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ON THE BOUNDARY CONTINUITY OF CONFORMAL MAPS

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Let the function f map the unit disk $\mathbf{D}$ conformally onto the domain G in $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. The prime end theory of Carathéodory gives a completely geometric characterization of the boundary behavior of f. Prime ends are defined in terms of crosscuts of G.

Our aim is to give a geometric description of the boundary behavior of f that refers only to the boundary ∂G and not to the domain itself. It can therefore be applied to any complementary domain of a connected closed set in $\hat{\mathbf{C}}$. Our description will however be incomplete because we will have to allow exceptional sets.

1. Introduction and results. We say that f has the *angular limit* $f(\zeta)$ at $\zeta \in \partial\mathbf{D}$ if

$$f(\zeta) = \lim_{z \to \zeta,\, z \in \Delta} f(z) \in \hat{\mathbf{C}}$$

exists for every Stolz angle Δ at ζ; we shall always denote by $f(\zeta)$ the angular limit if it exists. A theorem of Beurling [**1**] (see e.g. [**4**, p. 56] [**8**, p. 341, 344]) states that the angular limit $f(\zeta)$ exists for $\zeta \in B$ where $\operatorname{cap}(\partial\mathbf{D} \setminus B) = 0$ and furthermore that

$$\operatorname{cap}\{\zeta \in B: f(\zeta) = \omega\} = 0 \quad \text{for } \omega \in \hat{\mathbf{C}};$$

here cap denotes the logarithmic capacity.

We shall say that f is *continuous at* $\zeta \in \partial\mathbf{D}$ if f has a continuous extension to $\mathbf{D} \cup \{\zeta\}$, that is, if $f(z) \to f(\zeta)$ as $z \to \zeta$, $z \in \mathbf{D}$. Our first result states that discontinuity tends to imply injectivity.

THEOREM 1. *Let f map $\mathbf{D}$ conformally onto G. Then there is a partition*

$$\partial\mathbf{D} = A_0 \cup A_1 \cup A_2 \tag{1.1}$$

such that

(i) $\operatorname{cap} A_0 = 0$,

(ii) *the angular limit $f(\zeta)$ exists for every $\zeta \in A_1$, and f is one-to-one on A_1,*

(iii) *f is continuous at each $\zeta \in A_2$, and f is exactly two-to-one on A_2.*

Let E be a continuum in $\hat{\mathbf{C}}$. The point $\omega \in E$ will be called *accessible* if there exists a Jordan arc C with endpoint ω such that $C \cap E = \{\omega\}$. If G is a component of $\hat{\mathbf{C}} \setminus E$ we say that ω is *accessible from* G if there is a Jordan arc C ending at ω such that $C \subset G \cup \{\omega\}$; every accessible point is accessible from some component. If f maps $\mathbf{D}$ conformally onto the component G of $\hat{\mathbf{C}} \setminus E$, then every angular limit $f(\zeta)$ is accessible from G. Conversely, if $\omega \in E$ is accessible from G then there is at least one $\zeta \in \partial\mathbf{D}$ such that $\omega = f(\zeta)$ [**8**, p. 277].

We have to introduce another topological concept. We call $\omega \in E$ a *quasi-isolated accessible* point if there is a neighborhood V of ω such that ω is the *only* accessible point in the component of $E \cap \overline{V}$ containing ω. Thus the other accessible points of E cannot be connected to ω by a subcontinuum of small diameter.

As an example, we consider first the classical unsymmetric comb

$$E_1 = [-1 + i, 1 + i] \cup [0, i] \cup \bigcup_{n=1}^{\infty} \left[\frac{1}{n}, \frac{1}{n} + i\right]. \tag{1.2}$$

The point 0 is not quasi-isolated because, for $0 < r < 1$, its component of $E \cap \{|z| \leq r\}$ is $[0, ir]$ and all points on this segment are accessible. Consider now the symmetric comb

$$E_2 = E_1 \cup \bigcup_{n=1}^{\infty} \left[-\frac{1}{n}, -\frac{1}{n} + i\right]. \tag{1.3}$$

Then 0 is accessible but quasi-isolated because now no point of $(0, ir)$ is accessible.

Our next result is essentially topological.

THEOREM 2. *Let f map $\mathbf{D}$ conformally onto G. Then, for all $\zeta \in \partial\mathbf{D}$ with at most countably many exceptions, the function f is continuous at ζ, if and only if*

(i) *the angular limit $f(\zeta)$ exists, and*

(ii) *the accessible point $f(\zeta)$ of ∂G is not quasi-isolated.*

The classical comb shows that there may be exceptional values of ζ, and indeed our ideas about the boundary behavior of conformal maps seem to be strongly influenced by the exceptional cases.

COROLLARY 1. *Let f map $\mathbf{D}$ conformally onto G. Then there is a partition*

$$\partial\mathbf{D} = B_0 \cup B_1 \cup B_2 \tag{1.4}$$

such that

(i) $\operatorname{cap} B_0 = 0$,

(ii) $f(\zeta)$ *exists for* $\zeta \in B_1$ *and* $f(\zeta)$ *is a quasi-isolated accessible point of* ∂G,

(iii) f *is continuous at each* $\zeta \in B_2$.

This is a consequence of Theorem 2 because, for conformal maps, the set of $\zeta \in \partial \mathbf{D}$ where $f(\zeta)$ does not exist has zero capacity, by Beurling's theorem. The corollary is not true for arbitrary topological mappings; it is easy to construct a topological self-mapping of $\mathbf{D}$ that is nowhere continuous on $\partial \mathbf{D}$.

COROLLARY 2. *Let f be a conformal mapping of* $\mathbf{D}$ *onto G. If all points of ∂G are accessible then f is continuous on* $\partial \mathbf{D}$ *except possibly for a set of zero capacity.*

It is well-known that f is continuous in $\overline{\mathbf{D}}$ if every boundary point is accessible from G "from all sides." We have made a weaker assumption but then we have to allow for exceptions. If E_1 is again the classical comb defined by (1.2) then every boundary point of $G = \hat{\mathbf{C}} \setminus E_1$ is accessible but f is not continuous in $\overline{\mathbf{D}}$.

This corollary follows from Corollary 1 because there are no quasi-isolated points if every point of ∂G is accessible.

COROLLARY 3 (*B. Rodin*). *Let f map* $\mathbf{D}$ *conformally onto G and suppose that*

$$g(f(z)) = f(e^{2\pi i \alpha} z), \qquad \alpha \in \mathbf{R} \setminus \mathbf{Q}, \tag{1.5}$$

where g is continuous in $\overline{G}$. If all points of ∂G are accessible from G, then ∂G is a Jordan curve in $\hat{\mathbf{C}}$.

This result is of interest for Siegel disks in the theory of iterations. It is due to Rodin [**9**, Theorem 3]. The only new aspect is that his additional hypothesis that f is continuous for at least one $\zeta \in \partial \mathbf{D}$ follows automatically, by Corollary 2, from his assumption that each point of ∂G is accessible from G. Moeckel [**6**] has given an example of a function f satisfying (1.5) with a function g continuous in $\overline{G}$ such that every point of ∂G is accessible (though not always from G) and f has countably many discontinuities on $\partial \mathbf{D}$.

I want to thank Professor Burt Rodin for our discussions. His result was the starting point of the present investigation.

2. Proof of Theorem 1. The proofs are based on two remarkable topological countability theorems. A *triod* is the union of three Jordan arcs that begin at a common point but are otherwise disjoint. The following result is due to R. L. Moore [7].

MOORE TRIOD THEOREM. *Every disjoint collection of triods in the plane is countable.*

Let f be any function defined in $\mathbf{D}$. For $\zeta \in \partial\mathbf{D}$, the *left-hand cluster set* $C_L(\zeta)$ is defined by

$$C_L(\zeta) = \{w \in \hat{\mathbf{C}} \text{ there are } z_n \in \mathbf{D} \text{ with } z_n \to \zeta, \arg z_n \geq \arg \zeta, f(z_n) \to w\}. \tag{2.1}$$

The *right-hand cluster set* $C_R(\zeta)$ is defined similarly with $\arg z_n \leq \arg \zeta$ instead, and $C(\zeta) = C_L(\zeta) \cup C_R(\zeta)$ is the unrestricted cluster set. Note that f is continuous at ζ if and only if $C(\zeta)$ is a singleton. The following result is due to Collingwood [3] [4, p. 83].

COLLINGWOOD SYMMETRY THEOREM. *Let f be defined in $\mathbf{D}$. Then*

$$C_L(\zeta) = C_R(\zeta) = C(\zeta)$$

for all $\zeta \in \partial\mathbf{D}$ with at most countably many exceptions.

The point $\omega \in E$ is called a *cut point* of the continuum E if $E \setminus \{\omega\}$ is not connected. It follows from the plane separation theorem [10, p. 34] that $\omega \in E$ is a cut point of E if and only if there is a Jordan curve $J \subset \hat{\mathbf{C}}$ with $J \cap E = \{\omega\}$ that separates $E \setminus \{\omega\}$. If E bounds a domain G then $J \setminus \{\omega\}$ has to lie in G.

Proof of Theorem 1. Let A_0' denote the set of $\zeta \in \partial\mathbf{D}$ for which the angular limit $f(\zeta)$ does not exist. Beurling's theorem states that $\operatorname{cap} A_0' = 0$. Furthermore, let

$$A_2' = \{\zeta \in \partial\mathbf{D} \setminus A_0' : f(\zeta) \text{ is a cut point of } \partial G\} \tag{2.2}$$

and let $A_1 = \partial\mathbf{D} \setminus (A_0' \cup A_2')$.

We show first that (ii) holds. Suppose that f is not one-to-one on A_1. Then there exist $\zeta, \zeta^* \in A_1$ such that $f(\zeta) = f(\zeta^*) = \omega$. Then

$$J = f(\zeta S) \cup f(\zeta^* S), \qquad S \equiv [0,1], \tag{2.3}$$

is a Jordan curve that intersects ∂G only at ω. By Beurling's theorem, f has angular limits different from ω on both arcs of $\partial\mathbf{D}\setminus\{\zeta,\zeta^*\}$. Hence we conclude that there are points of ∂G in both components of $\hat{\mathbf{C}}\setminus J$. Therefore ω is a cut point of ∂G, contrary to our assumption $\zeta\in A_1\subset\partial\mathbf{D}\setminus A_2'$.

Let now $\zeta\in A_2'$. Then $\omega=f(\zeta)$ is a cut point of ∂G by (2.2), and there is a Jordan curve $J\subset G\cup\{\omega\}$ through ω that separates $\partial G\setminus\{\omega\}$. The open Jordan arc $Q=f^{-1}(J\setminus\{\omega\})\subset\mathbf{D}$ ends at definite points $\zeta_1,\zeta_2\in\partial\mathbf{D}$ [**8**, p. 267].

If $\zeta_1=\zeta_2$ then $Q\cup\{\zeta_1\}$ is a Jordan curve. Its inner domain H lies in $\mathbf{D}$, and $f(z)\to\omega$ as $z\to\zeta$, $z\in H$ by a theorem of Lehto and Virtanen [**5**]. Hence $f(H)$ is one of the components of $\hat{\mathbf{C}}\setminus J$. Since $f(H)\subset G$ and since J is to separate $\partial G\setminus\{\omega\}$, we conclude that the case $\zeta_1=\zeta_2$ is impossible. Since f has the angular limit ω at ζ_1 and at ζ_2 [**8**, p. 268] we thus see that there is at least one $\zeta^*\neq\zeta$ with $f(\zeta^*)=\omega$.

Let E_0 be the set of $\omega\in\partial G$ for which there are at least three points ζ_1 with angular limits $f(\zeta_j)=\omega$. For $\omega\in E_0$,

$$f(\zeta_1S_0)\cup f(\zeta_2S_0)\cup f(\zeta_3S_0),\qquad S_0\equiv[1/2,1],$$

is a triod because f is univalent in $\mathbf{D}$. If $\omega^*\in E_0$, $\omega^*\neq\omega$, then the corresponding triods are disjoint. Hence it follows from the Moore triod theorem that E_0 is countable. Hence

$$A_0''=\{\zeta\in\partial\mathbf{D}\setminus A_0':f(\zeta)\in E_0\}$$

has zero capacity by Beurling's theorem. If $\zeta\in A_2'\setminus A_0''$ then there is exactly one further $\zeta^*\in A_2'\setminus A_0''$ such that $f(\zeta^*)=f(\zeta)$.

Finally we define A_0 as $A_0'\cup A_0''$ together with all points $\zeta\in A_2\setminus A_0''$ such that either $C_L(\zeta)\neq C_R(\zeta)$ or $C_L(\zeta^*)\neq C_R(\zeta^*)$; by the Collingwood symmetry theorem, there are at most countably many such points. Hence $\operatorname{cap}A_0=0$ so that (i) holds. We define $A_2=A_2'\setminus A_0$. Then we have the partition $\partial\mathbf{D}=A_0\cup A_1\cup A_2$, and f is exactly two-to-one on A_2.

In order to establish (iii) we have to show that $C(\zeta)$ is a singleton for each $\zeta\in A_2$. Let ζ^* be the other point in A_2 with $f(\zeta^*)=f(\zeta)$ and consider the Jordan curve J defined by (2.3). Let H_L, H_R be the components of $\hat{\mathbf{C}}\setminus J$; we may assume that the points to the left of $f(\zeta S)$ lie in H_L. Then those to the right of $f(\zeta S)$ lie in H_R. Hence

$$C_L(\zeta)\subset\overline{H}_L,\qquad C_R(\zeta)\subset\overline{H}_R.$$

Since $C_L(\zeta)=C_R(\zeta)=C(\zeta)$ because of $\zeta\notin A_0$, we conclude that

$$C(\zeta)\subset\overline{H}_L\cap\overline{H}_R=J,$$

and since $C(\zeta)\subset\partial G$ and $J\cap\partial G=\{f(\zeta)\}$ it follows that $C(\zeta)=\{f(\zeta)\}$, and this completes the proof of Theorem 1.

3. Proof of Theorem 2. (a) Let first f be continuous at $\zeta \in \partial\mathbf{D}$. It is clear that the angular limit $f(\zeta)$ exists. Let V be a neighborhood of $f(\zeta)$. Then there is a disk around ζ such that its intersection U with $\mathbf{D}$ satisfies $f(U) \subset V$. Hence

$$F \equiv \overline{f(U)} \cap \partial G \subset \overline{U} \cap \partial G.$$

Since F is connected it follows that F lies in the component of $\overline{V} \cap \partial G$ that contains $f(\zeta)$.

Since f is conformal there exists $\zeta' \in \partial U \cap \partial\mathbf{D}$ such that the angular limit $f(\zeta')$ exists and is different from $f(\zeta)$, for instance by Beurling's theorem quoted above. Hence $f(\zeta')$ is also an accessible point in F. It follows that $f(\zeta)$ is not quasi-isolated.

(b) In order to prove the converse direction we may assume that $\infty \in G$ so that ∂G lies in $\mathbf{C}$. We shall not use that f is meromorphic so that f may be any topological mapping from $\mathbf{D}$ onto G.

Let A denote the set of all $\zeta \in \partial\mathbf{D}$ such that the angular limit $f(\zeta)$ exists and $f(\zeta)$ is not quasi-isolated. Let G_k denote the components of $\hat{\mathbf{C}} \setminus \partial G$. For $\zeta \in A$ and $n \in \mathbf{N}$, let $E_n(\zeta)$ denote the component of

$$\{w: |w - f(\zeta)| \le 1/n\} \cap \partial G$$

that contains $f(\zeta)$. Since ω is not quasi-isolated there is an accessible point $\omega_n(\zeta) \in E_n(\zeta)$ with $\omega_n(\zeta) \ne f(\zeta)$.

Let A_{nk} denote the set of $\zeta \in A$ such that $\omega_n(\zeta)$ is accessible from the component G_k; these sets need not be disjoint. Let

$$X = \{\zeta \in A: C_L(\zeta) \ne C_R(\zeta)\} \cup \bigcup_{A_{nk} \text{ singleton}} A_{nk}. \tag{3.1}$$

The first set is countable by the Collingwood symmetry theorem. Hence X is countable.

Let now $\zeta \in A \setminus X$. We shall show that f is continuous at ζ. Let $\Gamma = \{f(r\zeta): 1/2 \le r \le 1\}$. We have $\zeta \in A_{nk}$ for some $k = k(n)$ and there is a Jordan arc $\Gamma_n \subset G_k \cup \{\omega_n(\zeta)\}$ that ends at $\omega_n(\zeta)$. We distinguish two cases:

Case 1. Let first $\omega_n(\zeta)$ be accessible from G. Let $P_n \subset G$ be a Jordan arc connecting the other endpoints of Γ_n and Γ (without otherwise meeting Γ_n and Γ) and let

$$L_n = \Gamma \cup P_n \cup \Gamma_n \cup E_n(\zeta). \tag{3.2}$$

Since $\Gamma \cup P_n \cup \Gamma_n$ is a crosscut of $\hat{\mathbf{C}} \setminus E_n(\zeta)$ and since $E_n(\zeta)$ is a continuum, the points lying locally on the two sides of Γ belong to different components of $\hat{\mathbf{C}} \setminus L_n$, say H_n and H_n^*.

Case II. Let now $\omega_n(\zeta)$ be accessible from some component $G_k \neq G$. Since $\zeta \notin X$ we see from (3.1) that there exists $\zeta_n' \in A_{nk}$ with $\zeta_n' \neq \zeta_n$. Hence there are Jordan arcs Γ_n and Γ_n' that lie in G_k except for their endpoints $\omega_n(\zeta)$ and $\omega_n(\zeta')$. Let P_n be a Jordan arc in G_k that connects the other endpoints of Γ_n and Γ_n'. Furthermore let Q_n be a Jordan arc in G from $f(\zeta/2)$ to $f(\zeta_n'/2)$. We set $\Gamma_n^* = \{f(r\zeta_n')\colon 1/2 \le r \le 1\}$ and

$$(3.3)\quad L_n = (\Gamma_n \cup P_n \cup \Gamma_n') \cup (\Gamma \cup Q_n \cup \Gamma_n^*) \cup E_n(\zeta) \cup E_n(\zeta').$$

Since $E_n(\zeta)$ and $E_n(\zeta_n')$ are continua and since $\Gamma_n \cup P_n \cup \Gamma_n'$ and $\Gamma \cup Q_n \cup \Gamma_n^*$ are disjoint Jordan arcs connecting $E_n(\zeta)$ with $E_n(\zeta')$, the points lying locally on the two sides of Γ lie in different components of $\mathbf{C} \setminus L_n$, say H_n and H_n^*.

Now we consider both cases together. Let $j > 1$, let U_j and U_j^* be the "left" and "right" components of $\{z \in \mathbf{D}\colon |z - \zeta| < 1/j\} \setminus [\zeta/2, \zeta]$. Then $f(U_j)$ intersects H_n and $f(U_j^*)$ intersects H_n^* if we label the components H_n and H_n^* of $\mathbf{C} \setminus L_n$ accordingly. If j is large then $f(U_j)$ and $f(U_j^*)$ do not intersect L_n as we see from (3.2) or (3.3) because f is a homeomorphism from $\mathbf{D}$ onto G. Hence there exists j_n such that

$$(3.4)\qquad f(U_{j_n}) \subset H_n, \qquad f(U_{j_n}^*) \subset H_n^*.$$

It follows from (2.1) and the corresponding definition of $C_R(\zeta)$ and from (3.4) that, for $n = 1, 2, \dots,$

$$C_L(\zeta) \subset \overline{f(U_{j_n})} \subset \overline{H}_n, \qquad C_R(\zeta) \subset \overline{f(U_{j_n}^*)} \subset \overline{H}_n^*.$$

Since $\zeta \notin X$ we therefore obtain from (3.1) that

$$C(\zeta) = C_L(\zeta) \cap C_R(\zeta) \subset \overline{H}_n \cap \overline{H}_n^* \subset L_n.$$

Furthermore $C(\zeta) \subset \partial G$. Hence we conclude

$$(3.5)\qquad C(\zeta) \subset L_n \cap \partial G \subset E_n(\zeta) \quad \text{or} \quad \subset E_n(\zeta) \cup E_n(\zeta_n')$$

from (3.2) for Case I and from (3.3) for Case II, respectively.

In Case I, we immediately get from (3.5) that

$$\operatorname{diam} C(\zeta) \le \operatorname{diam} E_n(\zeta) \le 2/n;$$

this inequality holds also in Case II if $E_n(\zeta)$ and $E_n(\zeta_n')$ are disjoint because then (3.5) implies that the continuum $C(\zeta)$ lies in $E_n(\zeta)$. If $E_n(\zeta)$ and $E_n(\zeta_n')$ intersect then we obtain from (3.5) that

$$\operatorname{diam} C(\zeta) \leq \operatorname{diam}\left[E_n(\zeta) \cup E_n(\zeta_n')\right] \leq 4/n.$$

Since $C(\zeta)$ is independent of n we conclude in all cases that $C(\zeta)$ is a singleton so that f is continuous at ζ.

REFERENCES

[1] A. Beurling, *Ensembles exceptionnels*, Acta Math., **72** (1940), 1–13.
[2] C. Carathéodory, *Über die Begrenzung einfach zusammenhängender Gebiete*, Math. Ann., **73** (1913), 323–370.
[3] E. F. Collingwood, *Cluster sets of arbitrary functions*, Proc. Nat. Acad. Sci. U.S.A., **46** (1960), 1236–1242.
[5] O. Lehto and K. I. Virtanen, *Boundary behavior and normal meromorphic functions*, Acta Math., **97** (1957), 47–65.
[6] R. Moeckel, *Rotations of the closures of some simply connected domains*, preprint Univ. of Minnesota, 1984.
[7] R. L. Moore, *Concerning triods in the plane and the junction points of plane continua*, Proc. Nat. Acad. Sci. U.S.A., **14** (1928), 85–88.
[8] Ch. Pommerenke, *Univalent Functions*, Vandenhoeck & Ruprecht, Göttingen 1975.
[9] B. Rodin, *Intrinsic rotations of simply connected regions*, Complex Variables Theory Appl., **2** (1984), 319–326.
[10] G. T. Whyburn, *Topological Analysis*, Princeton Univ. Press 1958.

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