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ERGODIC CONTINUOUS SKEW PRODUCT ACTIONS OF AMENABLE GROUPS

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Given two compact, metric topological dynamical systems (Y, T, μ) and (Z, G, ν), where T and G are locally compact separable groups acting continuously on spaces, preserving finite ergodic measures μ and ν respectively, a continuous cocycle α on (Y, T, μ) defines a skew product T action on $Z \times Y$ by $(z, y) \cdot t \to (z\alpha(y, t), y \cdot t)$. We prove that for a large class of amenable groups T and, under some very general conditions on spaces Y, Z and G, residually many continuous cocycles lift various ergodic and mixing properties from Y to $Z \times Y$. Similar results are obtained for non-trivial compact group extensions of (Y, T, μ).

1. Introduction and definitions. (i) Given a compact metric space X, $C(X)$ will denote the set of all continuous complex valued maps on X. Given a Borel set $V \subseteq X$, $\mu|_V$ denotes the normalized restriction to V of a Borel probability μ on X. We let $C(X, Y)$ stand for the space of all continuous maps from X to Y. We always denote by d the metric on any space and without loss of generality assume that $d(\cdot, \cdot) < 1$. A *topological dynamical system* (t.d.s.) is a pair (Y, T) where T is a locally compact, separable (l.c.s.) group acting continuously on the right of the compact metric space Y, with action $(y, t) \to y \cdot t$. If in addition we have a T-invariant Borel probability μ on Y, we denote the system by (Y, T, μ). In this case one naturally gets for $t \in T$ a unitary representation U_t on $L^2(Y, \mu)$ defined by $U_t f(y) = f(y \cdot t)\ \forall f \in L^2(Y, \mu)$. The system (Y, T, μ) is *ergodic* if for each $t \in T$, $U_t f = f$ a.e. implies f is constant a.e.; *properly ergodic* if it is ergodic and $\mu(y \cdot T) = 0$, $\forall y \in Y$; and *uniquely ergodic* if μ is the only T-invariant Borel probability on Y. We call the system (Y, T, μ) *weakly mixing* iff $(Y \times Y, T, \mu \times \mu)$ is ergodic, where the action on $Y \times Y$ is the diagonal action. A *factor map* π: $(X, T, \mu) \to (Y, T, \nu)$ is a continuous onto map from X to Y such that $\pi(x \cdot t) = \pi(x) \cdot t$ and $\pi_* \mu = \nu$; we then say π: $X \to Y$ determines an *extension*. A *bi-transformation group* is a triple (G, X, T) where (X, T) is a t.d.s. and G is a compact group acting continuously and freely on the left of X, such that the G and T actions commute with each other. A bi-transformation group gives rise to an extension π: $(X, T) \to (Y, T)$ with $Y = G\backslash X$, π the quotient map, and the T action on Y is the quotient action. We call this extension a *group extension* with fiber G. Given a group extension $\pi\colon (X, T) \to (Y, T, \mu)$, the *Haar lift* of μ is the Borel measure $\tilde{\mu}$ on X defined by $\tilde{\mu}(f) = \int_Y(\int_{\pi^{-1}(y)} f(gx)\, d\eta(g))\, d\mu(y)$, $\forall f \in C(X)$, where η is the normalized Haar measure on fiber G.

(ii) Let (Y, T, μ) be a t.d.s. and G be a l.c.s. group. A *continuous cocycle* α is a continuous map $\alpha\colon Y \times T \to G$, satisfying the cocycle condition $\alpha(y, t_1t_2) = \alpha(y, t_1)\alpha(y \cdot t_1, t_2)$, $\forall y \in Y$, t_1, $t_2 \in T$. Let $Z(Y, T, G)$ denote the set of all continuous cocycles. (Hereafter, we will drop the word *continuous*.) Every $f \in C(Y, G)$ generates a cocycle $l^f\colon Y \times T \to G$ by setting $l^f(y, t) = f(y)^{-1}f(y \cdot t)$ $\forall y \in Y$ and $t \in T$. Such cocycles are called *coboundaries* and are denoted by $B(Y, T, G)$. The *trivial cocyle* (denoted by l) is a map $(y, t) \to e\forall(y, t) \in Y \times T$, where e is the identity element of G. If $\varphi \in Z(Y, T, G)$ and $l^f \in B(Y, T, G)$ set $\varphi \cdot l^f(y, t) = f(y)^{-1}\varphi(y, t)f(y \cdot t)$. It is easy to verify that $\varphi \cdot l^f \in Z(Y, T, G)$. We call $\varphi_1, \varphi_2 \in Z(Y, T, G)$ *cohomologous* if $\varphi_2 = \varphi_1 \cdot l^f$ for some $f \in C(Y, G)$. It can be shown that the set $Z(Y, T, G)$ is a Polish space with respect to the metric given by

$$D(\varphi_1, \varphi_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} D_n(\varphi_1, \varphi_2),$$

where

$$D_n(\varphi_1, \varphi_2) = \sup_{Y \times K_n} d(\varphi_1(y, t), \varphi_2(y, t)).$$

Here $(K_n)_{n=1}^{\infty} \subseteq T$ is a sequence of compact sets such that $K_n \subseteq \operatorname{int} K_{n+1}$, and $\bigcup_{n=1}^{\infty} K_n = T$. This metric also generates the compact-open topology.

(iii) Now we will describe the general set up used throughout this paper. Let (Y, T, μ) be a t.d.s. Let X denote either $Z \times Y$ (where (Z, G, ν) is another t.d.s.) or a group extension of Y with (compact) fiber group G. Let $\bar{\mu}$ be either $\nu \times \mu$ or the Haar lift $\tilde{\mu}$ on X. The factor map from X to Y will be denoted by π. In either case, we can assume that G acts on the left of X [when $X = Z \times Y$, the left G action is $g(z, y) = (zg^{-1}, y)$]. We also always have a right T action on X [when $X = Z \times Y$ this is $(z, y) \cdot t = (z, y \cdot t)$]. Given $\varphi \in Z(Y, T, G)$ *the skew product T action* on X is given by $x, t \to \varphi(\pi x, t)^{-1}x \cdot t$, $\forall x \in X$ and $\forall t \in T$. The t.d.s. obtained from this new action will be denoted by either (X, T_φ) or $(Z \times_\varphi Y, T)$ if $X = Z \times Y$.

(iv) We regard the skew-product action as a perturbation of the original T action on the extension X. The main result of this paper says that residually many such perturbations retain various dynamical properties of (Y, T, μ). Investigations of these sort of lifting results are not new in the context of group extensions and integer or real actions (i.e. when $T = \mathbf{Z}$ or $\mathbf{R}$). When $T = \mathbf{Z}$ and fiber group G is compact connected Lie group, R. Ellis [**4**] has shown that many cocycles lift minimality. Similar result for lifting topological weak-mixing is obtained by R. Peleg [**15**]. When G is compact connected abelian, Jones and Parry [**14**] have analogous results for lifting ergodicity and weak-mixing. Recently S. Glasner and B. Weiss [**6**] have obtained similar results for lifting unique ergodicity for integer actions, when the fiber Z is a homogeneous space of G and the group G is a Peano space. Using the same technique they proved a similar result when $Z = P^n(R)$-the projective n-space, and $G = \mathrm{SL}(n+1, R)$. The result of Glasner and Weiss is different from the previous results in the sense that their generic theorems hold in the compact-open closure of coboundaries rather than the class of all cocycles. In this paper, in addition to looking at more general ergodic properties, we also consider more general actions than Glasner and Weiss. Amenability of T plays the key role in our proofs. A Rokhlin type tower theorems of C. Series [**17**] and the existence of Følner sequences for such groups gives us a handle on constructing coboundaries with desired properties while remaining close to the trivial cocycle. Finally we emphasize that our perturbations are always with continuous cocycles and not simply measurable cocycles. We feel that these methods can be used in a variety of other situations such as differential equations [**5**], and we will give applications to affine extensions (see [**11**] for a summary) and smooth Anosov systems in future papers.

2. Statements of the main results and corollaries. From now on we will assume the notation of (iii) of §1. Set $H = L^2(X, \bar{\mu})$ and $B(H)$ be the set of bounded operators on H. Given $(W_n)_{n\in\mathbf{N}}$, $W \in B(H)$, $W_n \to_{\mathrm{w}} W$ ($W_n \to_{\mathrm{s}} W$) denotes W_n converges weakly (strongly) to W. We now need to define some bounded operators on H. Let Q and P be respectively the projections on the space of all G invariant and all T-invariant functions in $L^2(X, \bar{\mu})$ [here T-invariant means invariant under the unskewed T-action on X]. Let $(U_g)_{g\in G}$ and $(U_t)_{t\in T}$ be the unitary representations induced on H by the left G and right T actions. Given a $\varphi \in Z(Y, T, G)$, let $(U_t^\varphi)_{t\in T}$ be the unitary representation induced on H by the skew product T action on X, corresponding to φ. Note that $U_t^1 = U_t \, \forall t \in T$.

Next, we give a few more relevant definitions on groups. A l.c.s. group T is *amenable* if the space $L^\infty(T)$ of all essentially bounded Borel maps on T admits a T-invariant mean. This is equivalent to the existence of a *Følner sequence*, that is, a sequence $(K_n)_{n=1}^\infty$ of compact subsets of T such that $K_n \subseteq \operatorname{int} K_{n+1}, \bigcup_{n=1}^{\infty} K_n = T$, and

$$\lim_{n\to\infty} \frac{\rho(K_n t^{-1} \Delta k_n)}{\rho(K_n)} = 0, \quad \forall t \in T,$$

where ρ is a right-Haar measure on T.

We also say that a l.c.s. group G has *property* (A) if for each $\varepsilon > 0$, and each finite set $F \subseteq G$, $\exists$ a continuous map φ: $I = [0,1] \to G$ with $d(\varphi(I), G) < \varepsilon$, $\forall g \in F$. Note that if G is path connected or compact connected abelian then it has this property (A) (in the latter case this follows because the one-parameter subgroups are dense, see [**10**]).

Fix a Følner sequence $(K_n)_{n=1}^{\infty}$ in T and a right Haar measure ρ on T. Define operators V_n^{φ} by setting $V_n^{\varphi} = \int U_t^{\varphi}\, d\rho|_{K_n}(t)$. Finally if $\psi \in C(Y, G)$, let L_ψ be the unitary operator defined by $L_\psi f(x) = f(\psi(\pi x)^{-1} x)$ $\forall x \in X$, $\forall f \in L^2(X, \bar{\mu})$. The main theorem of this paper is the following

THEOREM (2.1). *Consider the extension* π: $(X, T, \bar{\mu}) \to (Y, T, \mu)$ *described before. Assume that*:

(i) T *is an amenable* R-*group*.

(ii) (Y, T, μ) *is free and properly ergodic.*

(iii) *the group* G *has property* (A). *Then* (1) *the set* $\{\varphi | \varphi \in \overline{B(Y, T, G)}$ *such that* $V_n^{\varphi} \to_{\mathrm{w}} PQ\}$ *is residual in* $\overline{B(Y, T, G)}$. (2) *In the case when* π: $X \to Y$ *is either a (compact) group extension or* $X = Z \times Y$ *with* (Z, G, ν) *is ergodic, the set* $\{\varphi | \varphi \in \overline{B(y, T, G)}$ *such that* $(X, T_\varphi, \bar{\mu})$ *is ergodic*$\}$ *is residual.* (3) *Further if* (Y, T, μ) *is weak-mixing, the set* $\{\varphi | \varphi \in \overline{B(Y, T, G)}$ *such that* $(X, T_\varphi, \bar{\mu})$ *is weak mixing*$\}$ *is residual.*

Note that when X is compact group extension of Y or $X = Z \times Y$ with (Z, G, ν) ergodic, the operator PQ is just projection on constants. In the general situation P and Q commute and PQ is again a projection operator. Commutativity of P and Q is based on the fact that T and G actions commute and P and Q can be weakly approximated by convex sums of T and G translates. (For non-amenable groups this fact is based on the existence of a unique invariant mean on the set of weakly almost periodic functions [**8**].)

For the precise definition of an R-group, see Lemma 3.9, it is known that a discrete solvable or connected amenable group is an R-group. Also, it can be seen that in the case of an abelian group extensions, one can use techniques from Jones and Parry [**14**] to get the result without assuming that T is an R-group. However, this technique does not seem to extend to general skew products or even non-abelian group extensions. Hence our method seems more suitable for general situation.

Now we state some corollaries and examples.

COROLLARY (2.2). *Consider the extension* π: $(Z \times Y, T, \nu \times \mu) \to (Y, T, \mu)$ *satisfying the assumptions of Theorem* 2.1, *and let* $T = \mathbf{Z}$. *Let* (Z, G) *be a minimal distal flow* (*see* [3]) *and* (Z, G, ν) *be ergodic. If* (Y, T, μ) *is a K-automorphism* (*Bernoullian*), *then the set* $\{\varphi \mid \varphi \in \overline{B(Y, T, G)}$ *such that* $(Z \times_\varphi Y, T, \nu \times \mu)$ *is a K-automorphism* (*Bernoullian*)$\}$ *is a residual.*

COROLLARY (2.3). *Consider the extension* π: $(X, T, \bar{\mu}) \to (Y, T, \mu)$ *satisfying assumptions of Theorem* (2.1). *Let* (Y, T, μ) *be uniquely ergodic and assume that either* X *is a group extension with* G *abelian, or* $X = Z \times Y$ *and* (Z, G, ν) *is uniquely ergodic with* G *amenable. Then the set* $\{\varphi \mid \varphi \in \overline{B(Y, T, G)}$ *such that* $(X, T_\varphi, \bar{\mu})$ *is uniquely ergodic*$\}$ *is residual.*

COROLLARY (2.4). *Consider either a group extension* π: $X \to Y$ *or* π: $Z \times Y \to Y$ *satisfying assumptions* (i) *and* (iii) *of Theorem* (2.1). *Further let* T *be discrete,* Y *be infinite,* (Y, T) *be free and minimal and* (Z, G) *be minimal distal. Then the set* $\{\varphi \mid \varphi \in \overline{B(Y, T, G)}$ *such that the corresponding skew-product action is minimal*$\}$ *is residual.*

We end this section with two examples.

EXAMPLE 1. Let (Y, T, μ) be any Bernoullian system (say an ergodic group automorphism or a shift). Let N be a connected, simply connected Nilpotent Lie group and Γ be any discrete cocompact subgroup. Setting $Z = N/\Gamma$ and $G = N$, it is known that (Z, G) is minimal distal. Hence by Corollary (2.2) there are continuous cocycles in to N for which nilmanifold extension $N/\Gamma \times_\alpha Y$ is Bernoullian. Moreover, since there are many minimal distal ergodic one-parameter actions (i.e., $G = \mathbf{R}$) on N/Γ, Corollary (2.2) can also be applied in this case.

EXAMPLE 2. Consider $\mathrm{SL}(2, \mathbf{R})$ and $\Gamma \subseteq \mathrm{SL}(2, \mathbf{R})$ be a discrete, cocompact subgroup. Consider the horocycle action of $G = \mathbf{R}$ on $Z = \mathrm{SL}(2, \mathbf{R})/\Gamma$ defined by

$$h_s(\Gamma_g) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} g\Gamma, \qquad s \in \mathbf{R}.$$

It is well known that the action of G on Z is minimal and uniquely ergodic, so Corollary (2.3) is applicable. In fact one can consider more general case of a horospherical extensions and these corollaries will still apply.

3. Proofs. Let $\langle\ ,\ \rangle$ and $\|\ \|$ stand for the inner product and norm in H. Given $f \in H$, $\varepsilon > 0$ and $m \in \mathbf{N}$, define $W(f, \varepsilon, m) = \{\varphi | \varphi \in \overline{B(Y, T, G)}$ such that $\exists M \in \mathbf{N}$, $M > m$ and $|\langle V_M^\varphi f, f\rangle - \langle PQf, f\rangle| < \varepsilon\}$. Note that M depends on f, ε, m and φ. The first part of Theorem (2.1) is an immediate consequence of the following propositions and the Baire-category theorem.

PROPOSITION (3.1). *Let $(f_j)_{j=1}^\infty$ be a dense subset of $L^2(X, \bar{\mu})$. Then*

$$\bigcap_{j\in\mathbf{N}} \bigcap_{n\in\mathbf{N}} \bigcap_{m\in\mathbf{N}} W(f_j, 1/n, m) = \left\{\varphi | \varphi \in \overline{B(Y, T, G)} \text{ such that } V_n^\varphi \underset{\mathrm{w}}{\to} PQ\right\}.$$

PROPOSITION (3.2). *Each $W(f, \varepsilon, m)$ is open in $\overline{B(Y, T, G)}$.*

PROPOSITION (3.3). *Each $W(f, \varepsilon, m)$ is dense in $\overline{B(Y, T, G)}$.*

Before proving these propositions we state an abstract ergodic theorem we need.

THEOREM (3.4). *Let (Y, T, μ) be a t.d.s., Σ be the weakly-closed convex hull of $\{U_t | t \in T\}$ and P be the projection operator on the set of all T-invariant Borel maps in $L^2(Y, \mu)$ and C be the projection on constants. Then*

(1) *$P \in \Sigma$ (see* **[8]**, *this is based on the fact that the space of all weakly almost periodic maps on T has an invariant mean).*

(2) *Further if T is amenable, $(K_n) \subseteq T$ is a Følner sequence and $V_n = \int U_t \, d\rho|_{K_n}(t)$, then*

(a) $V_n \to_{\mathrm{s}} P$ (see **[9]**).

(b) *If $\forall f \in L^2(Y, \mu)$, $\langle V_n f, f\rangle \to \langle Cf, f\rangle$ then $P = C$ and (Y, T, μ) is ergodic. {This is a generalization of the corresponding result for integer group action.}*

The proof of Proposition (3.1) follows from the following general lemma.

LEMMA (3.5). *Let H be a Hilbert space and $(B_n)_{n=1}^\infty$, C be bounded operators on H such that $\sup_{n\in\mathbf{N}}(\|B_n\|, \|C\|) < R$. For each $m, n \in \mathbf{N}$ set, $W(m, n) = \{x | x \in H$ such that $\exists M \in \mathbf{N}$, $M > m$ and $|\langle B_M x, x\rangle - \langle Cx, x\rangle| < 1/n\}$. If $W(m, n)$ is dense in H, $\forall m, n \in \mathbf{N}$, then $W(m, n) = H \; \forall m, n$.*

Proof. Fix $m, n \in \mathbf{N}$. Let $x \in H$. Pick a positive integer $k > 3n$ such that $3R(2\|x\| + 1) < k/n$. Since $W(m, k)$ is dense in H, $\exists y \in W(m, k)$

and $M > m$ such that $\|x - y\| < 1/k$ and $|\langle B_M y, y\rangle - \langle Cy, y\rangle| < 1/k < 1/3n$. Then

$$
\begin{aligned}
|\langle B_M x, x\rangle - \langle B_M y, y\rangle| &= |\langle B_M x, x - y\rangle - \langle B_M(y - x), y\rangle| \\
&\leq \|B_M\|(\|x\| + \|y\|)\|x - y\| \leq R(2\|x\| + \|x - y\|)\|x - y\| \\
&\leq R(2\|x\| + 1)\|x - y\| \leq R(2\|x\| + 1)\frac{1}{k} \leq \frac{1}{3n}.
\end{aligned}
$$

Similarly we can show that, $|\langle Cx, x\rangle - \langle Cy, y\rangle| < 1/3n$. Hence

$$
\begin{aligned}
|\langle B_M x, x\rangle - \langle Cx, x\rangle| &\leq |\langle B_M x, x\rangle - \langle B_M y, y\rangle| + |\langle B_M y, y\rangle - \langle Cy, y\rangle| \\
&\quad + |\langle Cy, y\rangle - \langle Cx, x\rangle| \\
&\leq \frac{1}{3n} + \frac{1}{3n} + \frac{1}{3n} = \frac{1}{n}.
\end{aligned}
$$

Thus $x \in W(m, n)$ and $W(m, n) = H$.

Let $\varphi \in W(f_j, 1/n, m)$ $\forall j, m, n$, set $H = L^2(X, \bar{\mu})$ and $B_n = V_n^\varphi$, the above lemma proves that, for each $f \in L^2(X, \bar{\mu})$ there is a subsequence $n_k \to \infty$ such that $\langle V_{n_k}^\varphi f, f\rangle \to \langle PQf, f\rangle$. Now Theorem (3.4)(2a) says V_n^φ is a weakly convergent sequence, hence $\langle V_n^\varphi f, f\rangle$ converges and converges to $\langle PQf, f\rangle$. This implies that $V_n^\varphi \to_{\mathrm{w}} PQ$ (this is a general fact about a sequence converging to a self adjoint projection on a complex Hilbert space). This proves Proposition (3.1). Proposition (3.2) is easy to verify. Now we prove Proposition (3.3) by collecting a series of lemmas. The following lemma is easy to verify.

LEMMA (3.6). *Let $\varphi \in Z(Y, T, G)$ and $\psi \in C(Y, G)$. Then*
(i) $U_t^\varphi L_\psi = L_\psi U_t^{\varphi \cdot l^\psi}$ $\forall t \in T$, *and* $V_n^\varphi L_\psi = L_\psi V_n^{\varphi \cdot l^\psi}$, $\forall n \in \mathbf{N}$
(ii) $L_\psi Q = QL_\psi = Q$ *and* $PQL_\psi = L_\psi PQ = PQ$.

LEMMA (3.7).
(a) *Let $f \in L^2(X, \bar{\mu})$, $\varepsilon > 0$ and $m \in \mathbf{N}$. Let $\varphi \in Z(Y, T, G)$ and $\psi \in C(Y, G)$. Then $\varphi \cdot l^\psi \in W(f, \varepsilon, m)$ iff $\varphi \in W(L_\psi f, \varepsilon, m)$.*
(b) *Given any $f \in L^2(X, \bar{\mu})$, assume that for any $\varepsilon > 0$, $m \in \mathbf{N}$ and $\delta > 0$, $\exists$ a $\psi \in C(Y, G)$ such that* (i) $D(l^\psi, l) < \delta$ *and* (ii) $l^\psi \in W(f, \varepsilon, m)$. *Then $\overline{W(g, \varepsilon, m)} = \overline{B(Y, T, G)}$, $\forall g \in L^2(X, \bar{\mu})$, $\forall \varepsilon > 0$ and $\forall m \in \mathbf{N}$.*

Proof. (a) Using Lemma (3.6) and that L_ψ is unitary we get,

$$
\left\langle V_n^{\varphi \cdot l^\psi} f, f\right\rangle = \left\langle L_\psi V_n^{\varphi \cdot l^\psi} f, L_\psi f\right\rangle = \left\langle V_n^\varphi L_\psi f, L_\psi f\right\rangle,
$$

and

$$\langle PQf, f\rangle = \langle L_\psi PQf, L_\psi f\rangle = \langle PQL_\psi f, L_\psi f\rangle.$$

This proves (a).

(b) Using part (a) and the fact that if $\varphi_n, \varphi \in Z(Y, T, G)$ and $\psi \in C(Y, G)$ are such that $D(\varphi_n, \varphi) \to 0$, then $D(\varphi_n \cdot l^\psi, \varphi \cdot l^\psi) \to 0$ as $n \to \infty$, we conclude that $\varphi \cdot l^\psi \in \overline{W(f, \varepsilon, m)}$ iff $\varphi \in \overline{W(L_\psi f, \varepsilon, m)}$, $\forall f, \varepsilon, m$. Hence $l \in \overline{W(f, \varepsilon, m)}$, $\forall f, \varepsilon, m$ implies $l^\psi \in \overline{W(f, \varepsilon, m)}$, $\forall f, \varepsilon, m$ and ψ, and the latter says $\overline{W(f, \varepsilon, m)} = \overline{B(Y, T, G)}$, $\forall f, \varepsilon, m$. Now (b) follows from this.

LEMMA (3.8). *Suppose that given any $f \in L^2(X, \bar{\mu})$, $\gamma > 0$ and $\delta > 0$, $\exists \psi \in C(Y, G)$ such that* (i) $D(l^\psi, l) < \delta$ *and* (ii) $\|PL_\psi f - PQf\| < \gamma$. *Then $W(f, \varepsilon, m)$ is dense in $\overline{B(Y, T, G)}$, $\forall f \in L^2(X, \bar{\mu})$, $\varepsilon > 0$, $m \in \mathbf{N}$.*

Proof. Let $f \in L^2(X, \bar{\mu})$, $\varepsilon > 0$ and $m \in \mathbf{N}$ be given. Pick $\psi \in C(Y, G)$ for f, $\gamma = \varepsilon/2\|f\|$ and δ, satisfying conditions (i) and (ii) above. Now consider

$$\begin{aligned}\left|\left\langle V_n^{l^\psi} f, f\right\rangle - \langle PQf, f\rangle\right| &= \left|\left\langle V_n L_\psi f, L_\psi f\right\rangle - \langle PQf, f\rangle\right| \\ &\le \left|\left\langle V_n L_\psi f, L_\psi f\right\rangle - \left\langle PL_\psi f, L_\psi f\right\rangle\right| + \left|\left\langle PL_\psi f, L_\psi f\right\rangle - \langle PQf, f\rangle\right|.\end{aligned}$$

The second term in the above expression satisfies

$$\begin{aligned}\left|\left\langle PL_\psi f, L_\psi f\right\rangle - \langle PQf, f\rangle\right| &= \left|\left\langle PL_\psi f, L_\psi f\right\rangle - \left\langle L_\psi PQf, L_\psi f\right\rangle\right| \\ &= \left|\left\langle PL_\psi f, L_\psi f\right\rangle - \left\langle PQf, L_\psi f\right\rangle\right| \cdots \quad (\text{since } L_\psi PQ = PQ) \\ &\le \|PL_\psi f - PQf\|\,\|f\| < \varepsilon/2 \quad (\text{since } L_\psi \text{ is unitary}).\end{aligned}$$

For the first term, since $V_n \to_{\mathrm{w}} P$, we can find $N_0 \in \mathbf{N}$, $N_0 > m$, such that $|\langle V_{N_0} L_\psi f, L_\psi f\rangle - \langle PL_\psi f, L_\psi f\rangle| < \varepsilon/2$. Thus $\exists N_0 \in \mathbf{N}$, $N_0 > m$, such that

$$\left|\left\langle V_{N_0}^{l^\psi} f, f\right\rangle - \langle PQf, f\rangle\right| < \varepsilon$$

and hence $l^\psi \in W(f, \varepsilon, m)$. Now use Lemma (3.7) to complete the proof.

Now we prove the crucial lemma that describes the construction of function ψ satisfying conditions (i) and (ii) of Lemma (3.8). We first construct this ψ when $X = Z \times Y$, (and this assumption is in force throughout the following lemma.) In this case we have $Pf(z, y) = \int_Y f(z, y)\, d\mu(y)$, $\forall f \in L^2(X, \bar{\mu})$.

LEMMA (3.9). *Given any* $f \in L^2(X, \bar{\mu})$, $\gamma > 0$ *and* $\delta > 0$, $\exists \psi \in C(Y, G)$ *such that* (i) $D(l^{\psi}, l) < \delta$ *and* (ii) $\|PL_{\psi} f - PQf\| < \gamma$.

Proof. Since $C(X)$ is dense in $L^2(X, \bar{\mu})$, standard approximation arguments allow us to assume that given f is continuous. Set, $M = \operatorname{Sup}_{x \in X} |f(x)|$ and γ' be a small positive number. First choose $P_0 \in \mathbf{N}$ such that

$$(1) \qquad \sum_{i > P_0} \frac{1}{2^i} < \frac{\delta}{2}$$

Next by Theorem (3.4) (1), we can select $(\lambda_i)_{i=1}^q \in [0,1]$ and $(g_i)_{i=1}^q \in G$ such that

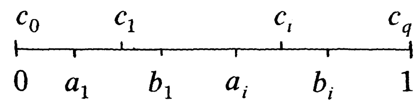
$$(2) \qquad \sum_{i=1}^{q} \lambda_i = 1 \quad \text{and} \quad \left\| \sum_{i=1}^{q} \lambda_i U_{g_i} f - Qf \right\| < \gamma'$$

We now claim that, we can find a continuous map h: $[0,1] \to G$ such that

$$(3) \qquad \left| \int_0^1 f(zh(s), y)\, ds - \sum_{i=1}^{q} \lambda_i f(zg_i, y) \right| < \gamma', \qquad \forall z \in Z \text{ and } y \in Y.$$

(Here ds is the Lebesgue measure on $[0,1]$.)

To prove the claim, set $c_0 = 0$, $c_i = \sum_{j=1}^{i} \lambda_j$, $1 \le i \le q$. Then pick $a_i, b_i \in [0,1]$ such that $a_i < c_i < b_i$ $\forall i < q$ and $\sum_{i=1}^{q}(b_i - a_i) < \gamma'/4M$. Let $a_q = 1$ and $b_0 = 0$.



Define $h = g_i$ on $[b_i, a_{i+1}]$, $\forall 0 \le i \le q-1$. Since G satisfies condition (A) one can easily extend h continuously to $[0,1]$. Since a_i's and b_i's are chosen to be very close to each other, one can readily verify (3).

(4) Since $h[0,1]$ is compact, select $\delta^* > 0$ such that if $g_1, g_2 \in h[0,1]$ with $d(g_1, g_2) < \delta^*$ then $d(g^{-1}g_1, g^{-1}g_2) < \delta/2P_0$ $\forall g \in h[0,1]$, and for this δ^* pick $\delta_1 > 0$ such that $\forall s_1, s_2 \in [0,1]$ if $|s_1 - s_2| < \delta_1$ then $d(h(s_1), h(s_2)) < \delta^*$.

Since T is amenable, we can choose a compact set $F \subseteq T$ with $\rho(F) > 1$ such that

$$(5) \qquad \frac{\rho(F \Delta F \cdot t^{-1})}{\rho(F)} < \delta_1,$$

$$\forall t \in K_{P_0} \left\{ \text{where } (K_n)_{n=1}^{\infty} \subseteq T \text{ is a Følner sequence} \right\}.$$

Now we have to use the fact that T is an R-group. First we give a precise definition. An open subset $S \subseteq T$ is called an *R-set* if (i) $\bar{S}$ is compact (ii) if T yields a measurable free right action on a Polish space Y and μ is a T-invariant probability measure on Y, then given any $\varepsilon > 0$, $\exists$ a set $V \subseteq Y$ such that (a) V and $V \cdot S = \bigcup_{s \in S} V \cdot s$ are measurable (b) $V \cdot s \cap V \cdot s' = \varnothing$ $\forall s, s' \in S$, $s \neq s'$ (c) $\mu(V \cdot S) > 1 - \varepsilon$. A l.c.s. group T is an *R-group* if given any two compact sets $E, F \subseteq T$ and $\delta > 0$, $\exists S \subseteq T$ such that (i) S is an R-set (ii) $E \subseteq S$ and (iii) S is $(F\text{-}\delta)$ invariant; $\rho\{s | s \in S, sF \subseteq S\} > (1 - \delta)\rho(S)$, where ρ is a right Haar measure on T.

In our situation, setting $F^{-1} \cup K_{P_0}$ to be E, F^{-1} to be F, γ'/M to be ε, and γ'/M to be δ we get a set $S \subseteq T$ and a Borel set $V \subseteq Y$ such that

(6) (i) S is open, $\bar{S}$ is compact (hence $0 < \rho(S) < \infty$)

(ii) $F^{-1} \cup K_{P_0} \subseteq S$

(iii) $\rho\{s | s \in S, sF^{-1} \subseteq S\} > (1 - \gamma'/M)\rho(S)$

(iv) (a) $V \cdot S$ is measurable

(b) $V \cdot s \cap V \cdot s' = \varnothing$ if $s \neq s'$, $\forall s, s' \in S$

(c) $\mu(V \cdot S) > 1 - \gamma'/M$.

Let ξ denote the probability on the transversal V, induced by μ (see **[17]** for definition). Since $\bar{S}$ is compact, select $\alpha > 0$ such that $y_1, y_2 \in Y$ and $d(y_1, y_2) < \alpha$ implies

$$|f(z, y_1 \cdot t) - f(z, y_2 \cdot t)| < \gamma', \quad \forall t \in \bar{S} \text{ and } \forall z \in Z \tag{7}$$

(8) Let $B_1, \dots, B_{R+1}$ be a partition of V in to Borel sets such that diameter $(B_i) < \alpha$, and $\xi(B_i) > 0$ $\forall 1 \leq i \leq R$ and $\xi(B_{R+1}) = 0$.

(9) Let l be the Lebesgue measure on $[0,1]$ and ψ_i: $B_i \to [0,1]$ be a Borel isomorphism taking $\xi|_{B_i}$ onto l ($\forall 1 \leq i \leq R$).

Since (Y, T, μ) is properly ergodic, ξ is non atomic (see **[17]**), such a choice of ψ_i is possible, **[1]**. Set $\psi_{R+1} = 0$ on B_{R+1}, and define $\tilde{\psi}$: $V \to [0,1]$ by setting $\psi|_{B_i} = \psi_i$ $\forall 1 \leq i \leq R + 1$. Next, define $\tilde{\psi}$: $V \cdot S \to [0,1]$ by setting $\tilde{\psi}(v \cdot s) = \tilde{\psi}(v)$, $\forall s \in S$ and $v \in V$. Since $V \cdot s_1 \cap V \cdot s_2 = \varnothing$, $\forall s_1 \neq s_2$, $\tilde{\psi}$ is well defined. Further $\tilde{\psi}$ is Borel measurable since $\tilde{\psi}(E)^{-1} = p((\psi^{-1}(E) \cap V) \times S)$ (where p: $V \times S \to V \cdot S$ is $p(v, s) = v \cdot s$ is a Borel isomorphism). Setting $\tilde{\psi} = 0$ outside $V \cdot S$ we get a Borel map $\tilde{\psi}$ from Y onto $[0,1]$. Applying Lusin's approximation theorem to $\tilde{\psi}$, we get a continuous map $\mathcal{O}$: $Y \to [0,1]$ such that,

$$(10)\quad \begin{cases} \mu\{y \mid y \in Y,\ \mathcal{O}(y) \neq \tilde{\psi}(y)\} < \beta^2\mu(V \cdot S), \text{ where } \beta > 0 \text{ is} \\ \text{such that (i) } \beta < \gamma'/\rho(S)M \text{ and (ii) } |t_1 - t_2| < 2\beta,\ t_1, \\ t_2 \in [0,1] \Rightarrow |f(zh(t_1), y) - f(zh(t_2), y)| < \gamma'\ \forall\ z \text{ and } y. \end{cases}$$

Now define $\tilde{\mathcal{O}}$: $Y \to [0,1]$ by setting

$$(11)\qquad \tilde{\mathcal{O}}(y) = \frac{1}{\rho(F)} \int_F \mathcal{O}(y \cdot t^{-1})\, d\rho(t).$$

Clearly $\tilde{\mathcal{O}}$ is continuous. Now we claim that if $y \in Y$ and $t_0 \in K_{P_0}$ then $|\tilde{\mathcal{O}}(y \cdot t_0) - \tilde{\mathcal{O}}(y)| < \delta_1$. To show this, note that

$$|\tilde{\mathcal{O}}(y \cdot t_0) - \tilde{\mathcal{O}}(y)| \leq \frac{1}{\rho(F)} \left| \int_F \mathcal{O}(y \cdot t_0 t^{-1})\, d\rho(t) - \int_F \mathcal{O}(y \cdot t^{-1})\, d\rho(t) \right|$$

$$= \frac{1}{\rho(F)} \left| \int_{F \cdot t_0^{-1}} \mathcal{O}(y \cdot t^{-1})\, d\rho(t) - \int_F \mathcal{O}(y \cdot t^{-1})\, d\rho(t) \right|$$

$$\leq \frac{1}{\rho(F)} \left| \int_{F \cap (F \cdot t_0^{-1})} \mathcal{O}(y \cdot t^{-1})\, d\rho(t) + \int_{F \cdot t_0^{-1} - F} \mathcal{O}(y \cdot t^{-1})\, d\rho(t) \right.$$

$$\left. - \int_{F \cap (F \cdot t_0^{-1})} \mathcal{O}(y \cdot t^{-1})\, d\rho(t) - \int_{F - F \cdot t_0^{-1}} \mathcal{O}(y \cdot t^{-1})\, d\rho(t) \right|$$

$$\leq \frac{1}{\rho(F)} \left\{ \left| \int_{F \cdot t_0^{-1} - F} \mathcal{O}(y \cdot t^{-1})\, d\rho(t) \right| + \left| \int_{F - F \cdot t_0^{-1}} \mathcal{O}(y \cdot t^{-1})\, d\rho(t) \right| \right\}$$

$$\leq \frac{\rho(F \cdot t_0^{-1} \Delta F)}{\rho(F)} < \delta_1, \quad \forall t_0 \in K_{p_0} \qquad \text{by (5)}.$$

This proves the claim.

Finally set $\psi = h \circ \mathcal{O}$. Then ψ: $Y \to G$ is continuous, and we now show that this ψ is the required map. Since

$$d(l^{\psi}(y,t), e) = d(\psi(y)^{-1}\psi(y \cdot t), e)$$
$$= d([h \circ \tilde{\mathcal{O}}(y)]^{-1}[h \circ \tilde{\mathcal{O}}(y \cdot t)], e),$$

our previous claim, along with the choice of δ_1, as in (4) implies that $D_i(l^{\psi}, l) < \delta/2P_0, \forall 1 \leq i \leq P_0$. Hence

$$D(l^{\psi}, l) \leq \sum_{i=1}^{P_0} \frac{D_i(l^{\psi}, l)}{2^i} + \sum_{i > P_0} \frac{1}{2^i} < \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

proving condition (i). Now we show that $\|PL_\psi f - PQf\| < \gamma$. We first consider,

$$PL_\psi f(z, y) = \int_Y f(z\psi(y), y)\, d\mu(y) = \int_{Y - V\cdot S} \cdots + \int_{V\cdot S} \cdots .$$

Now

$$\int_{V\cdot S} f(z\psi(y), y)\, d\mu\Big|_{V\cdot S}(y) = \int_{V\times S} f(z\psi(y\cdot t), y\cdot t)\, d\xi(y) \times d\rho\Big|_S(t).$$

Setting $S_0 = \{s \in S | sF^{-1} \subseteq S\}$ and observing from (6) that

(i) $\displaystyle \left|\int_{V\times(S - S_0)} f(z\psi(y\cdot t), y\cdot t)\, d\xi \times d\rho|_S\right| \le \frac{M\rho(S - S_0)}{\rho(S)} < \gamma'$,

(ii) $\mu(Y - V\cdot S) < \gamma'/M$, and (iii) V is a disjoint union of $(B_i)_{i=1}^{R+1}$, and $\xi(B_{R+1}) = 0$ we get

$$\left|PL_\psi f(z, y) - \mu(V\cdot S)\left[\sum_{i=1}^{R}\int_{B_i\times S_0} f(z\psi(y\cdot t), y_i\cdot t)\ d\xi \times d\rho|_S\right]\right| < 2\gamma'$$

$\forall z$ and $\forall y$.

Now pick $y_i \in B_i$ $(1 \le i \le R)$ and fix it throughout the rest of the proof. By (7) we then have

$$\left|PL_\psi f(z, y) - \mu(V\cdot S)\left[\sum_{i=1}^{R}\int_{B_i\times S_0} f(z\psi(y\cdot t), y_i\cdot t)\ d\xi \times d\rho|_S\right]\right| < 3\gamma'$$

$\forall z$ and $\forall y$.

Now we first prove the following technical lemma.

SUBLEMMA (3.10). *Let $R' = \{y | y \in Y$ such that $\mathcal{O}(y) \ne \tilde{\psi}(y)\}$, $R = R' \cap V\cdot S_0$ and $R^* = \{(y, t) | y \in V, t \in S_0$ such that $y\cdot t \in R\}$. Then*

(a) $R^* \subseteq V \times S_0$ is a Borel set and $\xi \times \rho|_S(R^*) < \beta^2$

(b) $\exists$ a Borel set $V_1 \subseteq V$ such that

(i) $\xi(V_1) < \beta\rho(S)$ and

(ii) $|\tilde{\mathcal{O}}(y\cdot t_0) - \tilde{\psi}(y)| < 2\beta$, $\forall y \in V - V_1$ and $\forall t_0 \in S_0$.

Proof. (a) Since p: $V \times S_0 \to V\cdot S_0$ $(p(v, s) = v\cdot s)$ is a Borel isomorphism and $p(R^*) = R$, R^* is a Borel set. Also

$$\xi \times \rho|_S(R^*) = \mu|_{V\cdot S}(R) \le \frac{\mu(R')}{\mu(V\cdot S)} < \beta^2 \quad \text{(by (10))}.$$

(b) Let $y \in Y$ and $R_y^* = \{t | t \in S_0, (y, t) \in R^*\}$. Set $V_1 = \{y | y \in V$ such that $\rho(R_y^*) > \beta\}$. Clearly R_y^* and V_1 are Borel sets. If $\xi(V_1) \geq \beta\rho(S)$, then considering

$$\begin{aligned}\xi \times \rho|_S(R^*) &= \int_{V \times S \cap R^*} d\xi \times d\rho|_S \\ &= \int_V \rho|_S(R_y^*)\, d\xi(y) \geq \int_{V_1} \rho|_S(R_y^*)\, d\xi(y) \\ &\geq \int_{V_1} \frac{\rho(R_y^*)}{\rho(S)} d\xi(y) > \frac{\beta\xi(V_1)}{\rho(S)} > \beta^2.\end{aligned}$$

This contradicts (a), hence $\xi(V_1) < \beta\rho(S)$. To show (ii) first observe that

$$\begin{aligned}|\tilde{\mathcal{O}}(y \cdot t_0) - \tilde{\psi}(y)| &= \left| \frac{1}{\rho(F)} \int_F \mathcal{O}(y \cdot t_0 t^{-1})\, d\rho(t) - \tilde{\psi}(y) \right| \\ &= \left| \frac{1}{\rho(F)} \int_{F \cdot t_0^{-1}} \mathcal{O}(y \cdot t^{-1})\, d\rho(t) - \tilde{\psi}(y) \right| \\ &= \left| \frac{1}{\rho(F)} \int_{F \cdot t_0^{-1} \cap R_y^*}^{\cdots} + \int_{F \cdot t_0^{-1} - R_y^*}^{\cdots} - \tilde{\psi}(y) \right| \\ &\leq \frac{\rho(R_y^*)}{\rho(F)} + \left| \frac{1}{\rho(F)} \int_{F \cdot t_0^{-1} - R_y^*} \mathcal{O}(y \cdot t^{-1})\, d\rho(t) - \tilde{\psi}(y) \right|\end{aligned}$$

Now $y \notin V_1$ and hence $\rho(R_y^*) \leq \beta$, also $\rho(F) > 1$ by (5). Hence

$$|\tilde{\mathcal{O}}(y \cdot t_0) - \tilde{\psi}(y)| \leq \beta + \left| \frac{1}{\rho(F)} \int_{F \cdot t_0^{-1} - R_y^*} \mathcal{O}(y \cdot t^{-1})\, d\rho(t) - \tilde{\psi}(y) \right|.$$

Now if $t \in F \cdot t_0^{-1}$ then $t^{-1} \in t_0 F^{-1} \subseteq S$. So if $y \cdot t^{-1} \notin R$ then

$$\mathcal{O}(y \cdot t^{-1}) = \tilde{\psi}(y \cdot t^{-1}) = \tilde{\psi}(y),$$

and

$$\begin{aligned}|\tilde{\mathcal{O}}(y \cdot t_0) - \tilde{\psi}(y)| &\leq \beta + \left| \frac{\rho\left(F \cdot t_0^{-1} - R_y^*\right)}{\rho(F)} \tilde{\psi}(y) - \tilde{\psi}(y) \right| \\ &\leq \beta + \left| \frac{\rho\left(F \cdot t_0^{-1} - R_y^*\right)}{\rho(F)} - 1 \right| \quad \left(\text{since } |\tilde{\psi}(y)| \leq 1\right).\end{aligned}$$

Since

$$\rho(F) = \rho\left(F \cdot t_0^{-1}\right) = \rho\left(F \cdot t_0^{-1} \cap R_y^*\right) + \rho\left(F \cdot t_p^{-1} - R_y^*\right),$$

we have

$$\left|\frac{\rho\left(F \cdot t_0^{-1} - R_y^*\right)}{\rho(F)} - 1\right| = \left|\frac{\rho\left(F \cdot t_0^{-1} - R_y^*\right) - \rho(F)}{\rho(F)}\right|$$

$$= \left|\frac{\rho\left(F \cdot t_0^{-1} \cap R_y^*\right)}{\rho(F)}\right| \leq \frac{\rho\left(R_y^*\right)}{\rho(F)} < \frac{\beta}{\rho(F)} < \beta \quad (\text{since } \rho(F) > 1).$$

This proves that $|\tilde{\mathcal{O}}(y \cdot t_0) - \tilde{\psi}(y)| < 2\beta$, thus proving the sublemma. □

Now we continue with the proof of Lemma (3.9). We already have shown,

$$\left|PL_\psi f(z, y) - \mu(V \cdot S) \sum_{i=1}^{R} \int_{B_i \times S_0} f(z\psi(y \cdot t), y_i \cdot t)\, d\xi \times d\rho|_S\right| < 3\gamma',$$

$$\forall z \in Z \text{ and } \forall y \in Y.$$

Now writing $B_i \times S_0$ as the union of $(B_i \cap V_1) \times S_0$ and $(B_i - V_1) \times S_0$ and applying (i) of the previous sublemma. We get

$$\left|\int_{B_i \times S_0} \cdots - \int_{(B_i - V_1) \times S_0} \cdots\right| < \gamma', \qquad \forall z \in Z \text{ and } \forall y \in Y, \quad 1 \leq i \leq R.$$

Further if $y \in B_i - V_1$, then by sublemma (3.10) $|\tilde{\mathcal{O}}(y \cdot t) - \tilde{\psi}(y)| < 2\beta$, and $\psi(y \cdot t) = h \circ \tilde{\mathcal{O}}(y \cdot t)$, $\tilde{\psi}(y) = \psi_i(y)$ on B_i. All these observations show that

$$\left|\int_{(B_i - V_1) \times S_0} f(z\psi(y \cdot t), y_i \cdot t)\, d\xi \times d\rho|_S \right.$$

$$\left. - \int_{(B_i - V_1) \times S_0} f(zh \circ \psi_i(y), y_i \cdot t)\, d\xi \times d\rho|_S\right| < \gamma',$$

$$\forall z \in Z \text{ and } \forall y \in Y, \forall 1 \leq i \leq R.$$

Moreover, since $\xi(V_1) < \gamma'/M$, we have

$$\left|\int_{(B_i - V_1) \times S_0} f(zh \circ \psi_i(y), y_i \cdot t)\, d\xi \times d\rho|_S \right.$$

$$\left. - \int_{B_i \times S_0} f(zh \circ \psi_i(y), y_i \cdot t)\, d\xi \times d\rho|_S\right| \leq \gamma', \quad \forall (z, y) \in Z \times Y.$$

These combine to yield

$$\left|PL_\psi f(z, y) - \mu(V \cdot S) \sum_{i=1}^{R} \int_{B_i \times S_0} f(zh \circ \psi_i(y), y_i \cdot t)\, d\xi \times d\rho|_S\right| < 6\gamma',$$

$$\forall (z, y) \in Z \times Y.$$

Now,

$$\begin{aligned}\int_{B_i \times S_0} & f(zh \circ \psi_i(y), y_i \cdot t)\ d\xi \times d\rho|_S \\ &= \xi(B_i) \int_{S_0} \left\{ \int_{B_i} f(zh \circ \psi_i(y), y_i \cdot t)\ d\xi|_{B_i}(y) \right\} d\rho|_S(t) \\ &= \xi(B_i) \int_{S_0} \int_0^1 f(zh(s), y_i \cdot t)\ dl(s)\ d\rho|_S(t).\end{aligned}$$

By our choice of h we have,

$$\left| \int_0^1 f(zh(s), y)\ dl(s) - \sum_{i=1}^{q} \lambda_i U_{g_i} f(z, y) \right| < \gamma', \quad \forall (z, y).$$

Hence

$$\text{(A)} \qquad \left| PL_\psi f(z, y) - a'(z, y) \right| < 7\gamma', \quad \forall (z, y) \in Z \times Y$$

where,

$$a'(z, y) = \mu(V \cdot S) \sum_{i=1}^{q} \sum_{j=1}^{R} \xi(B_j) \lambda_i \int_{S_0} U_{g_i} f(z, y_j \cdot t)\ d\rho|_S(t).$$

To complete the last step of the proof, again writing Y as $V \cdot S \cup (Y - V \cdot S)$ and transforming integration over $V \cdot S$ to that on $V \times S$, as before we can show that

$$\text{(B)} \quad \left| \int_Y f(z, y)\ d\mu - \mu(V \cdot S) \sum_{j=1}^{R} \xi(B_j) \int_{S_0} f(z, y_j \cdot t)\ d\rho|_S(t) \right| < 3\gamma' \qquad \forall (z, y) \in Z \times Y,$$

For simplicity letting $a(z, y) = \sum_{i=1}^{q} \lambda_i U_{g_i} f(z, y)$ we see that

$$\|PL_\psi f - PQf\| \le \|PL_\psi f - a'\| + \|a' - Pa\| + \|Pa - PQf\|.$$

Now $\|PL_\psi f - a'\| < 7\gamma'$ by inequality (A),

$$\|Pa - PQf\| \le \|P\|\ \|a - Qf\| < \gamma' \|P\| \quad \text{by (2)}$$

and

$$\begin{aligned}\|a' - Pa\|^2 &= \int_X |a' - Pa|^2\ d\bar{\mu} \le \int_X |a' - Pa|(|a'| + |Pa|)\ d\bar{\mu} \\ &\le 2M \int_X |a' - Pa| d\bar{\mu} < 2M(3\gamma') \quad \text{by inequality (B).}\end{aligned}$$

This shows that

$$\|PL_{\psi}f - PQf\| < 7\gamma' + \|P\|\gamma' + \sqrt{6M\gamma'} < \gamma,$$

if γ' is small enough. This completes the proof.

Proof in the nontrivial group extension case. Now let $(X, T, \bar{\mu})$ be a group extension of (Y, T, μ). Note that in this case $PQ = C$ the projection on constants. Since there is a Borel section from Y to X, one can assume, without loss of generality that $X = G \times Y$ and the T action on $G \times Y$ is given by $(g, y) \cdot t = (g\alpha(y, t), y \cdot t)$ where α: $Y \times T \to G$ is a Borel cocyle and $\bar{\mu} = \eta \times \mu$ where η is the normalized Haar measure on fiber G. [Note that the already proved case is when $\alpha = 1$]. Let $H = \text{Range}\,\alpha$ [see [**19**] for definition], then H is a closed subgroup and by changing α to a Borel cohomologous cocyle if necessary, we can assume that (i) $\alpha(y, t) \in H$ $\forall y, t$ and (ii) the action $(h, y) \cdot t = (h\alpha(y, t), y \cdot t)$ on $H \times Y$ is ergodic. (Here $\eta_H \times \mu$ is the measure on $H \times Y$, where η_H is the normalized Haar measure on H. (See [**19**] for details.) Define W: $L^2(X, \bar{\mu}) \to L^2(X, \bar{\mu})$ by $Wf = \int_H f(gh, y)\, d\eta_H(h)$.

Let

$$P_Y f(g, y) = \int_Y f(g, y)\, d\mu(y), \quad \forall f \in L^2(X, \bar{\mu}).$$

We claim that $WP_Y = P$. To see this consider operators $V_{K_n} = \int V_t^{\alpha}\, d\rho|_{K_n}$ where $V_t^{\alpha} f(g, y) = f(g\alpha(y, t), y \cdot t)$ and K_n is a Følner sequence. Abstract ergodic theorem (3.4) implies $V_{K_n} \to_s P$, we show that $V_{K_n} \to_s WP_Y$, proving our claim. Consider

$$\|V_{K_n} f - WP_Y f\|^2 = \int_Y \int_G |V_{K_n} f(g, y) - WP_Y(g, y)|^2\, d\eta \times \mu$$

$$= \int_Y \int_{G/H} \left(\int_H |V_{K_n} f(gh, y) - WP_Y f(gh, y)|^2\, d\eta_H(h) \right) d\beta_* \eta(\bar{g})\, d\mu(y)$$

where β: $G \to G/H$ is the quotient map and $\bar{g} = \beta(g)$

$$= \int_{G/H} \int_Y \left(\int_H |V_{K_n} f_g(h, y) - WP_Y f_g(h, y)|^2\, d\eta_H(h) \right) d\mu(y)\, d\beta_* \eta(\bar{g}),$$

where $f_g(h, y) = f(gh, y)$.

Note that

$$WP_Y f_g(h, y) = \int_H P_Y f_g(hh', y)\, d\eta_H(h')$$

$$= \int_H \int_Y f_g(h, y)\, d\mu(y)\, d\eta_H(h).$$

Since $(h, y) \cdot t = (h\alpha(y, t), y \cdot t)$ is ergodic action, and G is compact, given $\varepsilon > 0$, $\exists N_0 \in \mathbf{N}$ such that

$$n > N_0 \Rightarrow \int_Y \int_H \left|V_{K_n} f_g(h, y) - WP_Y f_g(h, y)\right|^2 d\eta_H \times d\mu < \varepsilon, \quad \forall g.$$

Hence

$$n > N_0 \Rightarrow \int_{G/H} \left(\int_Y \int_H \left|V_{K_n} f_g - WP_Y f_g\right|^2 d\eta_H \times d\mu \right) d\beta_* \eta(\bar{g}) < \varepsilon.$$

This proves $V_{K_n} \to_s WP_Y$.

Now $WC = C$, and

$$\|PL_\psi f - Cf\| = \|WP_Y L_\psi f - WCf\| \leq \|W\| \, \|P_Y L_\psi f - Cf\|.$$

Now $\|W\|$ depends only on the structure of the bundle $\pi: X \to Y$ and our Lemma (3.10) constructs a map $\psi \in C(Y, G)$ {given $\varepsilon > 0$ and $\delta > 0$} such that $D(l^\psi, l) < \delta$ and $\|P_Y L_\psi f - Cf\| < \varepsilon$. This completes the proof in group extension case.

In the case of part (2) of Theorem (2.1), $PQ = C$ the projection on constants. Now the proof follows from part (1) and Theorem (3.4)(2b). Part (3) can be considered as lifting ergodicity for the product extension $\pi \times \pi: X \times X \to Y \times Y$ by the cocycles of the form $\alpha \times \alpha$, $(\alpha \in \overline{B(Y, T, G)})$. It is not hard to see that all the previous lemmas and propositions will remain valid for this extension. Thus weak mixing can be lifted generically using exactly the same technique but for the product extension.

Now we turn to the proofs of the corollaries.

Proof of Corollary (2.2) Since (Z, G) is minimal distal, Furstenberg's structure theorem **[3]** gives a countable (because all spaces are separable metric) inverse system $\{(Z_n, G)\}_{n=0,1,\ldots,\infty}$ of t.d.s.'s with factor maps $\phi_{n,m}$: $Z_n \to Z_m$ $\forall \infty \geq n \geq m$ such that (i) Z_0 is a one point space (ii) $\phi_{n+1,n}$: $Z_{n+1} \to Z_n$ is an isometric extension (see [3]) and (iii) $(Z, G) = \operatorname{inv\,lim}(Z_n, G)$. Since $\phi_{n+1,n}$: $Z_{n+1} \to Z_n$ is an isometric extension, applying a structure theorem **[12]** for such extensions we get a metric bi-transformation group (K_{n+1}, Z'_{n+1}, G) and closed subgroups $H_{n+1} \subseteq K_{n+1}$ such that (1) $(K_{n+1} \setminus Z'_{n+1}, G) \equiv (Z_n, G)$, (2) $(H_{n+1} \setminus Z'_{n+1}, G) \equiv (Z_{n+1}, G)$, and (3) the quotient map: $H_{n+1} \setminus Z'_{n+1} \to K_{n+1} \setminus Z'_{n+1}$ identifies with $\phi_{n+1,n}$ $\forall n \in \mathbf{N}$. Set $\nu_n = (\phi_{\infty,n})_* \nu$, and select ergodic probabilities ν'_{n+1} on (Z'_{n+1}, G) such that $(\phi''_{n+1,n})_* \nu'_{n+1} = \nu_{n+1}$ $\forall n$, where $\phi''_{n+1,n}$: $Z'_{n+1} \to Z_{n+1}$ is the quotient map. In summary we get the following diagram.

$$
\begin{tikzcd}
 & & & (Z, G, \nu) \\
 & (Z_2', G, \nu_2') \arrow[r, "\phi''_{2,1}"] \arrow[d, "\phi'_{2,1}"'] & (Z_2, G, \nu_2) \arrow[dl, "\phi_{2,1}"] \arrow[ur, phantom, "\iddots"] & \\
(Z_1', G, \nu_1') \arrow[r, "\phi''_{1,0}"] \arrow[d, "\phi'_{1,0}"'] & (Z_1, G, \nu_1) \arrow[dl, "\phi_{1,0}"] & & \\
(Z_0, G, \nu_0) & & &
\end{tikzcd}
$$

Note that for each $\alpha \in Z(Y, T, G)$ the extension $\pi\colon (Z \times_\alpha Y, T) \to (Y, T)$ is distal and family $(Z_n \times_\alpha Y, T, \nu_n \times \mu)$ forms an inverse system with inverse limit $(Z \times_\alpha Y, T, \nu \times \mu)$. Also note that each $\phi'_{n+1,n} \times \mathrm{id}$: $(Z'_{n+1} \times_\alpha Y, T) \to (Z'_n \times_\alpha Y, T)$ is a group extension with fiber K_{n+1}. Finally set $U_n = \{\alpha | \alpha \in \overline{B(Y, T, G)}$ such that $(Z'_n \times_\alpha Y, T, \nu'_n \times \mu)$ is weak mixing$\}$. Theorem (2.1) implies each U_n and hence $\bigcap_{n=1}^{\infty} U_n$ is residual. Now if $\alpha \in \bigcap_{n=1}^{\infty} U_n$. Each $(Z'_n \times_\alpha Y, T, \nu'_n \times \mu)$ is weak mixing. Now a generalization of a theorem of R. K. Thomas **[18]** (and D. Rudolph **[16]**) says group extension of a K-automorphism (Bernoulli automorphism) is a K-automorphism (Bernoullian) iff it is weak-mixing. This proves Corollary (2.2). Here we remark that the theorems of Thomas and Rudolph are proved when the measure on the group extension is the Haar lift. However these theorems can be generalized to any ergodic measure on the group extension. To do this we use a structure theorem about ergodic measures on group extensions due to Keynes and Newton (Theorem (2.3) in **[12]**), which says "up to a Borel isomorphism" all ergodic measures on group extensions are Haar lifts. This completes the proof of Corollary (2.2).

Proof of Corollary (2.3). The proof is entirely analogous to that of Theorem (2.1) but now we carry out analysis on $C(X)$ with sup norm $\| \ \|_\infty$, rather than $L^2(X)$. First let $X = Z \times Y$, let all the operators be defined as before but now we think of them as operating on $C(X)$. One can easily show that $(Z \times_\alpha Y, T)$ is uniquely ergodic iff $\|V_n f - Cf\|_\infty \to 0$, $\forall f \in C(X)$ (C being the projection on constant). For $f \in C(X)$, $\varepsilon > 0$, and $m \in \mathbf{N}$, setting $W(f, \varepsilon, m) = \{\alpha | \alpha \in \overline{B(Y, T, G)}$ such that $\exists M \in \mathbf{N}$, $M > m$ and $\|V_M f - Cf\|_\infty < \varepsilon\}$ one can prove analogs of all the propositions through Lemma (3.8). This reduces the proof of Corollary (2.3) to constructing (given $\delta, \varepsilon > 0$) a $\psi \in C(Y, G)$ such that (i) $D(l^{\psi}, l) < \delta$ and (ii) $\|PL_{\psi}f - Cf\|_{\infty} < \varepsilon$. The construction of ψ is exactly as before the only difference being we need to replace inequality (2) [see proof of Lemma (3.9)] by the one with $\| \, \|_{\infty}$ norm rather than L^2 norm. To do this since G is amenable with Følner sequence say $(S_n) \subseteq_{n=1}^{\infty} G$, pick $N_0 \in \mathbf{N}$ large enough such that

$$\left\| \int_{S_n} f(zg, y)\, d\eta'|_{S_n}(g) - Qf(z, y) \right\|_{\infty} < \varepsilon, \quad \forall y,$$

(where η' is a right Haar measure on G). Then using usual approximation arguments we can get a finite set $(g_i)_{i=1}^{q} \in G$ such that

$$\left\| \sum_{i=1}^{q} \lambda_i U_{g_i} f - Qf \right\|_{\infty} < \varepsilon, \quad \forall y \in Y.$$

The proof in the case of non trivial group extension is reduced to lifting ergodicity by Corollary (2.2.6) of **[13]**. But this was proved in Theorem (2.1).

Proof of Corollary (2.4). Since T is amenable and (Z, G) is minimal distal, we can pick ergodic T and G invariant measures μ and ν on Y and Z respectively. Now note that proper ergodicity was used in the proof of Lemma (3.9) only to assure that the measure ξ on the transversal was non atomic, and this still holds because Y is infinite, T is discrete and (Y, T) is minimal. Thus Theorem (2.1) gives us a residual set of cocycles lifting ergodicity. Our claim is that this class of cocyles lift minimality too. To see this observe (i) by minimality, μ and ν give positive measures to non empty open sets and (ii) $Z \times_{\alpha} Y \to_{\pi} Y$ being a distal extension of a minimal system, $Z \times_{\alpha} Y$ is a disjoint union of its minimal sets. Since $(Z \times_{\alpha} Y, \nu \times \mu)$ is ergodic, condition (i) implies that there are points with dense orbit and hence $(Z \times_{\alpha} Y, T)$ is minimal. Again the group extension case is similar. □

REMARK. As this paper was being written, a paper of M. Hermann [7] was brought to our notice. There he proves some generic theorems when the base space is a circle. The notion of the L^{∞}-fixed point property used in [7] turns out to be useful in our situation also. A t.d.s. (Z, G, ν) where ν is any σ-finite invariant measure, is said to have a L^{∞}-fixed point property if any weakly compact T-invariant set $K \subseteq L^\infty(Z, \nu)$ contains a fixed point. Adopting the technique used by Hermann to construct the map h: $[0,1] \to G$ in our proof of Lemma (3.9) and replacing the entire L^2 analysis by a similar L^∞ analysis, one can generalize Theorem (2.1), to the situation when (Z, G, ν) has a L^∞-fixed point property. No further details will be provided here due to space limitation. A special case of particular interest is when G is path connected and amenable, $Z = G$ and action of G on Z is by left multiplication. In this case Theorem (2.1) implies that given a properly ergodic free t.d.s. (Y, T, μ) with T-an amenable R-group, $\exists$ a continuous cocyle α: $Y \times T \to G$ such that $(G \times_\alpha Y, \eta \times \mu)$ is ergodic, (η being a left Haar measure on G). This answers, in topological setting, a question raised by R. Zimmer, namely is every amenable group range of some Borel cocycle on an ergodic dynamical system.

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