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A NEW APPROACH TO THE KREIN-MILMAN THEOREM

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In this paper we give a new definition of extreme points for which we get a generalization of the Krein-Milman theorem within the general context of locally convex spaces over valued fields.

Some generalizations of the theorem of Krein-Milman were developed in the seventies in order to include other types of topological vector spaces apart from the usual ones (e.g. Kalton's papers within the context of locally p-convex spaces). However, A. F. Monna says in 1974 that no way is known to attack problems such as the Krein-Milman theorem in ultrametric analysis (i.e. when the real or complex field is substituted for another valued field).

In order to give a theorem of Krein-Milman which includes the case of locally p-convex spaces ($p \in (0,1]$) and the ultrametric case, we propose a new definition of extreme points. The latter definition agrees with the usual one in case the ground field is R or C and it allows us to give a non-archimedean Krein-Milman theorem.

We are going to consider vector spaces E over any complete non-trivially valued field K. For $K = R$, C, and $p \in (0,1]$ we say that $A \subset E$ is p-convex if $\lambda A + \mu A \subset A$ for all $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$. For a non-archimedean valued field K two different kinds of convex sets will be considered: $A \subset E$ is said to be M-convex (convex à la Monna) if $\lambda A + \mu A + \nu A \subset A$ for all $\lambda, \mu, \nu \in K$ such that $|\lambda|, |\mu|, |\nu| \leq 1$ and $\lambda + \mu + \nu = 1$; and for $a \in E$, the set $A \subset E$ is said to be a-convex if A is M-convex and $a \in A$. More details over these kinds of convex sets we will use in the sequel are in [3] (for p-convex sets) and [5] (for the non-archimedean case).

In the sequel we will use the term "convex" to indicate any of the different kinds of convex sets; also $E_c(A)$ stands for the corresponding convex hull of A.

1. Semiconvexity. Extreme points. The following definition is very close to the weak-convexity of Monna ([5] p. 28).

DEFINITION 1. Let E be a vector space over a valued field K. A subset A of E is said to be semiconvex if $\lambda A + (1 - \lambda)A \subset A$ for every λ of K satisfying $|\lambda| < 1$.

Notice that if $K = R$, C every semiconvex set is 1-convex and if K is non-archimedean every M-convex set and every a-convex set are semiconvex.

DEFINITION 2. Let E be a vector space over a valued field K and A a subset of E. A non-empty part S of A is said to be an extreme set of A if the following properties are verified:

(i) S is semiconvex.

(ii) If $x_1,\dots,x_n \in A$ and $E_c\{x_1,\dots,x_n\} \cap S \neq \varnothing$, then there exists an index $i \in \{1,\dots,n\}$ such that $x_i \in S$.

It is easy to verify that if A is convex, then the property (ii) is equivalent to $A - S$ is convex.

DEFINITION 3. Let E be a vector space over K and A a subset of E. A point $x \in A$ is said to be an extreme point of A if it belongs to some minimal element of

$$E_A = \{\, S \subset A | S \text{ is an extreme set of } A \}.$$

Next, we prove that if $K = R$, C this definition gives the same extreme points as the usual ones for every p-convex compact set A of a separated locally p-convex space. For that, we denote by $E_p(A)$ the set of p-extreme points of A according to the definition of Kalton [**4**]. (Notice that this definition is slightly different from the corresponding definition of Jarchow [**3**], however they agree for closed p-convex sets). Also $F_p(A)$ indicates the set of extreme points of A corresponding to our Definition 3 for p-convex sets.

THEOREM 1. *Let E be a Hausdorff locally p-convex space over $K = R$, C and let $p \in (0, 1]$. If A is a non-empty compact p-convex set of E, then*:

(1) *Every minimal element of E_A consists of one point.*

(2) $F_p(A) = E_p(A)$.

Proof. (1) Let S be a minimal element of E_A and suppose x, y are different points of S. As $\bar{S}$ is semiconvex and closed, $\lambda\bar{S} + (1 - \lambda)\bar{S} \subset \bar{S}$ for every $\lambda \in K$ such that $|\lambda| \leq 1$. Consequently $2y - x \in \bar{S}$; and also for $n = 1, 2\dots$ $z_n = (n + 1)y - nx \in \bar{S}$. So the sequence $(z_n) \subset \bar{S}$ verifies $\lim(z_n/n) = y - x \neq 0$ and $\bar{S}$ is not bounded.

(2) This follows from (1) and the fact that $x \in E_p(A)$ if and only if $A - \{x\}$ is p-convex (see **[2]** p. 96 for $p = 1$ and **[4]** for any p).

2. Non-Archimedean extreme points. Throughout the rest of the paper, E will indicate a topological vector space over a field K, endowed with a non-trivial non-archimedean valuation. We are going to restrict ourselves to the case of K local (i.e. locally compact) (otherwise there do not exist any compact convex set with more than one point **[5]** p. 40).

Theorems in this section are proved for 0-convex sets; however with minor changes they remain true for the other kinds of convexities over E.

THEOREM 2. *Let E be a topological vector space over K and assume that E' separates points of E. Then, every convex and compact subset A of E has extreme points.*

Before proving the theorem we need the following lemma:

LEMMA 1. *Let E be a topological vector space over K and let A, B be convex sets in E with $B \subset A$. If the interior of B in A is non-empty, then B is clopen in A.*

Proof. Take $x_0 \in B$ such that B is a neighborhood of x_0 in A. Now, if $x \in B$, then $B = B - x_0 + x$; hence B is a neighborhood of x in A and B is open in A. Also, if $y \in A - B$, then $B \cap (y + B) = \varnothing$ and consequently $A - B$ is open in A.

Proof (*of Theorem* 2). Let A be with more than one point and define

$$CE_A = \{ S \subset A \mid S \text{ is a closed extreme set of } A \}.$$

CE_A is non-empty and a standard application of Zorn's lemma shows that CE_A has some minimal element.

Let S_0 be one such minimal element. First we prove that $S_0 \neq A$.

For that, choose $f \in E'$ such that $f(A)$ is not reduced to be a single point and define

$$S_f = \left\{ s \in A \mid |f(s)| = \sup_{x \in A} |f(x)| \right\}.$$

It is easy to verify that $S_f \in CE_A$ and that $S_f \neq A$. Then, $S_0 \neq A$.

Let $S \in E_A$ such that $S \subset S_0$. Applying Lemma 1 to A and $A - S$, we deduce that S is closed in E. Thus $S_0 = S$ and S_0 is a minimal element of E_A.

COROLLARY 1. *Under the assumptions of the Theorem 2, every closed extreme subset of A contains extreme points of A.*

If $A \subset E$ we will denote by $\mathrm{Ext}(A)$ the set of extreme points of A. In the following theorem we use the terminology of [**1**].

THEOREM 3 (*Non-archimedean Krein-Milman theorem*). *Let E be a Hausdorff locally convex space over K. If A is a non-empty convex compact set of E, then $A = \overline{E}_c(\mathrm{Ext}(A))$.*

Proof. For $x_0 \in A - \overline{E}_c(\mathrm{Ext}(A))$, let H be a closed hyperplane which separates x_0 and $\overline{E}_c(\mathrm{Ext}(A))$, and let $f(x) = \alpha$ an equation of H ($f \in E'$). As $S_f \in CE_A$ (see the preceding theorem), we can choose $x \in \mathrm{Ext}(A) \cap S_f$. Also, $\overline{E}_c(\mathrm{Ext}(A))$ is in one side of H, so $|f(y)| < |\alpha|$ for every $y \in \overline{E}_c(\mathrm{Ext}(A))$. It follows that $|f(a)| \leq |f(x)| < |\alpha|$ for every $a \in A$.

Thus, A and $\overline{E}_c(\mathrm{Ext}(A))$ are in the same side of H and, therefore, $x_0 \notin A$.

The main difference between the real or complex case and the non-archimedean case is contained in the following theorem.

THEOREM 4. *Let E be a Hausdorff topological vector space over K and let A be a convex set in E with more than one point. Then, every extreme set of A cannot be reduced to a single point.*

Proof. Suppose A to be absorbing (otherwise replace E by the linear hull of A).

First suppose that the interior of A is non-empty (i.e. A is clopen). If S is an extreme set of A with only one point, then $A - S$ is an open convex set in A. Thus (Lemma 1) $A - S$ is clopen in A and, consequently, S is open.

Now assume A to be bounded and let τ_p be the topology on E defined by the Minkowski's functional of A. If τ is the original topology in E we have $\tau \leq \tau_p$ and the interior of A with respect to τ_p is non-empty. Now apply the first result of this proof.

Finally, let A be any set in the hypotheses of the theorem. If $S = \{x\}$ is an extreme set of A, take $y \in A - \{0\}$ such that $x \in \{\lambda y \mid |\lambda| \leq 1\} = A_y$. Therefore, S ought to be an extreme set of the bounded convex set A_y.

3. An expression of the extreme points. First we are going to consider the case of 0-convex compact sets. Such a subset A of a Hausdorff topological vector space E can be expressed in the following way:

$$A = \left\{ \sum_{i \in I} x_i e_i \,\middle|\, |x_i| \leq 1 \right\} \tag{1}$$

where $(e_i)_{i \in I}$ is a topologically independent family of elements of A. The expression of each element of A as a sum $\sum_{i \in I} x_i e_i$ is unique, and the convergence of the sums is in the sense of the Cauchy's filter ([**1**] p. 152). If $x = \sum_{i \in I} x_i e_i$ we put $\langle x, e_i \rangle = x_i$.

We denote $\mathrm{Ext}_0(A)$ the set of the extreme points of A for the 0-convexity. Also, p_A denotes the Minkowski's functional of A in the linear hull of A.

THEOREM 5. *Let E be a Hausdorff topological vector space over K and let A be a* 0*-convex compact set of E. If $(e_i)_{i \in I}$ is a family of points of A satisfying* (1), *then the following properties for a point $x \in A$ are equivalent*:

(i) $x \in \mathrm{Ext}_0(A)$.
(ii) $\sup_{i \in I} |\langle x, e_i \rangle| = 1$.
(iii) *There exists $i_0 \in I$ such that* $|\langle x, e_{i_0} \rangle| = 1$.
(iv) $p_A(x) = 1$.

Proof. For $i \in I$, consider

$$D_i = \{ x \in A \mid |\langle x, e_i \rangle| = 1 \}.$$

D_i is a 0-extreme set of A. Now we wish to prove that D_i is minimal. Otherwise, consider T to be a proper subset of D_i which is a 0-extreme set of A. Take $x \in D_i - T$ and define $y \in A$ in the way $\langle y, e_j \rangle = x_j$ for $j \neq i$ and $\langle y, e_i \rangle = 0$. Obviously $y \in A - T$, and being $A - T$ 0-convex, then $\langle x, e_i \rangle^{-1}(x - y) = e_i \in A - T$. Pick $t \in T$ and define $z \in A - T$ in the way $\langle z, e_j \rangle = t_j$ for $j \neq i$ and $\langle z, e_i \rangle = 0$. Finally, we have $t = \langle t, e_i \rangle e_i + z$ which contradicts the assumption that $A - T$ is 0-convex. This proves (iii) $\Rightarrow$ (i).

The equivalence (ii) $\Leftrightarrow$ (iii) is obvious because the valuation over K is discrete.

For the equivalence (ii) $\Leftrightarrow$ (iv), it is straightforward to verify that for a point $x \in A$, $p_A(x) = \sup_{i \in I} |\langle x, e_i \rangle|$.

For (i) $\Rightarrow$ (ii), consider an $x \in A$ such that $\sup_{i \in I} |\langle x, e_i \rangle| < 1$. Choose $\mu \in K$ with $|\mu| > 1$ such that $\mu x \in A$. If S is a proper 0-extreme set of A which contains x, then $\mu x \in S$. Also, $0 = \lambda \mu x + (1 - \lambda)x \in S$ (with $\lambda = -1/(\mu - 1)$) which contradicts that $A - S$ is 0-convex. Hence, $x \notin \mathrm{Ext}_0(A)$.

REMARKS. (1) The latter theorem holds for a compact a-convex set A. In fact, under the conditions of the Theorem 5, the following properties are equivalent: (i) $x \in \mathrm{Ext}_a(A)$ (ii) $\sup_{i \in I} |\langle x - a, e_i \rangle| = 1$ (iii) There exists $i_0 \in I$ such that $|\langle x - a, e_{i_0} \rangle| = 1$ (iv) $p_{A-a}(x - a) = 1$ where

$\mathrm{Ext}_a(A)$ indicates the set of a-extreme points of A and $(e_i)_{i\in I}$ satisfies (1) for the 0-convex set $A - a$.

(2) However, if we consider M-convex sets, the result we get is trivial. If we put $\mathrm{Ext}_M(A)$ to indicate the set of M-extreme points of an M-convex set A, and with the assumptions on E of Theorem 5 we have:

COROLLARY 2. $\mathrm{Ext}_M(A) = A$.

Proof. It follows from the fact that $\{x \in A \mid |\langle x - a, e_i\rangle| = 1\}$ is a minimal element of E_A (for this convexity) for all $a \in A$, $i \in I$.

(3) Our Theorem 5 is quite similar to the Theorem 2 of Kalton's paper [**4**], which establishes that every point of a compact p-convex $(0 < p < 1)$ subset A of a Hausdorff topological vector space E can be expressed in the way $x = \sum a_n x_n$ with $a_n \geq 0$, $\sum a_n^p = 1$ and (x_n) being a sequence of distinct p-extreme points of A.

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