CONCERNING HYPERSPACES OF CERTAIN PEANO CONTINUA AND STRONG REGULARITY OF WHITNEY MAPS

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Let X be a Peano continuum and let $\mathfrak{H} = 2^X$ (resp., $C(X)$) be the space of all nonempty closed subsets (resp., subcontinua) of X with Hausdorff metric. If $\mathfrak{H} = C(x)$, assume that X contains no free arc. Then the following are shown.

(1) If ω is an admissible Whitney map for $\mathfrak{H}$, then

$$\omega|\omega^{-1}((0, \omega(X))): \omega^{-1}((0, \omega(X))) \to (0, \omega(X))$$

is a trivial bundle map with Hilbert cube fibers.

(2) If X is the Hilbert cube Q, then there is a strongly admissible Whitney map ω for $\mathfrak{H}$ such that $\omega|\omega^{-1}([0, \omega(X))) \to [0, \omega(X))$ is a trivial bundle map with Hilbert cube fibers.

(3) If X is the n-sphere S^n ($n = 1, 2, \ldots,$), then there is a Whitney map ω for 2^X such that for some $t_0 \in (0, \omega(X))$, $\omega|\omega^{-1}((0, t_0])$: $\omega^{-1}((0, t_0]) \to (0, t_0]$ is a trivial bundle map with $X \times Q$ fibers. If X is the n-sphere S^n ($n = 2, 3, \ldots,$), there is a Whitney map ω for $C(X)$ such that for some $t_0 \in (0, \omega(X))$, $\omega|\omega^{-1}((0, t_0])$ is a trivial bundle map with $X \times Q$ fibers.

1. **Introduction.** Throughout this paper, the word *compactum* means a compact metric space. A connected compactum is a *continuum*. A *Peano continuum* is a locally connected continuum. If x and y are points of a metric space, $d(x, y)$ denotes the metric from x to y. For any subsets A and B of a metric space, let $d(A, B) = \inf\{d(a, b)|a \in A, b \in B\}$. Also, let $d_H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}$. d_H is called the *Hausdorff metric*. The hyperspaces of a continuum X are the spaces $2^X = \{A \subset X|A$ is nonempty and compact$\}$ and $C(X) = \{A \in 2^X|A$ is connected$\}$ which are metrized with the Hausdorff metric d_H. Let $F_1(X) = \{\{x\}|x \in X\}$. A *Whitney map* for a hyperspace $\mathfrak{H}$ of a continuum X is a continuous function $\omega\colon \mathfrak{H} \to [0, \omega(X)]$ such that $\omega(\{x\}) = 0$ for each $\{x\} \in F_1(X)$, and if $A, B \in \mathfrak{H}$ and $A \subsetneq B$, then $\omega(A) < \omega(B)$ (see [8, 13]). The notion of Whitney map is a convenient and important tool for hyperspace theory. If ω is a Whitney map for $\mathfrak{H} = 2^X$ or $C(X)$ and $0 \le t < \omega(X)$, then $\omega^{-1}(t)$ is called a *Whitney level*. Note that Whitney levels are coverings of X which, as t gets close to zero, converge to $\omega^{-1}(0) = F_1(X) \cong X$. It is of interest to obtain information about the

structure of Whitney levels and to determine those properties which are preserved by the convergence of positive Whitney levels to the zero level. Throughout this paper, by Q we mean the Hilbert cube $\prod_{i=1}^{\infty}[-1,1]$. In [4], Curtis and Schori proved that for any Peano continuum X, 2^X is a Hilbert cube and if X contains no free arc, $C(X)$ is also a Hilbert cube. Recently, Goodykoontz and Nadler introduced a very important notion "admissible Whitney map" to study the structure of certain Whitney maps. Let $\mathfrak{H} = 2^X$ or $C(X)$. A Whitney map ω for $\mathfrak{H}$ is an *admissible Whitney map* [7] if there is a homotopy $h\colon \mathfrak{H} \times [0,1] \to \mathfrak{H}$ satisfying the following two conditions;

(1) for all $A \in \mathfrak{H}$, $h(A,1) = A$, $h(A,0) \in F_1(X)$, and

(2) if $\omega(h(A,t)) > 0$ for some $A \in \mathfrak{H}$, $t \in (0,1]$, then $\omega(h(A,s)) < \omega(h(A,t))$ whenever $0 \le s < t \le 1$.

Such a homotopy $h\colon \mathfrak{H} \times [0,1] \to \mathfrak{H}$ is called an *ω-admissible deformation* for $\mathfrak{H}$. In [7], it is shown that if X is either a compact starshaped subset of a Banach space or a dendrite, then there exist admissible Whitney maps for 2^X and $C(X)$. Also, if X is any smooth dendroid, then every Whitney map for $C(X)$ is admissible. By using this notion, Goodykoontz and Nadler proved the following very interesting results.

(1.1) THEOREM ([7]). *Let X be a Peano continuum and let ω be an admissible Whitney map for $\mathfrak{H} = 2^X$ or $C(X)$. If $\mathfrak{H} = C(X)$, assume that X contains no free arc. Then for any $t \in (0, \omega(X))$, $\omega^{-1}(t)$, $\omega^{-1}([0,t])$ and $\omega^{-1}([t, \omega(X)])$ are Hilbert cubes.*

(1.2) THEOREM ([7]). *If ω is an admissible Whitney map for $\mathfrak{H} = 2^X$ or $C(X)$, then ω is an open map with FAR fibers.*

The purpose of this paper is to give more precise information by using infinite-dimensional topology. The ideas and techniques in this paper essentially depend on [7]. Our main results are the following: Let X be a Peano continuum and let $\mathfrak{H} = 2^X$ or $C(X)$. If $\mathfrak{H} = C(X)$, assume that X contains no free arc.

(1) If ω is an admissible Whitney map for $\mathfrak{H}$, then $\omega|\omega^{-1}((0, \omega(X)))\colon$ $\omega^{-1}((0, \omega(X))) \to (0, \omega(X))$ is a trivial bundle map with Hilbert cube fibers.

(2) If X is the Hilbert cube Q, then there is a strongly admissible Whitney map ω for $\mathfrak{H}$ such that $\omega|\omega^{-1}([0, \omega(X)))\colon \omega^{-1}([0, \omega(X))) \to [0, \omega(X))$ is a trivial bundle map with Hilbert cube fibers.

(3) If X is the n-sphere S^n ($n = 1, 2, \ldots,$), then there is a Whitney map ω for 2^X such that for some $t_0 \in (0, \omega(X))$, $\omega|\omega^{-1}((0, t_0])$: $\omega^{-1}((0, t_0]) \to (0, t_0]$ is a trivial bundle map with $X \times Q$ fibers. If X is the n-sphere S^n ($n = 2, 3, \ldots$), there is a Whitney map ω for $C(X)$ such that for some $t_0 \in (0, \omega(X))$, $\omega|\omega^{-1}((0, t_0])$: $\omega^{-1}((0, t_0]) \to (0, t_0]$ is also a trivial bundle map with $X \times Q$ fibers.

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2. Strongly regular mappings and Whitney maps. In this section, we show that if X is a Peano continuum and ω is an admissible Whitney map for $\mathfrak{H} = 2^X$ or $C(X)$, then the restriction $\omega|\omega^{-1}((0, \omega(X)])$: $\omega^{-1}((0, \omega(X)]) \to (0, \omega(X)]$ of ω to $\omega^{-1}((0, \omega(X)])$ is a strongly regular mapping with AR fibers. Also, we introduce the notion "strongly admissible Whitney map" and prove that if ω: $\mathfrak{H} \to [0, \omega(X)]$ is a strongly admissible Whitney map, then ω is a strongly regular mapping with AR fibers.

(2.1) DEFINITION (cf. [7, (2.1)]). Let X be a continuum and let $\mathfrak{H}$ denote either one of the hyperspaces 2^X or $C(X)$. A Whitney map ω for $\mathfrak{H}$ is called a *strongly admissible Whitney map* for $\mathfrak{H}$ provided that there is a homotopy h: $\mathfrak{H} \times [0, 1] \to \mathfrak{H}$ satisfying the following conditions;
(1) $h(\{x\}, s) = \{x\}$ for each $\{x\} \in F_1(X)$ and $0 \le s \le 1$,
(2) $h(A, 1) = A, h(A, 0) \in F_1(X)$ for each $A \in \mathcal{H}$ and
(3) if $\omega(h(A, t)) > 0$ for some $A \in \mathfrak{H}$ and $t \in (0, 1]$, then $\omega(h(A, s)) < \omega(h(A, t))$ for each $0 \le s < t \le 1$.

By definitions, every strongly admissible Whitney map is an admissible Whitney map, but the converse of this assertion is not true. Note that the existence of a strongly admissible Whitney map for $\mathfrak{H}$ implies that $F_1(X) \cong X$ is a strong deformation retract of $\mathfrak{H}$. Hence we have

(2.2) PROPOSITION. *Suppose that X is a Peano continuum and ω is a strongly admissible Whitney map for $\mathfrak{H} = 2^X$ or $C(X)$. Then X is an AR.*

Proof. By [14], $\mathfrak{H}$ is an AR, which implies that X is an AR.

(2.3) THEOREM (*cf.* [7, (2.14)]). *If X is a compact convex subset of a Banach space B, then there are strongly admissible Whitney maps for $\mathfrak{H} = 2^X$ and $C(X)$.*

Proof. Let ω be the Whitney map for $\mathfrak{H}$ as defined in [13, p. 275]. Since X is an AR, there is a retraction r: $\mathfrak{H} \to F_1(X) \cong X$. For each $A \in \mathfrak{H}$, define a homotopy h_A: $A \times [0,1] \to X$ by $h_A(a, s) = sa + (1 - s)r(A)$ for each $a \in A$ and $0 \le s \le 1$. Also, define a function h: $\mathfrak{H} \times [0,1] \to \mathfrak{H}$ by

$$h(A, s) = \left\{ h_A(a, s) | a \in A \right\} \qquad \text{for each } A \in \mathfrak{H} \text{ and } 0 \le s \le 1.$$

It is easily seen that h is continuous. The definition of h implies that h satisfies the conditions (1) and (2) of Definition (2.1). Also, by [7, (2.13)], h satisfies the condition (3) of Definition (2.1). This completes the proof.

(2.4) THEOREM (*cf.* [7, (2.16)]). *If X is any dendrite, then there are strongly admissible Whitney maps for $\mathfrak{H} = 2^X$ and $C(X)$.*

Proof. Let $p \in X$. Let $\le_p$, d and D be as in [7, (2.16)] and let ω be the Whitney map for $\mathfrak{H} = 2^X$ or $C(X)$ as in [13, p. 275]. Since X is an AR, there is a retraction r: $\mathfrak{H} \to F_1(X) \cong X$. For each $A \in \mathfrak{H}$, define a function h_A: $A \times [0,1] \to X$ satisfying the following conditions;
  (1) $h_A(a, s)$ is contained in the arc from a to $r(A)$, and
  (2) if $a \in A$, then $s \cdot D(a, h_A(a, s)) = (1 - s) \cdot D(h_A(a, s), r(A))$.
Then we can easily see that h_A is continuous. Also, define a function h: $\mathfrak{H} \times [0,1] \to \mathfrak{H}$ by
  (3) $h(A, s) = \{ h_A(a, s) | a \in A \}$ for each $A \in \mathfrak{H}$ and $0 \le s \le 1$.
Then h is continuous. Also, by the definition of h and [7, (2.13)], we can see that h satisfies the conditions (1), (2) and (3) of Definition (2.1). Hence ω is a strongly admissible Whitney map for $\mathfrak{H}$.

A map p: $E \to B$ between metric spaces is a *strongly regular mapping* ([1] and [6]) if it is a proper map and for each $b_0 \in B$ and $\varepsilon > 0$ there is a neighborhood $U(b_0)$ of b_0 in B such that if $b \in U(b_0)$, there exist maps g_{bb_0}: $p^{-1}(b) \to p^{-1}(b_0)$ and $g_{b_0 b}$: $p^{-1}(b_0) \to p^{-1}(b)$ such that g_{bb_0}, $g_{b_0 b}$ move points no more than ε and $g_{bb_0} g_{b_0 b}$, $g_{b_0 b} g_{bb_0}$ are homotopic to the identity maps on $p^{-1}(b_0)$, $p^{-1}(b)$ via homotopies which move points no more than ε, respectively. Note that every strongly regular mapping is an open map.

Then we have the following

(2.5) THEOREM. *Let X be a Peano continuum and let $\mathfrak{H} = 2^X$ or $C(X)$.*
  (i) *If ω is an admissible Whitney map for $\mathfrak{H}$, then $\omega | \omega^{-1}((0, \omega(X)])$: $\omega^{-1}((0, \omega(X)]) \to (0, \omega(X)]$ is a strongly regular mapping with AR fibers.*

(ii) *If ω is a strongly admissible Whitney map for $\mathfrak{H}$, then ω: $\mathfrak{H} \to$ $[0, \omega(X)]$ is a strongly regular mapping with AR fibers.*

Proof. Suppose that ω is an admissible Whitney map for $\mathfrak{H}$ and h is an ω-admissible deformation. By [7, (2.9)], $\omega^{-1}(t)$ is a compact AR for each $0 < t \leq \omega(X)$. Let $t \in (0, \omega(X))$. Define θ_t: $\omega^{-1}([t, \omega(X)]) \to [0, 1]$ by

$$\theta_t(A) \equiv \text{the unique number in } [0, 1] \text{ such that } \omega\big(h(A, \theta_t(A))\big) = t$$

$$\text{(see } [7, (2.7)]\text{)}.$$

Define r_t: $\omega^{-1}([t, \omega(X)]) \to \omega^{-1}(t)$ by

$$r_t(A) = h(A, \theta_t(A)) \qquad \text{(see } [7, (2.7)]\text{)}.$$

Now let $t \in (0, \omega(X)]$. Let K_d be as in [7, (1.2)]. Define Ψ_t: $\omega^{-1}([0, t]) \to [0, \infty)$ by

$$\Psi_t(A) \equiv \text{the smallest number in } [0, \infty) \text{ such that}$$

$$\omega\big(K_d(A, \Psi_t(A))\big) = t.$$

Define R_t: $\omega^{-1}([0, t]) \to \omega^{-1}(t)$ by

$$R_t(A) = K_d(A, \Psi_t(A)).$$

Now suppose that $0 < t_1 < t_2 \leq \omega(X)$. Define mappings $g_{t_1 t_2}$: $\omega^{-1}(t_1) \to \omega^{-1}(t_2)$ and $g_{t_2 t_1}$: $\omega^{-1}(t_2) \to \omega^{-1}(t_1)$ by

$$g_{t_1 t_2} = R_{t_2}\big|\omega^{-1}(t_1) \quad \text{and} \quad g_{t_2 t_1} = r_{t_1}\big|\omega^{-1}(t_2).$$

To show that $g_{t_2 t_1} \circ g_{t_1 t_2}$ is homotopic to the identity map on $\omega^{-1}(t_1)$, define H_1: $\omega^{-1}(t_1) \times [0, 1] \to \omega^{-1}(t_1)$ by

$$H_1(A, s) = r_{t_1}\Big(K_d\big(A, s \cdot \Psi_{t_2}(A)\big)\Big).$$

To show that $g_{t_1 t_2} \circ g_{t_2 t_1}$ is homotopic to the identity map on $\omega^{-1}(t_2)$, define H_2: $\omega^{-1}(t_2) \times [0, 1] \to \omega^{-1}(t_2)$ by

$$H_2(A, s) = R_{t_2}\Big(h\big(A, 1 - s(1 - \theta_{t_1}(A))\big)\Big).$$

By using maps $g_{t_1 t_2}$, $g_{t_2 t_1}$ and homotopies H_1, H_2, we can easily see that $\omega|\omega^{-1}((0, \omega(X)])$: $\omega^{-1}((0, \omega(X)]) \to (0, \omega(X)]$ is a strongly regular mapping. Finally, if ω is a strongly admissible Whitney map for $\mathfrak{H}$, the case when $t_1 = 0$ can be handled in the analogous fashion. This completes the proof.

3. Trivial bundle maps and Whitney maps. A map $p: E \to B$ is a *trivial bundle map with F fibers* if there is a homeomorphism $h: E \xrightarrow{\approx} B \times F$ such that the following diagram is commutative;

$$E \xrightarrow[\approx]{h} B \times F$$

$$p \searrow \quad \swarrow q$$

$$B$$

where $q: B \times F \to B$ is the projection map, i.e., $q(b, f) = b$ for $(b, f) \in B \times F$.

Then we have the following

(3.1) THEOREM. *Let X be a Peano continuum and let $\mathfrak{H} = 2^X$ or $C(X)$. If $\mathfrak{H} = C(X)$, assume that X contains no free arc.*

(i) *If ω is an admissible Whitney map for $\mathfrak{H}$, then $\omega|\omega^{-1}((0, \omega(X)))$: $\omega^{-1}((0, \omega(X))) \to (0, \omega(X))$ is a trivial bundle map with Hilbert cube fibers.*

(ii) *If X is the Hilbert cube Q, then there is a strongly admissible Whitney map for $\mathfrak{H}$ such that $\omega|\omega^{-1}([0, \omega(X)))$: $\omega^{-1}([0, \omega(X))) \to [0, \omega(X))$ is a trivial bundle map with Hilbert cube fibers.*

To prove (3.1) we need the following result of Chapman and Ferry.

(3.2) THEOREM ([3]). *If $p: E \to B$ is a strongly regular mapping with compact Q-manifold fibers and $\dim B < \infty$, then p is a locally trivial bundle map.*

Proof of (3.1). *Case* (i). It follows from (2.5) that $\omega|\omega^{-1}((0, \omega(X)))$: $\omega^{-1}((0, \omega(X))) \to (0, \omega(X))$ is a strongly regular mapping. By (1.1), $\omega^{-1}(t)$ is a Hilbert cube for each $0 < t < \omega(X)$. Note that $\dim(0, \omega(X)) = 1$. Hence by (3.2), $\omega|\omega^{-1}((0, \omega(X)))$ is a locally trivial bundle map with Hilbert cube fibers. Since $(0, \omega(X))$ is contractible, $\omega|\omega^{-1}((0, \omega(X)))$ is a trivial bundle map.

Case (ii). We may assume that Q is a compact convex subset of the Hilbert space l_2. By (2.3), there is a strongly admissible Whitney map ω for $\mathfrak{H}$. (1.1), (2.5) and (3.2) imply that $\omega|\omega^{-1}([0, \omega(X)))$: $\omega^{-1}([0, \omega(X))) \to [0, \omega(X))$ is a trivial bundle map with Hilbert cube fibers. This completes the proof.

Note that every trivial bundle map is a strongly regular mapping. In the statement (i) of (2.5) (resp., (3.1)), we can not conclude that $\omega|\omega^{-1}([0, \omega(X)))$ is a strongly regular mapping (resp., a trivial bundle

map). In fact, we have the following

(3.3) PROPOSITION. *Let X be a Peano continuum and let ω be an admissible Whitney map for $\mathfrak{H} = 2^X$ or $C(X)$. If $\omega|\omega^{-1}([0, \omega(X)))$: $\omega^{-1}([0, \omega(X))) \rightarrow [0, \omega(X))$ is a strongly regular mapping, then X is an AR. In particular, if X is the topological cone over a locally connected compactum Y which is not an ANR, then each admissible Whitney map for $\mathfrak{H}$ is not a strongly regular mapping.*

Proof. By [7, (2.9)], $\omega^{-1}(t)$ is a compact AR for each $0 < t \leq \omega(X)$. If $\omega|\omega^{-1}([0, \omega(X)))$ is a strongly regular mapping, for any $\varepsilon > 0$ there is a positive Whitney level $\omega^{-1}(t)$ $(t > 0)$ such that X is ε-homotopically dominated by $\omega^{-1}(t)$. By Dowker's result [Homotopy extension theorems, Proc. London Math. Soc. **6** (1959), p. 105], X is an ANR. Since X is contractible, X is an AR.

Naturally, the following question is raised.

(3.4) *Question.* If X is a compact AR, is there a strongly admissible Whitney map for 2^X or $C(X)$?

Finally, we shall prove the following

(3.5) THEOREM. *Let X be the n-sphere S^n ($n = 1, 2, \ldots,$). Then there is a Whitney map ω for $\mathfrak{H} = 2^{S^n}$ ($n = 1, 2, \ldots,$) or $C(S^n)$ ($n = 2, 3, \ldots,$) such that for some $t_0 \in (0, \omega(X))$, $\omega|\omega^{-1}([0, t_0])$: $\omega^{-1}([0, t_0]) \rightarrow [0, t_0]$ is a strongly regular mapping. Moreover, $\omega|\omega^{-1}((0, t_0])$: $\omega^{-1}((0, t_0]) \rightarrow (0, t_0]$ is a trivial bundle map with $X \times Q$ fibers.*

To prove (3.5), we need Toruńczyk's characterization of compact Q-manifolds.

(3.6) THEOREM ([13, Theorem 1]). *Let X be a compact ANR. If the identify map on X is a uniform limit of Z-maps, then X is a compact Q-manifold* (for the definition of Z-map, see [13]).

Proof of (3.5). Assume that $X = S^n = \{ x = (x_1, x_2, \ldots, x_{n+1}) \in R^{n+1} | \|x\| = 1 \} \subset R^{n+1}$. Let d be the metric on S^n such that for each $x = (x_1, x_2, \ldots, x_{n+1})$ and $y = (y_1, y_2, \ldots, y_{n+1}) \in S^n$,

$$d(x, y) = \arccos[x_1 y_1 + x_2 y_2 + \cdots + x_n y_n + x_{n+1} y_{n+1}].$$

Note that if $d(x, y) < \pi$ and $0 \leq s \leq 1$, there is only one point $h_s(x, y)$ $\in S^n$ such that

(1) $d(x, h_s(x, y)) + d(h_s(x, y), y) = d(x, y)$, and

(2) $d(x, h_s(x, y)) = s \cdot d(x, y)$ (e.g., see [2, p. 327]).

Since $F_1(X) \cong X$ is an ANR, there is a neighborhood U of $F_1(X)$ in $\mathfrak{H}$ and a retraction $r: U \to F_1(X)$. Choose a positive number $t_1 \in (0, \omega(X))$ such that

(3) $\omega^{-1}([0, t_1]) \subset U$ and

(4) $d(r(A), a) < \pi/2$ for each $A \in \omega^{-1}([0, t_1])$ and $a \in A$.

Define a function $h_A: A \times [0, 1] \to X$ by

(5) $h_A(a, s) = h_s(r(A), a)$ for each $a \in A$ and $0 \leq s \leq 1$.

Then h_A is continuous. Also, define a function $h: \omega^{-1}([0, t_1]) \times [0, 1] \to \mathfrak{H}$ by

(6) $h(A, s) = \{h_A(a, s) | a \in A\}$ for each $A \in \omega^{-1}([0, t_1])$, $0 \leq s \leq 1$.

Then h is continuous and the definition of h and [7, (2.13)] imply that h satisfies the following conditions.

(7) $h(\{x\}, s) = \{x\}$ for each $\{x\} \in F_1(X), 0 \leq s \leq 1$,

(8) $h(A, 1) = A$, $h(A, 0) = \{r(A)\} \in F_1(X)$ for each $A \in \omega^{-1}([0, t_1])$ and

(9) if $A \in \omega^{-1}([0, t_1])$ and $\omega(h(A, s)) > 0$ for some $0 < s \leq 1$, then $\omega(h(A, s)) > \omega(h(A, s'))$ for each $0 \leq s' < s \leq 1$.

Choose a positive number t_0 ($< t_1$). By (9), we have

(10) $h(\omega^{-1}([0, t_0]) \times [0, 1]) \subset \omega^{-1}([0, t_0])$.

also, by using the homotopy K_d (see [7, (1.2)]) and by the same way as in [7, (2.7)], $\omega^{-1}(t)$ is a retract of $\omega^{-1}([0, t_1))$ for each $0 \leq t \leq t_0$. Since $\omega^{-1}([0, t_1))$ is an ANR, $\omega^{-1}(t)$ is also an ANR. By using (3.6) and the same argument as in [7, §3, 4], we can see that $\omega^{-1}(t)$ is a compact Q-manifold for each $0 < t \leq t_0$. Also, by using the homotopy h, we can see that $\omega | \omega^{-1}([0, t_0]): \omega^{-1}([0, t_0]) \to [0, t_0]$ is a strongly regular mapping (see (2.5)). Hence by (3.2), $\omega | \omega^{-1}((0, t_0])$ is a trivial bundle map with compact Q-manifold fibers. Let $\omega^{-1}(t) = M$ for some $t \in (0, t_0)$. Consider the map $g = q \circ ((\omega | \omega^{-1}([0, t_0])) \times 1_q): \omega^{-1}([0, t_0]) \times Q \to [0, t_0] \times Q \to [0, t_0]$, where $1_Q: Q \to Q$ denotes the identity map on Q and q: $[0, t_0] \times Q \to [0, t_0]$ is the projection map. Then g is also a strongly regular mapping with compact Q-manifold fibers, hence g is a trivial bundle map with $M \times Q$ fibers. In particular, $M \cong M \times Q \cong g^{-1}(0) \cong S^n \times Q$. This completes the proof.

(3.7) REMARK. In the proof of (3.5), we cannot conclude that $\omega | \omega^{-1}((0, \omega(X)))$ is a trivial bundle map with $S^n \times Q$ fibers. In [7, (4.15)], Goodykoontz and Nadler showed the following fact: Let $X = S^1$ be the

unit circle and let ω be the Whitney map for 2^X as defined in the proof (3.5). Then there is a positive number t_2, $t_2 < \omega(X)$, such that $\omega^{-1}(t_2)$ has a finite-dimensional nonempty open subset of $\omega^{-1}(t_2)$. Hence $\omega^{-1}(t_2)$ is not a Q-manifold.

(3.8) COROLLARY. *There is a Whitney map* ω *for* $\mathfrak{H} = 2^{S^n \times Q}$ *or* $C(S^n \times Q)$, $(n = 1, 2, \ldots)$ *such that for some* $t_0 \in (0, \omega(S^n \times Q))$, $\omega|\omega^{-1}([0, t_0])$ *is a trivial bundle map with* $S^n \times Q$ *fibers.*

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