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SOME PROPERTIES OF ALMOST RIMCOMPACT SPACES

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A 0-space is a completely regular Hausdorff space possessing a compactification with zero-dimensional remainder. In a previous paper the class of almost rimcompact spaces was introduced and shown to be intermediate between the classes of rimcompact spaces and 0-spaces. In this paper some properties of almost rimcompact spaces and of 0-spaces are developed. If X is a space whose non-locally compact part has compact boundary, then X is a 0-space if and only if X is almost rimcompact. Neither perfect images or perfect preimages of rimcompact spaces need be 0-spaces. However, if the perfect preimage of an almost rimcompact space is a 0-space, then that perfect preimage is almost rimcompact. Subspaces and products are considered.

1. Introduction and known results. The characterization of those completely regular Hausdorff spaces possessing a compactification with zero-dimensional remainder has been considered by various researchers (see for example [7], [8] and [10]). Such a compactification will be called *0-dimensional at infinity* (denoted by O.I.); a *0-space* is any space possessing a O.I. compactification. Recall that a space is *rimcompact* if it has a basis of open sets with compact boundaries ([7]). Each rimcompact space X possesses a compactification which has a basis of open sets whose boundaries are contained in X ([9], [10]), hence X is a 0-space; the converse is not true ([10]). In [2] we introduced a natural generalization of rimcompactness called almost rimcompactness and obtained the following characterization: a space X is almost rimcompact if and only if X possesses a compactification KX in which each point of $KX \setminus X$ has a basis (in KX) of open sets whose boundaries are contained in X. (If KX is such a compactification of X, we say that $KX \setminus X$ is *relatively 0-dimensionally embedded* in KX.) Hence each almost rimcompact space is a 0-space; we showed in [2] that the converse is not true.

In this paper we discuss the properties of almost rimcompact spaces and of 0-spaces. In §2 we show that if the non-locally compact part of X has compact boundary, then X is a 0-space if and only if X is almost rimcompact. Such a space need not be rimcompact. In §3 we show that any closed subspace of a 0-space (respectively, almost rimcompact space) is a 0-space (respectively, almost rimcompact). This statement does not hold for open subspaces. In §4 we indicate that neither perfect images nor perfect preimages of rimcompact spaces need be 0-spaces. However, if the perfect preimage of an almost rimcompact space is a 0-space, then that perfect preimage is almost rimcompact.

In the remainder of this section, we present our notation and terminology and some known results. All spaces are assumed to be completely regular and Hausdorff. The notions used from set theory are standard. The symbol ω_α is used to denote the αth cardinal. For any set X, $|X|$ denotes the cardinality of X. A *map* is a continuous surjection. A function $f\colon X \to Y$ is *closed* is whenever F is a closed subset of X, then $f[F]$ is a closed subset of Y. A closed function $f\colon X \to Y$ is *perfect* if for each $y \in Y$, $f^{\leftarrow}(y)$ is compact.

The family $\mathscr{K}(X)$ of (equivalence classes of) compactifications of X is partially ordered in the usual way: $JX \leq KX$ if there is a map $f\colon KX \to JX$ such that $f(x) = x$ for all $x \in X$; KX is *equivalent* to JX if f is a homeomorphism. For background information on compactifications the reader is referred to [**1**] or [**4**]. The maximum element of $K(X)$, the *Stone-Čech compactification* of X, is denoted by βX. In the sequel, if $KX \in \mathscr{K}(X)$, the natural map from βX into KX is denoted by Kf.

The following is an easy consequence of 3.2.1 of [**3**].

1.1. Proposition (*Taimanov's theorem*). *Let KX and KY be compactifications of X and Y respectively, and f be a map from X into Y. There is a map $f'\colon KX \to KY$ such that $f'|_X = f$ if and only if for $A, B \subset Y$, $\mathrm{Cl}_{KY}A \cap \mathrm{Cl}_{KY}B = \varnothing$ implies $\mathrm{Cl}_{KX}f^{\leftarrow}[A] \cap \mathrm{Cl}_{KX}f^{\leftarrow}[B] = \varnothing$.*

The next result follows from 1.5 of [**6**].

1.2. Proposition. *Let X, Y, KX, KY and f be as in* 1.1. *If f is perfect, and if f' exists, then $f'(KX \setminus X) = KY \setminus Y$.*

We often call $KX \setminus X$ the *remainder* of KX. For any space X, the *residue* of X (denoted by $R(X)$) is the set of points at which X is not locally compact. If KX is any compactification of X, then $\mathrm{Cl}_{KX}(KX \setminus X) = R(X) \cup (KX \setminus X)$.

The first of the following results combines Theorems 1 and 4 of [**5**]; the second is 6.7 of [**4**].

1.3. PROPOSITION. *Let* $\{X_\alpha: \alpha \in A\}$ *be a set of pseudocompact spaces. Then*:

(i) *If* $\prod\{X_\alpha: \alpha \in A\}$ *is pseudocompact, then* $\beta[\prod\{X_\alpha: \alpha \in A\}] = \prod\{\beta X_\alpha: \alpha \in A\}$.

(ii) *If* X_α *is locally compact for all but one* $\alpha \in A$, *then* $\prod\{X_\alpha: \alpha \in A\}$ *is pseudocompact.*

1.4. PROPOSITION. *If* X *is any space, and* $X \subset T \subset \beta X$, *then* $\beta T = \beta X$.

If U is an open subset of X, and $\delta X \in \mathscr{K}(X)$, then $Ex_{\delta X}U$ is defined to be $\delta X \setminus \mathrm{Cl}_{\delta X}(X \setminus U)$. The set $Ex_{\delta X}U$ is often called the *extension* of U in δX. It is an easy exercise to verify (i), (ii), and (iii) of the following proposition. Statement (iv) is implicit in the proof of Lemma 2 of [**10**].

1.5. PROPOSITION. *Let* $\delta X \in \mathscr{K}(X)$.

(i) *If* W *is open in* δX, *then* $W \subset Ex_{\delta X}(W \cap X)$.

(ii) *If* U *and* V *are open in* X, *then* $Ex_{\delta X}(U \cap V) = (Ex_{\delta X}U) \cap (Ex_{\delta X}V)$.

(iii) *If* U *is open in* X, *then* $(Ex_{\delta X}U) \cap X = U$, *hence* $\mathrm{Cl}_{\delta X}U = \mathrm{Cl}_{\delta X}Ex_{\delta X}U$.

(iv) *If* U *and* V *are open in* X, *then*

$$Ex_{\delta X}(U \cup V) \setminus (Ex_{\delta X}U \cup Ex_{\delta X}V) \subset \mathrm{Cl}_{\delta X}U \cap \mathrm{Cl}_{\delta X}V.$$

If U is any open subset of X, then it follows from 1.5(i) that $Ex_{\delta X}U$ is the largest open subset of δX whose intersection with X is the set U. The collection $\{Ex_{\delta X}U: U$ is an open subset of $X\}$ of open sets of δX is easily seen to be a basis for the topology of δX.

If $B \subset X$, the *boundary* of B in X, denoted by $\mathrm{bd}_X B$, is defined to be the set $\mathrm{Cl}_X B \cap \mathrm{Cl}_X(X \setminus B)$. A compactification δX of X is a *perfect compactification* of X if for each open subset of U of X, $\mathrm{Cl}_{\delta X}(\mathrm{bd}_X U) = \mathrm{bd}_{\delta X}(Ex_{\delta X}U)$. According to the corollary to Lemma 1 of [**10**], βX is a perfect compactification of X.

The equivalence of (i), (ii), and (iii) of the following proposition appear in Theorems 1 and 2 of [**10**].

1.6. PROPOSITION. *Let* $\delta X \in \mathscr{K}(X)$. *The following are equivalent.*

(i) δX *is a perfect compactification of* X.

(ii) *If* U *and* V *are disjoint open sets of* X, *then* $Ex_{\delta X}(U \cup V) = Ex_{\delta X}U \cup Ex_{\delta X}V$.

(iii) *For each* $p \in \delta X$, $(\delta f)^{\leftarrow}(p)$ *is a connected subset of* βX.

The *connected component* C_x of $x \in X$ is the union of all connected subspaces of X containing x. A space X is *totally disconnected* if $C_x = \{x\}$ for each $x \in X$. The *quasi-component* of $x \in X$ is the intersection of all closed-and-open (denoted *clopen*) subsets of X containing x. A space X is *zero-dimensional* (denoted 0-dimensional) if X has a basis of clopen sets. A space X is *strongly* 0-*dimensional* if any two disjoint zerosets of X are contained in disjoint clopen subsets of X.

For a detailed discussion of the disconnectedness of remainders of compactifictions see [**2**]. Any 0-space X has a maximum O.I. compactification (which we denote by $F_0 X$) which is also a minimum perfect compactification of X. For each $p \in F_0 X \setminus X$, the set $(F_0 f)^{\leftarrow}(p)$ is the connected compact quasi-component (in $\beta X \setminus X$) of each element of $(F_0 f)^{\leftarrow}(p)$.

The maximum O.I. compactification of a rimcompact space X is called the Freudenthal compactification of X, and is denoted by FX. If X is 0-dimensional then $FX = \beta_0 X$, where $\beta_0 X$ denotes the maximum 0-dimensional compactification of X.

Following the terminology of [**9**] and [**10**], we say that an open set U of X is π-*open* in X if $\mathrm{bd}_X U$ is compact. The intersection and union of finitely many π-open sets are π-open, as is the complement of the closure of a π-open set.

1.7. DEFINITIONS. (i) If $F_1, F_2 \subset X$, then F_1 and F_2 are π-*separated* in X if there is a π-open set U of X such that $F_1 \subset U$, and $\mathrm{Cl}_X U \cap F_2 = \varnothing$. We shall often write "$\{x\}$ and F are π-separated" as "x and F are π-separated". We say that F_1 is π-*contained* in $X \setminus F_2$ if F_1 and F_2 are π-separated.

(ii) If F is closed in X, U is open in X, and $F \subset U$, then F is *nearly* π-*contained* in U if there is a compact subset K of F so that whenever F' is a closed subset of F, and $F' \cap K = \varnothing$, F' is π-contained in U.

(iii) A space X is *nearly rimcompact* if whenever U is open in X, and $x \in U$, there is an open set W of X such that $x \in W$ and $\mathrm{Cl}_X W$ is nearly π-contained in U.

(iv) A space X is *quasi-rimcompact* if for any $x \in X$, there is a compact set K_x of X, so that whenever F is a closed subset of X and $F \cap K_x = \varnothing$, then x and F are π-separated.

(v) A space X is *almost rimcompact* if X is nearly rimcompact and quasi-rimcompact.

The following is 2.18 of [**2**].

1.8. THEOREM. *For any space X, the following are equivalent.*

(i) *X is almost rimcompact.*

(ii) *X is a 0-space, and F_0X has relatively 0-dimensionally embedded remainder.*

(iii) *X has a compactification with relatively 0-dimensionally embedded remainder.*

(iv) *X is quasi-rimcompact, and has a compactification with totally disconnected remainder.*

The following is justified in 3.5 of [**2**].

1.9. EXAMPLE. Let Y be any 0-dimensional non-strongly 0-dimensional space, and let KY be any perfect compactification of Y. If $X = (KY \times (\omega_1 + 1)) \setminus (Y \times \{\omega_1\})$, then X is almost rimcompact. X is rimcompact if and only if $KY = \beta_0 Y$.

2. 0-spaces whose residues have compact boundary. We begin by listing some straightforward results concerning π-open subsets of X and related open subsets of compactifications of X.

2.1. DEFINITION. Let $KX \in \mathscr{K}(X)$, and let W be open in KX. If $\mathrm{bd}_{KX} W \subset X$, W is said to be a *small boundary* (denoted by sb) subset of KX.

2.2. LEMMA. *Let $KX \in \mathscr{K}(X)$.*

(i) *The intersection (union) of finitely many sb open subsets of KX is an sb open subset of KX.*

If W is an sb open subset of KX, then

(ii) *$W \cap X$ is π-open in X.*

(iii) *$W = Ex_{KX}(W \cap X)$.*

(iv) *$KX \setminus \mathrm{Cl}_{KX} W$ is sb in KX.*

(v) *If U is π-open in X, and KX is a perfect compactification of X, then $\mathrm{Cl}_{KX} U \cap (KX \setminus X) = Ex_{KX} U \cap (KX \setminus X)$; that is, $Ex_{KX} U$ is an sb open subset of KX.*

The straightforward proof of 2.2 is left to the reader.

We consider separately the cases where X is nowhere locally compact, and where X has compact residue.

2.3. LEMMA. *Suppose that X is nowhere locally compact, and that KX is a O.I. compactification of X. Then $KX \setminus X$ is relatively 0-dimensionally embedded in KX.*

Proof. Suppose that $p \in KX \setminus X$, and that $p \in W$, where W is an open subset of KX. Since $KX \setminus X$ is 0-dimensional, there is a clopen subset V of $KX \setminus X$ such that $p \in V \subset W \cap (KX \setminus X)$, and $\mathrm{Cl}_{KX} V \subset W$. Let U be any open subset of KX such that $U \cap (KX \setminus X) = V$. Since $KX \setminus X$ is dense in KX, $\mathrm{Cl}_{KX} U = \mathrm{Cl}_{KX}(U \cap (KX \setminus X)) = \mathrm{Cl}_{KX} V$. Then

$$(\mathrm{Cl}_{KX} U) \cap (KX \setminus X) = \mathrm{Cl}_{KX} V \cap (KX \setminus X) = \mathrm{Cl}_{KX \setminus X} V = V.$$

It follows that $\mathrm{bd}_{KX} U = \mathrm{Cl}_{KX} U \setminus U \subset X$, hence U is an sb open subset of KX. Since $\mathrm{Cl}_{KX} U \subset W$, p has a basis in KX of sb open sets of KX. Thus $KX \setminus X$ is relatively 0-dimensionally embedded in KX. □

We make the following easily proved result explicit.

2.4. LEMMA. *Suppose that S, T are closed subsets of X, and that $S \cap (T \cup R(X)) = \varnothing$. If S is compact, then there is an open set U of X such that $\mathrm{Cl}_X U$ is compact, $S \subset U$, and $T \cap \mathrm{Cl}_X U = \varnothing$.*

2.5. LEMMA. *Let X be a space, and let $KX \in \mathscr{K}(X)$. Suppose that T is a closed subset of KX, that W is a compact clopen subset of $\mathrm{Cl}_{KX}(KX \setminus X)$ and that $T \cap W = \varnothing$. Then there is an sb open set U of KX such that $\mathrm{bd}_{KX} U \subset X \setminus R(X)$, $W = U \cap \mathrm{Cl}_{KX}(KX \setminus X)$, and $T \cap \mathrm{Cl}_{KX} U = \varnothing$.*

Proof. If W is a compact clopen subset of $\mathrm{Cl}_{KX}(KX \setminus X)$, then $W' = \mathrm{Cl}_{KX}(KX \setminus X) \setminus W$ is a compact clopen subset of $\mathrm{Cl}_{KX}(KX \setminus X)$. There are disjoint open sets U_1, U_1' of KX such that $W \subseteq U_1$, $W' \subseteq U_1'$ and $\mathrm{Cl}_{KX} U_1 \cap \mathrm{Cl}_{KX} U_1' = \varnothing$. Then $\mathrm{bd}_{KX} U_1 \subseteq X \setminus R(X)$, hence U_1 is an sb open subset of KX. Also, $U_1 \cap \mathrm{Cl}_{KX}(KX \setminus X) = W$. Since $T \cap W = \varnothing$, it follows that $T \cap \mathrm{Cl}_{KX}(KX \setminus X) \cap \mathrm{Cl}_{KX}(U_1 \cap X) = \varnothing$, hence $T \cap \mathrm{Cl}_{KX}(U_1 \cap X)$ is a compact set contained in $X \setminus R(X)$. According to 2.4, there is an open set V of X such that $\mathrm{Cl}_X V$ is a compact subset of $X \setminus R(X)$, and $T \cap \mathrm{Cl}_{KX}(U_1 \cap X) \subset V$. Let $U_2 = KX \setminus \mathrm{Cl}_X V$. Then U_2 is an sb open set of KX by 2.2 (iv), and $W \subset U_2$. If $U = U_1 \cap U_2$, then U is an sb open set of KX by 2.2 (i), and $W = U \cap \mathrm{Cl}_{KX}(KX \setminus X)$. Also $\mathrm{bd}_{KX} U \subset \mathrm{bd}_{KX} U_1 \cup \mathrm{bd}_{KX} U_2 \subset X \setminus R(X)$. Since $T \cap \mathrm{Cl}_{KX} U \subset T \cap \mathrm{Cl}_{KX} U_1 \cap \mathrm{Cl}_{KX} U_2 = \varnothing$, the statement is proved. □

Let X be a space. In the sequel, $L(X)$ denotes the locally compact part of X; that is $L(X) = X \setminus R(X)$. Note that if $KX \in \mathscr{K}(X)$, then $L(X) = KX \setminus \mathrm{Cl}_{KX}(KX \setminus X)$, and that

$$L(KX \setminus X) = (KX \setminus X) \setminus R(KX \setminus X) = KX \setminus [X \cup \mathrm{cl}_{KX} R(X)].$$

The following is easy to prove.

2.6. LEMMA. *If X is a space, $KX \in \mathcal{K}(X)$, and W is a compact clopen subset of either $L(KX \setminus X)$ or $KX \setminus X$, then W is a (compact) clopen subset of* $\mathrm{Cl}_{KX}(KX \setminus X)$.

2.7. LEMMA. *Suppose that X is a space in which $R(X)$ is compact. If KX is a O.I. compactification of X, then $KX \setminus X$ is relatively 0-dimensionally embedded in KX.*

Proof. Suppose that T is closed in KX, and that $p \in (KX \setminus X) \setminus T$. As $R(X)$ is compact, there is an open set U of KX such that $p \in U$, while $[T \cup R(X)] \cap \mathrm{Cl}_{KX} U = \varnothing$. Since $U \cap (KX \setminus X)$ is open in $KX \setminus X$, and $KX \setminus X$ is locally compact and 0-dimensional, there is a compact clopen set W of $KX \setminus X$ such that $p \in W \subset U$. Then $W \cap T = \varnothing$, so by 2.5 and 2.6 there is an sb open set V of KX such that $V \cap \mathrm{Cl}_{KX}(KX \setminus X) = W$ and $T \cap \mathrm{Cl}_{KX} V = \varnothing$. Then $p \in V$, and $V \cap T = \varnothing$. Thus each point of $KX \setminus X$ has a basis in KX of open sets whose boundaries lie in X. That is, $KX \setminus X$ is relatively 0-dimensionally embedded in KX. □

2.8. THEOREM. *If X is a space in which $\mathrm{bd}_X R(X)$ is compact, then the following are equivalent.*

(i) *X is a 0-space.*

(ii) *X is almost rimcompact.*

(iii) *X is a 0-space, and $F_0 X \setminus X$ is relatively 0-dimensionally embedded in $F_0 X$.*

(iv) *If KX is any O.I. compactification of X in which $\mathrm{Cl}_{KX}(\mathrm{int}_X R(X)) \cap \mathrm{Cl}_{KX}(X \setminus R(X)) \subset X$, then $KX \setminus X$ is relatively 0-dimensionally embedded in KX.*

Proof. It follows from 1.8 that (iii) implies (ii) and (ii) implies (i).

(i) implies (iv). Suppose that KX is a O.I. compactification of X in which $\mathrm{Cl}_{KX}(\mathrm{int}_X R(X)) \cap \mathrm{Cl}_{KX}(X \setminus R(X)) \subset X$. We claim that

$$KX \setminus X \subset Ex_{KX}(\mathrm{int}_X R(X)) \cup Ex_{KX}(X \setminus R(X)).$$

As $X \setminus [\mathrm{int}_X R(X) \cup (X \setminus R(X))] = \mathrm{bd}_X R(X)$, which is a compact subset of X,

$$KX \setminus X \subset Ex_{KX}[\mathrm{int}_X R(X) \cup (X \setminus R(X))].$$

If U and V are open subsets of X, and

$$p \in Ex_{KX}(U \cup V) \setminus (Ex_{KX} U \cup Ex_{KX} V),$$

then by 1.5 (iv), $p \in \mathrm{Cl}_{KX} U \cap \mathrm{Cl}_{KX} V$. As

$$\mathrm{Cl}_{KX}\big(\operatorname*{int}_X R(X)\big) \cap \mathrm{Cl}_{KX}(X \setminus R(X)) \subset X,$$

it follows that $KX \setminus X \subset Ex_{KX}(\mathrm{int}_X R(X)) \cup Ex_{KX}(X \setminus R(X))$, and the claim is proved.

Note that $\mathrm{Cl}_X \mathrm{int}_X R(X)$ is a nowhere locally compact space. For if V is any non-empty open subset of $\mathrm{Cl}_X \mathrm{int}_X R(X)$, there is an open set U of X such that

$$U \cap \mathrm{Cl}_X \operatorname*{int}_X R(X) = V.$$

Then $U \cap \mathrm{int}_X R(X)$ is a non-empty open subset of X. Since $\mathrm{int}_X R(X)$ is nowhere locally compact, $\mathrm{Cl}_X(U \cap \mathrm{int}_X R(X))$ is not compact. Then $\mathrm{Cl}_X V$, which is the closure in $\mathrm{Cl}_X \mathrm{int}_X R(X)$ of V, is not compact. Thus no point of $\mathrm{Cl}_X \mathrm{int}_X R(X)$ has a basis (in $\mathrm{Cl}_X \mathrm{int}_X R(X)$) of compact closed neighbourhoods, and $\mathrm{Cl}_X \mathrm{int}_X R(X)$ is nowhere locally compact.

As $\mathrm{Cl}_{KX} \mathrm{int}_X R(X)$ is a O.I. compactification of $\mathrm{Cl}_X \mathrm{int}_X R(X)$, it follows from 2.3 that $\mathrm{Cl}_{KX} \mathrm{int}_X R(X) \setminus \mathrm{Cl}_X \mathrm{int}_X R(X)$, which by our claim is just $[Ex_{KX} \mathrm{int}_X R(X)] \cap [KX \setminus X]$, is relatively 0-dimensionally embedded in $\mathrm{Cl}_{KX} \mathrm{int}_X R(X)$. Let $p \in [Ex_{KX} \mathrm{int}_X R(X)] \cap [KX \setminus X]$. We show that p has a basis in KX of open sets whose boundaries lie in X. Suppose that $p \in KX \setminus T$, where T is a closed subset of KX. Since $p \notin \mathrm{Cl}_{KX}(X \setminus R(X))$, there is an open subset U_1 of KX such that $p \in U_1$ and $\mathrm{Cl}_{KX} U_1 \cap [\mathrm{Cl}_{KX}(X \setminus R(X)) \cup T] = \varnothing$. Then U_1 is open in $Ex_{KX} \mathrm{int}_X R(X)$, and hence in $\mathrm{Cl}_{KX} \mathrm{int}_X R(X)$. It follows that there is an sb (with respect to $\mathrm{Cl}_{KX} \mathrm{int}_X R(X)$) open set U_2 of $\mathrm{Cl}_{KX} \mathrm{int}_X R(X)$ such that $p \in U_2 \subset U_1$. As $U_1 \subset Ex_{KX} \mathrm{int}_X R(X)$, it follows that U_2 is open in KX. Since $\mathrm{Cl}_{KX} U_2 \cap \mathrm{Cl}_{KX}(X \setminus R(X)) = \varnothing$, U_2 is an sb open subset of KX which contains p and has empty intersection with T.

The subset $\mathrm{Cl}_X(X \setminus R(X))$ of X is a space with compact residue, so by 2.7, $\mathrm{Cl}_{KX}(X \setminus R(X))$ is a O.I. compactification of X with a relatively 0-dimensionally embedded remainder. If

$$p \in \mathrm{Cl}_{KX}(X \setminus R(X)) \setminus \mathrm{Cl}_X(X \setminus R(X))$$

(which by our earlier claim equals $Ex_{KX}(X \setminus R(X)) \cap (KX \setminus X))$, then $p \notin \mathrm{Cl}_{KX} R(X)$. It follows from an argument similar to that in the preceding paragraph that p has a basis in KX of sb open sets of KX. Thus each point of $KX \setminus X$ has a basis of sb open sets of KX, hence $KX \setminus X$ is relatively 0-dimensionally embedded in KX.

(iv) implies (iii). Since $F_0 X$ is a perfect compactification of X, and $\mathrm{bd}_X R(X)$ is compact, by 2.2 (v) and 1.5 (ii),

$$\begin{aligned}&\mathrm{Cl}_{F_0 X}(\mathrm{int}_X R(X)) \cap \mathrm{Cl}_{F_0 X}(X \setminus R(X)) \cap (F_0 X \setminus X)\\ &\qquad = Ex_{F_0 X}\mathrm{int}_X R(X) \cap Ex_{F_0 X}(X \setminus R(X)) \cap (F_0 X \setminus X) = \varnothing .\end{aligned}$$

Thus $F_0 X$ satisfies the condition imposed on KX in (iv) and hence $F_0 X \setminus X$ is relatively 0-dimensionally embedded in $F_0 X$. □

The hypotheses of 2.8 do not imply that X is rimcompact. If in 1.9, Y is chosen to be a locally compact 0-dimensional space which is not strongly 0-dimensional, and βY is chosen as the perfect compactification of Y, then $X = (\beta Y \times (\omega_1 + 1)) \setminus (Y \times \{\omega_1\})$ is an almost rimcompact non-rimcompact space in which $R(X)$ is compact.

3. Subsets, supersets and products. We outline a construction that we will use to produce many of our examples.

A collection of infinite subsets of $\mathcal{N}$ is called *almost disjoint* if the intersection of the two distinct members is finite. Zorn's lemma implies that there exists a maximal collection of almost disjoint infinite subsets of $\mathcal{N}$. In the following $\mathcal{R}$ will denote a maximal such collection. The following topology on $\mathcal{N} \cup \mathcal{R}$ is credited to Isbell in [4]. Each point of $\mathcal{N}$ is isolated, and $\lambda \in \mathcal{R}$ has as an open base $\{\{\lambda\} \cup (\lambda \setminus F)$: F is a finite subset of $\mathcal{N}\}$. It is noted in 5I of [4] that such spaces $\mathcal{N} \cup \mathcal{R}$ are first countable, locally compact, 0-dimensional and pseudocompact. The following is 2.1 of [**12**].

3.1. PROPOSITION. *Any compact metric space without isolated points is homeomorphic to the remainder* $\beta(\mathcal{N} \cup \mathcal{R}) \setminus \mathcal{N} \cup \mathcal{R}$ *for a suitably chosen maximal almost disjoint collection* $\mathcal{R}$.

As indicated in [**12**], 3.1 holds for any first-countable, separable, compact T_2 space. We do not make use of this more general statement.

In the sequel, when we choose a maximal almost disjoint collection $\mathcal{R}$ such that $\beta(\mathcal{N} \cup \mathcal{R}) \setminus \mathcal{N} \cup \mathcal{R}$ is homeomorphic to a compact metric space X having no isolated points, we identify points of

$$\beta(\mathcal{N} \cup \mathcal{R}) \setminus \mathcal{N} \cup \mathcal{R}$$

with points of X in the obvious manner, and consider $\beta(\mathcal{N} \cup \mathcal{R}) \setminus \mathcal{N} \cup \mathcal{R}$ to be the space X.

The next example shows that, as might be expected, it is not true that if a space X is rimcompact, and $X \subset T \subset \beta X$, then T is necessarily a 0-space.

3.2. EXAMPLE. Choose $\mathscr{R}$ so that $\beta(\mathscr{N} \cup \mathscr{R}) \setminus \mathscr{N} \cup \mathscr{R} = I$, where I denotes the unit interval. Let $X = \mathscr{N} \cup \mathscr{R}$, and $T = \mathscr{N} \cup \mathscr{R} \cup \{1\}$. Then X is rimcompact. However, the single connected component of $\beta T \setminus T = \beta X \setminus T$ is $[0, 1)$, which is not compact. Thus T is not a 0-space. □

It is clear that if X is a 0-space, and $X \subset T \subset F_0 X$, then T is a 0-space. Recall that if $X \subset Y \subset \beta X$, then $\beta Y = \beta X$. The following indicates that the expected relationship between $F_0 X$ and $F_0 T$ holds.

3.3. THEOREM. *If X is a 0-space, and $X \subset T \subset F_0 X$, then T is a 0-space and $F_0 X = F_0 T$. If X is almost rimcompact (respectively, rimcompact) then T is almost rimcompact (respectively, rimcompact).*

Proof. Clearly $F_0 X$ is a O.I. compactification of T. Suppose that KT is a O.I. compactification of T such that $KT \geq F_0 X$. Then KT is a compactification δX of X. Recall that $\delta f\colon \beta X \to \delta X$ denotes the natural map. Define $g\colon \delta X \to F_0 X$ to be the natural map. Then $g \circ (\delta f) = F_0 f$. Suppose that $p \in F_0 X \setminus T$. Since $F_0 X$ is a perfect compactification of X, by 1.6, $(F_0 f)^{\leftarrow}(p) = (g \circ \delta f)^{\leftarrow}(p)$ is a connected subset of βX. Then $(\delta f)[(F_0 f)^{\leftarrow}(p)] = g^{\leftarrow}(p)$ is a connected subset of KT contained in $KT \setminus T$. Since $KT \setminus T$ is 0-dimensional, $|g^{\leftarrow}(p)| = 1$. It follows that $KT = F_0 X$, and hence $F_0 X = F_0 T$.

If each point of $F_0 X \setminus X$ has a basis of open sets of $F_0 X$ whose boundaries are contained in X, then each point of $F_0 X \setminus T$ has a basis of open sets of $F_0 X = F_0 T$ whose boundaries are contained in T. Thus if X is almost rimcompact, T is almost rimcompact. A similar statement holds if X is rimcompact. □

It is tempting to attempt to shorten the proof of the preceding theorem by immediately claiming that KT as chosen is a O.I. compactification of X. However, since the union of two 0-dimensional spaces need not be 0-dimensional, it is not immediately clear that $KT \setminus X$ is 0-dimensional, and further argument of the sort provided in the proof is necessary.

We note in passing the following special case for 3.3. If X is a 0-space, and $X \cup \mathrm{Cl}_{F_0 X} R(X) \subset T \subset F_0 X$, then since $X \cup \mathrm{Cl}_{F_0 X} R(X)$ is almost rimcompact by 2.7, T is almost rimcompact.

We now consider subspaces of 0-spaces. It is an easy exercise to prove that an open or a closed subspace of a rimcompact space is rimcompact. This contrasts with the fact that while a closed subspace of an almost rimcompact space is almost rimcompact, an open subspace of an almost rimcompact space need not even be a 0-space.

3.4. THEOREM. *If F is a closed subset of a 0-space (respectively, an almost rimcompact space) X, then F is a 0-space (respectively, almost rimcompact).*

Proof. If F is closed in a 0-space X, and KX is any O.I. compactification of X, then $\mathrm{Cl}_{KX}F$ is a O.I. compactification of F. Thus F is a 0-space.

Suppose that $KX \setminus X$ is relatively 0-dimensionally embedded in KX. We show that $\mathrm{Cl}_{KX}F \setminus F$ is relatively 0-dimensionally embedded in $\mathrm{Cl}_{KX}F$. Suppose that T is a closed subset of $\mathrm{Cl}_{KX}F$ and $p \in (\mathrm{Cl}_{KX}F \setminus F) \setminus T$. Then T is closed in KX. Since $KX \setminus X$ is relatively 0-dimensionally embedded in KX, there is an sb open set U of KX such that $p \in U$ and $(\mathrm{Cl}_{KX}U) \cap T = \varnothing$. Consider the open set $U \cap \mathrm{Cl}_{KX}F$ of $\mathrm{Cl}_{KX}F$. The boundary in $\mathrm{Cl}_{KX}F$ of $U \cap \mathrm{Cl}_{KX}F$ is

$$\begin{aligned}\mathrm{Cl}_{KX}(U \cap \mathrm{Cl}_{KX}F) \setminus U \cap \mathrm{Cl}_{KX}F &\subset [\mathrm{Cl}_{KX}(U \cap \mathrm{Cl}_{KX}F) \setminus U] \cap \mathrm{Cl}_{KX}F \\ &\subset [(\mathrm{Cl}_{KX}U) \setminus U] \cap \mathrm{Cl}_{KX}F \subset \mathrm{bd}_{KX}U \cap \mathrm{Cl}_{KX}F \\ &\subset X \cap \mathrm{Cl}_{KX}F = F.\end{aligned}$$

Then $U \cap \mathrm{Cl}_{KX}F$ is an sb open subset of $\mathrm{Cl}_{KX}F$ and a neighbourhood (in $\mathrm{Cl}_{KX}F$) of p, while $T \cap (\mathrm{Cl}_{KX}F) \cap U = \varnothing$. Thus each point of $\mathrm{Cl}_{KX}F \setminus F$ has a basis of sb open sets of $\mathrm{Cl}_{KX}F$. Hence $\mathrm{Cl}_{KX}F \setminus F$ is relatively 0-dimensionally embedded in $\mathrm{Cl}_{KX}F$. It follows from 1.8 that F is almost rimcompact. □

3.5. EXAMPLE. Choose $\mathscr{R}$ to be a maximal almost disjoint collection of infinite subsets of $\mathscr{N}$ such that $\beta(\mathscr{N} \cup \mathscr{R}) \setminus (\mathscr{N} \cup \mathscr{R})$ is homeomorphic to I. Let $Z = [\beta(\mathscr{N} \cup \mathscr{R}) \times (\omega_1 + 1)] \setminus [(\mathscr{N} \cup \mathscr{R}) \times \{\omega_1\}]$, and $X = Z \setminus \{(\frac{1}{2}, \omega_1)\}$. Then X is an open subset of Z. As demonstrated in 3.8 of [2], X is not a 0-space, while according to 1.9, Z is almost rimcompact. □

For completeness we include the following example which illustrates that the product of two rimcompact spaces need not be a 0-space. We mention that it is straightforward to show that a space possessing a compactification with countable remainder is rimcompact.

3.6. EXAMPLE. Choose $\mathscr{R}$ to be a family such that $\beta(\mathscr{N} \cup \mathscr{R}) \setminus \mathscr{N} \cup \mathscr{R} = I$. Let P, Q denote the irrationals and rationals in I, respectively. If $Y = \mathscr{N} \cup \mathscr{R} \cup P$, then $\beta Y \setminus Y = Q$, hence Y is rimcompact. According to 1.3 $\beta((\mathscr{N} \cup \mathscr{R}) \times (\mathscr{N} \cup \mathscr{R})) = \beta(\mathscr{N} \cup \mathscr{R}) \times \beta(\mathscr{N} \cup \mathscr{R})$, so by 1.4 $\beta(Y \times Y) = \beta Y \times \beta Y$. Let $Z = \beta(Y \times Y) \setminus (Y \times Y)$. If $q \in Q$, let $C_{(q,q)}$ denote the connected component of (q, q) in Z. We show that $C_{(q,q)}$ is not compact, hence $Y \times Y$ is not a 0-space. Now $q \times I$ is a connected subset of Z. For each $q' \in Q$, $I \times q'$ is a connected subset of Z which intersects $q \times I$, hence $\bigcup_{q' \in Q}(I \times q') \subseteq C_{(q,q)}$. The smallest compact connected set containing $\bigcup_{q' \in Q}(I \times q')$ is $I \times I$. However, $(I \times I) \cap (Y \times Y) \neq \varnothing$, hence $C_{(q,q)}$ is not compact. □

4. Images and preimages. Continuous images and preimages of rimcompact spaces need not be rimcompact, even if the map is perfect. In fact, since any completely regular space is the image of an extremally disconnected space (i.e., a space in which disjoint open sets have disjoint closures) under a perfect irreducible map (see [**11**]), the perfect image of a rimcompact space need not even be a 0-space. The next example shows that the perfect preimage of a rimcompact space need not be a 0-space. However, we show in 4.3 that if the perfect preimage of an almost rimcompact space is a 0-space, then that preimage is almost rimcompact.

4.1. EXAMPLE. Let $Y = I \times \{0, 1, 1/2, 1/3, \ldots\}$, and

$$X = [Y \times (\omega_1 + 1)] \setminus [I \times \{1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots\} \times \{\omega_1\}].$$

It is shown in 3.7 of [**2**] that X is not a 0-space. Let

$$f: I \times \{0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots\} \times (\omega_1 + 1) \to \{0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots\} \times (\omega_1 + 1)$$

be the projection map. Then f is closed, since I is compact. Let

$$S = [\{0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots\} \times (\omega_1 + 1)] \setminus [\{1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots\} \times \{\omega_1\}].$$

Since $f^{\leftarrow}(y) = I \times \{y\}$, for $y \in S$, f is a perfect map from X onto S. The space S, being a subspace of $\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\} \times (\omega_1 + 1)$, is 0-dimensional (and hence rimcompact). □

The following is 1.2 of [**6**].

4.2. LEMMA. *Let f: $X \to Y$ be a perfect map. If S is a compact subset of Y, then $f^{\leftarrow}[S]$ is a compact subset of X.*

4.3. THEOREM. *Let f: $X \to Y$ be a perfect map. If X is a 0-space, and Y is almost rimcompact, then X is almost rimcompact.*

Proof. We show that X is quasi-rimcompact. It then follows from 1.8 that X is almost rimcompact. If $x \in R(X)$, let $K_x = f^{\leftarrow}[K]$, where K is the compact subset of Y witnessing the fact that Y is quasi-rimcompact at $f(x)$. According to 4.2, K_x is a compact subset of X. Suppose that F is a closed subset of X such that $F \cap K_x = \varnothing$. Then $K \cap f[F] = \varnothing$. Since f is a closed map, it follows from our choice of K that there is a π-open subset W of Y such that $f(x) \in W \subseteq \mathrm{Cl}_Y W \subset Y \setminus f[F]$. As f is a perfect map, and $\mathrm{bd}_Y W$ is compact, according to 4.2 $f^{\leftarrow}[\mathrm{bd}_Y W]$ is compact. Since $\mathrm{bd}_X f^{\leftarrow}[W] \subset f^{\leftarrow}[\mathrm{bd}_X W]$, $f^{\leftarrow}[W]$ is a π-open subset of X. Also, $x \in f^{\leftarrow}[W]$, and $F \cap \mathrm{Cl}_X f^{\leftarrow}[W] = \varnothing$. Thus x and F are π-separated. Hence X is quasi-rimcompact, and the theorem is proved. □

In 4.3, X and Y can be chosen so that X is not rimcompact and Y is rimcompact.

4.4. Example. Choose $\mathscr{R}$ to be a family such that $\beta(\mathscr{N} \cup \mathscr{R}) \setminus \mathscr{N} \cup \mathscr{R}$ is homeomorphic to I. Then $F(\mathscr{N} \cup \mathscr{R}) = \omega(\mathscr{N} \cup \mathscr{R})$, the one-point compactification of $\mathscr{N} \cup \mathscr{R}$. If

$$X = [\beta(\mathscr{N} \cup \mathscr{R}) \times (\omega_1 + 1)] \setminus [(\mathscr{N} \cup \mathscr{R}) \times \{\omega_1\}],$$

then according to 1.9, X is almost rimcompact but is not rimcompact. Let

$$f\colon \beta(\mathscr{N} \cup \mathscr{R}) \times (\omega_1 + 1) \to \omega(\mathscr{N} \cup \mathscr{R}) \times (\omega_1 + 1)$$

be the natural map, and let

$$Z = [\omega(\mathscr{N} \cup \mathscr{R}) \times (\omega_1 + 1)] \setminus [(\mathscr{N} \cup \mathscr{R}) \times \{\omega_1\}].$$

If $z \in Z$, then $f^{\leftarrow}(z) = \{z\}$ or $f^{\leftarrow}(z) = I \times \{p\}$ for some $p \in (\omega_1 + 1)$. Also $f^{\leftarrow}[Z] = X$, so $f|_X$ is a perfect map from X into Z. The space Z is 0-dimensional (and hence rimcompact). □

It is well known that if $f\colon X \to Y$ is a map, where X and Y are 0-dimensional, then f extends to $g \in C(FX, FY) = C(\beta_0 X, \beta_0 Y)$. The following generalizes this fact.

4.5. Theorem. *Suppose that X is a space, Y is 0-dimensional and KX is a perfect compactification of X. If $f\colon X \to Y$ is a map, then f extends to $g \in C(KX, \beta_0 Y)$.*

Proof. Subsets C and D of Y have disjoint closures in $\beta_0 Y$ if and only if C and D are contained in disjoint clopen subsets U and $Y \setminus U$ of Y respectively. Since $f^{\leftarrow}[U]$, $f^{\leftarrow}[Y \setminus U]$ are then disjoint clopen subsets of

X, and KX is a perfect compactification of X, it follows that $\mathrm{Cl}_{KX}f^{\leftarrow}[U] \cap \mathrm{Cl}_{KX}f^{\leftarrow}[Y \setminus U] = \varnothing$. Then $\mathrm{Cl}_{KX}f^{\leftarrow}[C] \cap \mathrm{Cl}_{KX}f^{\leftarrow}[D] = \varnothing$; thus by 1.1, f extends to $g \in C(KX, \beta_0 Y)$. □

4.6. DEFINITION. A map f: $X \to Y$ is *monotone* if $f^{\leftarrow}(y)$ is connected for each $y \in Y$.

The following answers a question communicated verbally to R. G. Woods (Topology Conference, 1980) by D. Bellamy.

4.7. THEOREM. *Let f: $X \to Y$ be a monotone quotient map, and let KX, KY be perfect compactifications of X and Y respectively. If f extends to $g \in C(KX, KY)$, then g is monotone.*

Proof. Suppose that there is $p \in KY$ such that $g^{\leftarrow}(p)$ is not connected. Write $g^{\leftarrow}(p) = G_1 \cup G_2$, where G_1 and G_2 are disjoint closed subsets of $g^{\leftarrow}(p)$. Since $g^{\leftarrow}(p)$ is compact, G_1 and G_2 are disjoint compact subsets of KX; hence there are open sets U_1 and U_2 of X such that $G_i \subset Ex_{KX}U_i$ $(i = 1,2)$ and $\mathrm{Cl}_{KX}U_1 \cap \mathrm{Cl}_{KX}U_2 = \varnothing$. Since g is a closed map, there is an open set V of Y such that $g^{\leftarrow}(p) \subset g^{\leftarrow}[V] \subset Ex_{KX}U_1 \cup Ex_{KX}U_2$. Let $W_i = g^{\leftarrow}[V] \cap U_i = f^{\leftarrow}[V \cap Y] \cap U_i$ $(i = 1,2)$. Then W_1 and W_2 are disjoint open subsets of X, and $W_1 \cup W_2 = f^{\leftarrow}[V \cap Y]$. Since $f^{\leftarrow}(y)$ is connected for each $y \in Y$, $W_i = f^{\leftarrow}[V_i]$ for some subset V_i of Y $(i = 1,2)$. Since f is a quotient map, V_i is open in Y $(i = 1,2)$. Then $V \cap Y = V_1 \cup V_2$, while $V_1 \cap V_2 = \varnothing$. It follows from 1.5 (i) and (ii), and 1.6 that $p \in Ex_{KY}V = Ex_{KY}V_1 \cup Ex_{KY}V_2$, while $Ex_{KY}V_1 \cap Ex_{KY}V_2 = \varnothing$. Suppose without loss of generality that $p \in Ex_{KY}V_1$. Since $g^{\leftarrow}[Ex_{KY}V_1]$ is an open subset of KX containing $f^{\leftarrow}[V_1]$,

$$g^{\leftarrow}(p) \subset g^{\leftarrow}[Ex_{KY}V_1] \subset Ex_{KX}f^{\leftarrow}[V_1] = Ex_{KX}W_1 \subset Ex_{KX}U_1,$$

which contradicts the fact that $g^{\leftarrow}(p) \cap Ex_{KX}U_2 \neq \varnothing$. Thus $g^{\leftarrow}(p)$ is connected for each $p \in KY$. □

4.8. COROLLARY. *Suppose that X is a 0-space and Y is 0-dimensional. If there is a perfect monotone map from X into Y, then X is almost rimcompact and $F_0X \setminus X$ is homeomorphic to $FY \setminus Y$.*

Proof. Let f: $X \to Y$ be a perfect monotone map. Then f extends to $g \in C(F_0X, FY)$ by 4.5. Since f is perfect, $g^{\leftarrow}[FY \setminus Y] = F_0X \setminus X$. As f is monotone, it follows from 4.7 that $g^{\leftarrow}(y)$ is connected for each $y \in FY \setminus Y$. Since $F_0X \setminus X$ is 0-dimensional, and $g^{\leftarrow}(y) \subset F_0X \setminus X$,

$|g^{\leftarrow}(y)| = 1$. Thus $g|_{F_0X\setminus X}$: $F_0X\setminus X \to FY\setminus Y$ is a closed continuous one-to-one map, hence g is a homeomorphism. The fact that X is almost rimcompact follows from 4.3. □

Example 4.1 shows that the perfect monotone preimage X of a 0-dimensional space need not be a 0-space, while Example 4.4 shows that even if X is a 0-space, X need not be rimcompact.

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