_1\bar{\xi}_2\right)\right]\\ &+2\phi v\left(\sum_3^n|\xi_i|^2\right)\\ &\ge 0 \quad \text{for all } \xi \in \mathbf{C}^n,\end{aligned}$$

where $v = |\lambda^2-\mu^2|^2 + 1 - 2(\lambda^2+\mu^2)$ $(0 < v < 1)$, $\phi = 1 - v^{1/2}$, and $|\lambda^2 - \mu^2| < 1$. Simple but long calculations show that the above inequality holds. □

COROLLARY 5.5. $B_{\mathrm{I}}^2 \le (m+n)nK_{\mathrm{I}}^2$, $B_{\mathrm{II}}^2 \le (n+1)nK_{\mathrm{II}}$,

$$B_{\mathrm{III}}^2 \le (n-1)[n/2]K_{\mathrm{III}}^2, \qquad B_{\mathrm{IV}}^2 \le 2nK_{\mathrm{IV}}^2.$$

Proof. From Theorem 4.4 and Proposition 5.2, we have, for example,

$$B_{\mathrm{I}}^2(0, \xi) \leq (m+n)nK_{\mathrm{I}}^2(0, \xi) \quad \text{for all } \xi \in \mathbf{C}^n.$$

Using the homogeneity of R_j and invariantness of the Bergman and Kobayashi metrics, we have the conclusion. □

REMARKS. For the classical domains R_j, C, P^*, and K-metrics all coincide. The K-metric $K_{\mathrm{I}}(0, \xi)$ is given by

$$K_{\mathrm{I}}^2(0, \xi) = \max\{\text{eigenvalues of } \xi^*\xi\}$$

(cf. [**13**]). Hence,

$$K_{\mathrm{I}}^2(0, \xi) \geq n^{-1}\operatorname{Trace} \xi^*\xi = (m+n)^{-1}n^{-1}B_{\mathrm{I}}^2(0, \xi) = \alpha B_{\mathrm{I}}^2(0, \xi).$$

On the other hand, the holomorphic sectional curvature κ of B_{I}^2 satisfies $-4/(m+n) \leq \kappa \leq -4/(m+n)n$, hence the constant k in Lemma 4.2 is equal to 4 α and we have $\alpha B_{\mathrm{I}}^2 \leq K_{\mathrm{I}}^2$. For R_{II} and R_{III}, we can do similarly. With respect to R_{IV},

$$K_{\mathrm{IV}}^2(0, \xi) = \|\xi\|^2 + \left(\|\xi\|^4 - |\xi'\xi|^2\right)^{1/2} \geq \|\xi\|^2 = 1/2nB_{\mathrm{IV}}^2(0, \xi),$$

and the holomorphic sectional curvature κ of B_{IV}^2 satisfies $-4/n \leq \kappa \leq -2/n$. Hence $\alpha B_{\mathrm{IV}}^2 \leq K_{\mathrm{IV}}^2$ $(\alpha = 1/2n)$ (cf. [**13**].)

REFERENCES

[1] J. Burbea, *On metrics and distorsion theorem*, Ann. of Math. Studies. No. 100 (1981), Princeton Univ. Press, 65–92.

[2] J. D'Atri, *Holomorphic sectional curvature of bounded homogeneous domains and related questions*, Trans. Amer. Math. Soc., **256** (1979), 405–412.

[3] K. Diederich and J. Fornaess, *Comparison of the Bergman and Kobayashi metric*, Math. Ann., **254** (1980), 257–260.

[4] L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, Van Nostrand (1966).

[5] L. K. Hua, *On the theory of automorphic functions of a matrix variable*, Amer. J. Math., **66** (1944), 470–480.

[6] S. Kobayashi, *Geometry of bounded domains*, Trans. Amer. Math. Soc., **92** (1959), 267–290.

[7] ______, *Hyperbolic manifolds and holomorphic mappings*, Pure and Appl. Math., **2**, M. Dekker (1970).

[8] ______, *Intrinsic distances, measures and geometric function theory*, Bull. Amer. Math. Soc., **82** (1976), 357–416.

[9] W. Rudin, *Function Theory in the Unit Ball of* $\mathbf{C}^n$, Grundlehren der Math. Wiss. 241, Springer Verlag, 1980.

[10] N. Sibony, *A class of hyperbolic manifolds*, Ann. of Math., Studies No. 100, Princeton Univ. Press.

[11] M. Skwarczynski, *Biholomorphic invariants related the Bergman functions*, Dissertationes Math., **173** (1980).

[12] M. Suzuki, *The holomorphic curvature of intrinsic metrics*, Math. Rep. Toyama Univ., **4** (1981), 107–114.

[13] _____, *The intrinsic metrics on the circular domains in* $\mathbf{C}^n$, Pacific J. Math., **112** (1984), 249–256.

[14] _____, *The intrinsic metrics on the domains in* $\mathbf{C}^n$, Math. Rep. Toyama Univ., **6** (1983).

Received July 18, 1983.

FACULTY OF SCIENCE
TOYAMA UNIVERSITY
GOFUKU, TOYAMA
JAPAN