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PATH PARTITIONS AND PACKS OF ACYCLIC DIGRAPHS

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In memory of Ernst Straus

Let G be an acyclic directed graph with $|V(G)| \geq k$. We prove that there exists a colouring $\{C_1, C_2, \dots, C_m\}$ such that for every collection $\{P_1, P_2, \dots, P_k\}$ of k vertex disjoint paths with $|\bigcup_{j=1}^{k} P_j|$ a maximum, each colour class C_i meets $\min\{|C_i|, k\}$ of these paths. An analogous theorem, partially interchanging the roles of paths and colour classes, has been shown by Cameron [4] and Saks [17] and we indicate a third proof.

1. Introduction. Let $G = (V, E)$ be a directed graph containing no loops or multiple edges. A *path* P in G is a sequence of distinct vertices $(v_1, v_2, \dots, v_l)$ such that $(v_i, v_{i+1}) \in E$, $i = 1, 2, \dots, l-1$. The set of vertices $\{v_1, v_2, \dots, v_l\}$ of a path $P = (v_1, v_2, \dots, v_l)$ will be denoted by $V(P)$. The *cardinality* of P, denoted by $|P|$, is $|V(P)|$.

A family $\mathscr{P}$ of paths is called a *path-partition* of G if its members are vertex disjoint and $\bigcup\{V(P): P \in \mathscr{P}\} = V$. For each nonnegative integer k, the *k-norm* $|\mathscr{P}|_k$ of a path partition $\mathscr{P} = \{P_1, \dots, P_m\}$ is defined by

$$|\mathscr{P}|_k = \sum_{i=1}^{m} \min\{|P_i|, k\}.$$

A partition which minimizes $|\mathscr{P}|_k$ is called *k-optimum*. For example, a 1-optimum partition is a partition P containing a minimum number of paths.

A *partial k-colouring* is a family $\mathscr{C}^k = \{C_1, C_2, \dots, C_t\}$ of at most k disjoint independent sets C_i called *colour classes*. The cardinality of a partial k-colouring $\mathscr{C}^k = \{C_1, C_2, \dots, C_t\}$ is $|\bigcup_{i=1}^{t} C_i|$, and $\mathscr{C}^k$ is said to be *optimum* if $|\bigcup_{i=1}^{t} C_i|$ is as large as possible. A path partition $\mathscr{P} = \{P_1, P_2, \dots, P_m\}$ and a partial k-colouring $\mathscr{C}^k$ are *orthogonal* if every path P_i in $\mathscr{P}$ meets $\min\{|P_i|, k\}$ different colour classes of $\mathscr{C}^k$.

Berge [2] made the following conjecture:

Conjecture 1. Let G be a directed graph and let k be a positive integer. Then for every k-optimum path partition $\mathscr{P}$, there exists a partial k-colouring $\mathscr{C}^k$ orthogonal to $\mathscr{P}$.

Let $\pi_k(G)$ be the k-norm of a k-optimum path partition in G, and let $\alpha_k(G)$ be the cardinality of an optimum partial k-colouring in G. A weaker conjecture by Linial **[14]** is as follows:

Conjecture 2. Let G be a directed graph and let k be a positive integer. Then,

$$\alpha_k(G) \geq \pi_k(G).$$

If Conjecture 1 holds, then every path P in a k-optimum path partition $\mathscr{P}$ meets at least $\min\{|P|, k\}$ vertices of some partial k-colouring $\mathscr{C}^k$. Hence, $\alpha_k(G) \geq \sum_{P \in \mathscr{P}} \min\{|P|, k\} = \pi_k(G)$, and Conjecture 2 holds.

For $k = 1$, Conjecture 2 holds by the Gallai-Milgram theorem **[9]**. Linial **[13]** showed that the proof of the Gallai-Milgram theorem also yields Conjecture 1 for this case.

For transitive graphs, Conjecture 2 is given for $k = 1$ by Dilworth's theorem **[6]**, and for all k by the theorem of Greene and Kleitman **[10]**. It is easy to deduce from it that Conjecture 1 also holds for such graphs. Linial **[14]** and Cameron **[3]** independently showed that Conjecture 2 holds for all acyclic graphs. Conjecture 1 was proved for such graphs in **[1]**. Cameron **[4]** and Saks **[17]** have shown that an even stronger version of Conjecture 1 holds for all acyclic graphs:

THEOREM 1. *Let G be a directed acyclic graph, and let k be a positive integer. Then there exists a partial k-colouring $\mathscr{C}^k$ which is orthogonal to every k-optimum path partition $\mathscr{P}$ of G.*

We indicate a proof of Theorem 1 in §3. This proof is different from the ones in **[4]** and **[17]** and was found independently.

It is possible to 'dualize' the notions of path partition and partial k-colouring, by interchanging the roles of 'path' and 'independent set' in the definitions and theorems above.

A *colouring* $\mathscr{C}$ is a partition of V into disjoint independent sets. For each non-negative integer k, the k-norm $|\mathscr{C}|_k$ of a colouring $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$ is defined as:

$$|\mathscr{C}|_k = \sum_{i=1}^{m} \min\{|C_i|, k\}.$$

A colouring which minimizes $|\mathscr{C}|_k$ is called k-optimum. For example, a 1-optimum colouring is a colouring with χ colours, where χ is the chromatic number of G.

The analogue of a partial k-colouring for paths, is a *path k-pack*, defined to be a family $\mathscr{P}^k = \{P_1, P_2, \ldots, P_t\}$ of at most k disjoint paths

P_i. The cardinality of a path k-pack $\mathcal{P}^k = \{P_1, P_2, \ldots, P_t\}$ is $|\bigcup_{i=1}^t P_i|$, and $\mathcal{P}^k$ is *optimum* if $|\bigcup_{i=1}^t P_i|$ is as large as possible. A colouring $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ and a path k-pack $\mathcal{P}^k$ are *orthogonal* if every colour class C_i in $\mathcal{C}$ meets $\min\{|C_i|, k\}$ different paths of $\mathcal{P}^k$.

As a dual analogue of Conjecture 1, we suggest the following:

Conjecture 3. Let G be a directed graph and let k be a positive integer. Then for every optimum path k-pack $\mathcal{P}^k$, there exists a colouring $\mathcal{C}$ orthogonal to $\mathcal{P}^k$.

Let $\chi_k(G)$ be the k-norm of a k-optimum colouring in G, and let $\lambda_k(G)$ be the cardinality of an optimum path k-pack in G. The dual of Conjecture 2 would be:

Conjecture 4. (Linial [**14**]). Let G be a directed graph and let k be a positive integer. Then,

$$\lambda_k(G) \geq \chi_k(G).$$

It is not difficult to see that Conjecture 3 implies Conjecture 4. For $k = 1$, Conjecture 4 is given by the Gallai-Roy theorem [**7, 15**] and Conjecture 3 is also valid in this case, by the proof of the Gallai-Roy theorem.

For transitive graphs, Conjecture 4 is true by Greene's theorem [**9**] and Conjecture 3 can be deduced from it. Hoffman [**12**] and Saks [**16**] have independently proved Conjecture 4 for all acyclic graphs.

In this paper we prove the following stronger version of Conjecture 3 for all acyclic graphs:

THEOREM 2. *Let G be a directed acyclic graph and let k be a positive integer. Then there exists a colouring $\mathcal{C}$ orthogonal to every optimum path k-pack $\mathcal{P}^k$.*

2. Proof of Theorem 2. If V can be covered by k or fewer vertex disjoint paths, then making each vertex a colour class satisfies Theorem 2. So assume otherwise. Let $|V| = n$, and label the vertices $1, 2, \ldots, n$. We shall use the linear program defined in [**12**]:

Let $C = (c_{ij})$, $i, j = 0, 1, \ldots, n$, be defined by

$c_{i0} = 0$ for all i; $\quad c_{0j} = 1$ for all $j > 0$

$c_{ii} = 0$ for all i

if $i > 0, j > 0$, and $i \neq j$, then $c_{ij} = 1$ if $(i, j) \in E$

$\qquad = \text{not defined}$ if $(i, j) \notin E$.

Consider the transportation problem:

I.

$$\text{maximize} \sum_{\substack{i=0\\j=0}}^{n} c_{ij}x_{ij} \tag{2.1}$$

where $x_{ij} \geq 0$ for all i, j, except that x_{ij} is not defined if $i > 0, j > 0, i \neq j$ and $(i, j) \notin E$.

$$\sum_{j=0}^{n} x_{0j} = \sum_{i=0}^{n} x_{i0} = k \tag{2.2}$$

$$\sum_{j=0}^{n} x_{ij} = 1 \quad \text{for } i > 0; \qquad \sum_{i=0}^{n} x_{ij} = 1 \quad \text{for } j > 0. \tag{2.3}$$

Every path k-pack $\mathscr{P}^k = \{P_1, P_2, \ldots, P_t\}$, $t \leq k$, corresponds to a feasible solution of (2.1)–(2.3), x, defined in the following way:

$$x_{00} = k - t$$

if $j > 0$,

$$\begin{aligned} x_{0j} &= 1 \quad \text{if } j \text{ is the start of one of } P_1, \ldots, P_t \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

if $i > 0$,

$$\begin{aligned} x_{i0} &= 1 \quad \text{if } i \text{ is the end of one of } P_1, \ldots, t_t \\ &= 0 \quad \text{otherwise} \end{aligned}$$

if $i > 0$,

$$\begin{aligned} x_{ii} &= 1 \quad \text{if } i \notin V(P_1) \cup \cdots \cup V(P_t) \\ &= 0 \quad \text{if } i \in V(P_1) \cup \cdots \cup V(P_t) \end{aligned}$$

if $i > 0, j > 0, i \neq j$, then

$$\begin{aligned} x_{ij} &= 1 \quad \text{if } (i, j) \text{ is an edge of } P_r \text{ for some } r = 1, \ldots, t \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

It can be shown that every vertex of (2.1)–(2.3) is integral and corresponds to a path k-pack of G. Hence, an integral optimum solution of (2.1)–(2.3) corresponds to an optimum path k-pack, and conversely.

Consider the dual problem:

II.
$$\min k(u_0 + v_0) + \sum_{i=1}^{n} u_i + \sum_{j=1}^{n} v_j$$

where

(2.4) $$u_i + v_j \geq c_{ij} \quad \text{for all } i, j.$$

Complementary slackness conditions for I and II are

(2.5) $$x_{ij} > 0 \Rightarrow u_i + v_j = c_{ij} \quad \text{for all } i, j.$$

Since the matrix of equations (2.4) is totally unimodular, the l.p. attains its minimum at integral u's and v's. We may subtract u_0 from each u_i and v_i, $i = 0, 1, \ldots, n$, to get an integral optimum solution with

(2.6) $$u_0 = 0.$$

We are now ready to define our colour classes. The "interesting" classes —the S_r defined below—get their names from the values of variables. Let

$$W = \{i > 0 : u_i + v_i = 0\}$$
$$S_r = \{i \in W : v_i = r\}$$

and

$$T_j = \{j\}, \quad \text{where } j \notin W.$$

Let $s = \max\{v_i | i \in W\}$. (We shall show later that $s = v_0$.) We shall establish that $\mathscr{C} = \{S_1, S_2, \ldots, S_s, T_1, T_2, \ldots, T_n\}$ is a colouring of G which satisfies the theorem.

To show $\mathscr{C}$ is a colouring, we need only prove that each S_r is an independent set. Suppose not. Then there exist $i, j \in S_r$, $(i, j) \in E$. But $u_i + v_i = 0$, $u_i + v_j \geq 1$ imply $v_j - v_i \geq 1$, so $v_i = v_j = r$ is impossible.

By our stipulations at the beginning of the proof, an optimum path k-pack contains k paths. Let $\mathscr{P}^k = \{P_1, P_2, \ldots, P_k\}$ be optimum. We must show that:

(i) each $T_j = \{j\}$ is on some path of $\mathscr{P}^k$ and

(ii) each S_r meets all paths of $\mathscr{P}^k$.

To prove (i), note that $j \in T_j$ means $u_j + v_j > 0$, implying by (2.5) that $x_{jj} = 0$. Since $\sum_k x_{jk} = 1$, we must have $x_{jl} = 1$ for some l, so j is in some path of $\mathscr{P}^k$.

To prove (ii), we first observe that

(2.7) $$v_0 \geq s.$$

To show (2.7) we use (2.4):

$$u_i + v_0 \geq c_{i0} = 0 \quad \forall i \in W$$
$$u_i + v_i = 0 \qquad \forall i \in W.$$

From the last two equations we deduce that $v_0 \geq v_i$ $\forall i \in W$, and (2.7) follows.

Next, let P be a path of $\mathscr{P}^k$, and for ease of notation, assume the path is $(1, 2, \ldots, l)$. Then

$$x_{01} = x_{12} = \cdots = x_{l-1l} = x_{l0} = 1.$$

By (2.5), $u_0 + v_1 = 1$, so by (2.6)

$$v_1 = 1. \tag{2.8}$$

Similarly, by (2.5), $u_l + v_0 = 0$, and by (2.4), $u_l + v_l \geq 0$, so

$$v_l \geq v_0. \tag{2.9}$$

From $u_j + v_j \geq 0$ and $u_j + v_{j+1} = 1$ it follows that

$$\begin{cases} \text{for } j = 1, 2, \ldots, l-1, \quad v_{j+1} - v_j \leq 1, & \text{with equality if and} \\ & \text{only if } u_j + v_j = 0. \end{cases} \tag{2.10}$$

Together, (2.8)–(2.10) show that $S_1, S_2, \ldots, S_{v_0 - 1}$ all meet P. All that remains to be shown is that S_{v_0} meets P.

From the proof of (2.9), we see that if $u_l + v_l = 0$, then also $v_l = v_0$ and l is in S_{v_0} and in P. If $u_l + v_l > 0$, then $v_l > v_0$. From (2.10) it follows that there is some $j < l$ with $u_j + v_j = 0$ and $v_j = v_l - 1 \geq v_0$. By (2.7), this means $v_j = v_0$, j is in S_{v_0} and j is on P. This completes the proof.

Another proof of the theorem can be deduced from [5] and [**11**]. It is worth noting that Theorem 2 is not true for general directed graphs, as we shall show in §4.

3. An outline of a proof of Theorem 1. The proof uses ideas similar to the ones used in the proof of Theorem 2.

Let $C = (c_{ij})$, i, $j = 0, \ldots, n$, be defined by

$$c_{i0} = 0 \quad \text{for all } i; \qquad c_{0j} = k \quad \text{for all } j, 0 \tag{3.1}$$

$$c_{ii} = 1 \quad \text{for all } i > 0$$

if $i > 0, j > 0$ and $i \neq j$ then

$$c_{ij} = 0 \quad \text{if } (i, j) \in E$$

$$= \text{not defined if } (i, j) \notin E.$$

Consider the following linear program:

I′.

$$\text{minimize} \sum_{i=0, j=0}^{n} c_{ij} x_{ij}$$

$$(3.2)\qquad \begin{cases} \text{where } x_{ij} \geq 0 \text{ for all } i,j, \text{ except that } x_{ij} \text{ is not defined} \\ \text{if } i > 0, j > 0, i \neq j, (i, j) \notin E. \end{cases}$$

$$(3.3)\qquad \sum_{j=0}^{n} x_{0j} = \sum_{i=0}^{n} x_{i0} = n$$

$$(3.4)\qquad \sum_{j=0}^{n} x_{ij} = 1 \quad \text{for all } i > 0; \qquad \sum_{i=0}^{n} x_{ij} = 1 \quad \text{for } j > 0.$$

Let $\mathscr{P}$ be a path partition, and let $\mathscr{P}^0$ denote the set of all paths in $\mathscr{P}$ of cardinality at most k, and $\mathscr{P}^+$ denote the set of paths in $\mathscr{P}$ of cardinality at least k. Paths of cardinality k are assigned arbitrarily to $\mathscr{P}^0$ or $\mathscr{P}^+$. We define the following matrix

$$X(\mathscr{P}) = (x_{ij}) \text{ corresponding to } \mathscr{P}:$$

$$\begin{array}{lll} & x_{00} = n - |\mathscr{P}^+| & \\ \text{if } j > 0, & x_{0j} = 1 & \text{if } j \text{ is the start of some path in } \mathscr{P}^+. \\ & \phantom{x_{0j}} = 0 & \text{otherwise} \\ \text{if } i > 0, & x_{i0} = 1 & \text{if } i \text{ is the end of some path in } \mathscr{P}^+. \\ & \phantom{x_{i0}} = 0 & \text{otherwise} \\ \text{if } i > 0, & x_{ii} = 1 & \text{if } i \text{ belongs to some path in } \mathscr{P}^0. \\ & \phantom{x_{ii}} = 0 & \text{otherwise} \\ & x_{ij} = 1 & \text{if for some } P \in \mathscr{P}^+, (i, j) \text{ is an edge of } P. \\ & \phantom{x_{ij}} = 0 & \text{otherwise.} \end{array}$$

As in §2, it can be shown that in this correspondence, every integral optimal solution of (3.2)–(3.4) corresponds to a k-optimum path partition, and conversely.

Consider the dual problem.

II$'$.

$$(3.5)\qquad \begin{array}{l} \text{maximize } n(u_0 + v_0) + \sum_{i=1}^{n} u_i + \sum_{i=1}^{n} v_j \\ \text{where } u_i + v_j \leq c_{ij} \quad \text{for all } i, j. \end{array}$$

We may assume that there exists an integral optimum solution of II$'$ satisfying $u_0 = v_0 = 0$, $u_i \leq 0$ and $0 \leq v_i \leq k$.

We associate a partial k-colouring $\mathscr{C}^k = \{C_1, C_2, \ldots, C_k\}$ to such a solution in the following way. Let

$$C_r = \{i > 0 \colon 1 - u_i = v_i = r\}.$$

Using the complementary slackness conditions it can be proved (as in §2) that $\mathscr{C}^k$ is orthogonal to every k-optimum path partition.

4. Some counterexamples. Let G be a poset, and let $\mathscr{P}$, and $\mathscr{C}^k$ be a path partition, and a partial k-colouring of G, respectively. Since every path P in $\mathscr{P}$ meets at most $\min\{|P|, k\}$ vertices of $\mathscr{C}^k$, we have

$$|\mathscr{C}^k| \leq \sum_{P \in \mathscr{P}} \min\{|\mathscr{P}|, k\}. \tag{4.1}$$

If $\mathscr{P}$ and $\mathscr{C}$ are orthogonal, then equality holds and $\mathscr{P}$ is k-optimum and $\mathscr{C}$ is optimum. Thus, the following extension of Conjecture 1 is valid for G.

THEOREM 1′. *For every k-optimum path partition $\mathscr{P}$, there exists an optimum partial k-colouring $\mathscr{C}^k$ orthogonal to $\mathscr{P}$.*

However, if G is not a poset, Theorem 1′ may not be valid, as demonstrated in the following example, for $k = 1$ (see Figure 1). The set $S = \{1, 3, 6\}$ denotes the unique optimum independent set. $\mathscr{P} = \{(1, 2, 3, 5, 6), (4)\}$ is a 1-optimum path partition not orthogonal to S.

In a similar manner, the following extension of Conjecture 3 holds for all posets G.

THEOREM 3′. *For every optimum path k-pack $\mathscr{P}^k$, there exists a k-optimum colouring $\mathscr{C}$ orthogonal to $\mathscr{P}^k$.*

Theorem 3′ may not be valid for graphs other than posets, as shown in the following counterexample for $k = 1$ (see Figure 2).

The path $P = (1, 2, 3, 4)$ is a longest path, and $\chi(G) = 3$. But any 3-colouring colours P in two different colours, as shown in Figure 2.

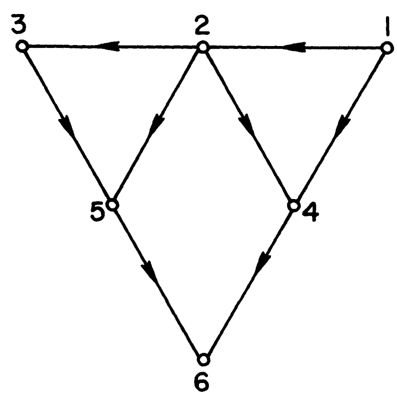

FIGURE 1

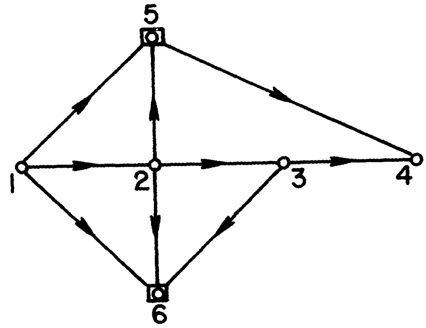

FIGURE 2

Another variant of Conjecture 1 is:

THEOREM 1″. *For every optimum partial k-colouring $\mathscr{C}^k$, there exists a path partition $\mathscr{P}$, orthogonal to $\mathscr{C}^k$.*

It can be proved that Theorem 1″ is valid for posets, but not in general. For $k = 1$, we have the following counterexample (see Figure 3).

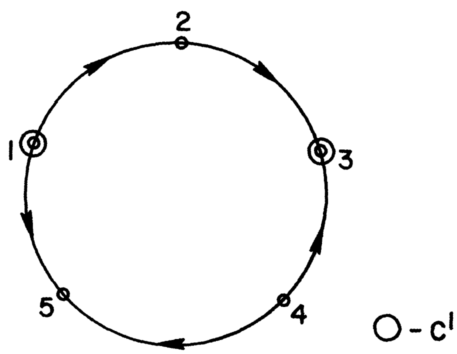

FIGURE 3

No path partition is orthogonal to $\mathscr{C}^1 = \{(1,3)\}$ in G. A similar variant on Conjecture 3 is

THEOREM 3″. *For every k-optimum colouring $\mathscr{C}$ there exists a path k-pack orthogonal to $\mathscr{C}$.*

As in Theorem 1″, this theorem is valid for posets, but not for all graph, as demonstrated in Figure 4.

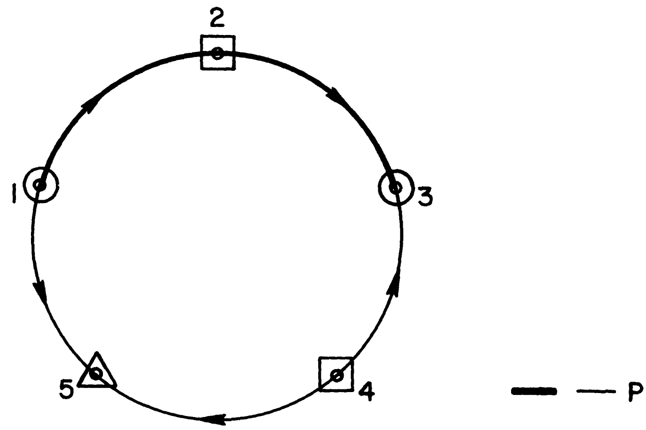

FIGURE 4

The path $P = (1, 2, 3)$ is a unique longest path but it is not orthogonal to the colouring $\mathscr{C} = \{(1,3), (2,4), (5)\}$.

Finally, we show that neither Theorem 1 nor Theorem 2 is true in general for all graphs.

Let $G = (V, E)$ be defined by (see Figure 5)

$$V = \{P_1, P_2, P_3, P_4, P_5, Q, R\}$$

and

$$E = \{(P_i, P_j) \text{ where } i < j\} \cup \{(P_3, Q), (Q, R), (R, P_3)\}.$$

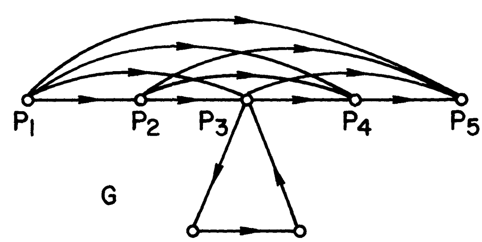

FIGURE 5

It can be verified that for any maximum independent set S in G, there exists a path partition which is not orthogonal to S. Also, there is no way of colouring G so that all longest paths (there are three of them) meet all colours. Hence G serves as a counterexample for $k = 1$ for Theorem 1 as well as for Theorem 2, when considered for general graphs.

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