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THE FEYNMAN INTEGRAL OF QUADRATIC POTENTIALS DEPENDING ON TWO TIME VARIABLES

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We show that the double integral of certain quadratic potentials depending on two time variables is in a Banach algebra $\mathscr{S}$ of functions on Wiener space all of whose members have an analytic Feynman integral. Corollaries are given insuring (a) that $\mathscr{S}$ contains a rather broad class of functions involving double integrals of potentials depending on two time parameters, and (b) the existence of the Fresnel integral for such functions.

1. Introduction. Let $C_1[0,\tau]$ denote Wiener space; that is, the space of continuous functions x on $[0,\tau]$ such that $x(0)=0$. In Feynman's original paper [**13**, §13] and again in the book of Feynman and Hibbs [**14**, §§3–10] on path integrals, attention is drawn to functions on Wiener space of the form

$$G(x)=\exp\left\{\int_0^\tau\int_0^\tau W(s_1,s_2;x(s_1),x(s_2))\,ds_1\,ds_2\right\}. \tag{1.1}$$

Feynman obtained such functions by formally integrating out the oscillator coordinates in a system involving a harmonic oscillator interacting with a particle moving in a potential. The double dependence on time occurs because, as Feynman and Hibbs explain [**14**, p. 71], "The separation of past and future can no longer be made. This happens because the variable x at some previous time affects the oscillator which, at some later time reacts back to affect x."

By combining the results of this paper and another paper now in preparation, we are able to show that a rather broad class of functions of the form (1.1) above can be included within the transform approach to the Feynman integral; see Corollary 4.5 below.

Quadratic potentials appear frequently in the quantum theory and certain quadratic potentials involving double integrals will be our primary concern in this paper. Specifically we treat functions on Wiener space of the form

(1.2) $F(x)$

$$=\exp\left\{-\int_0^\tau\int_0^\tau\langle A(s_1,s_2)(x(s_1),x(s_2)),(x(s_1),x(s_2))\rangle\,ds_1\,ds_2\right\}$$

where $\{A(s_1, s_2)\colon (s_1, s_2) \in [0, \tau]^2\}$ is a commutative family of 2 by 2 real, symmetric, nonnegative definite matrices such that the eigenvalues $p_1(s_1, s_2)$ and $p_2(s_1, s_2)$ have square roots which are of bounded variation on the rectangle $Q = [0, \tau]^2$. We show that such functions F are in the Banach algebra $\mathcal{S}$ of functions on Wiener space which was introduced by Cameron and Storvick in **[6]**. It follows immediately from a theorem of Cameron and Storvick [**6**, Theorem 5.1] that F has an analytic Feynman integral.

We will precisely define the space $\mathcal{S}$ further on, but roughly speaking, $\mathcal{S}$ consists of functions on Wiener space which are stochastic transforms of finite Borel measures on $L_2[0, \tau]$. Let H be the space of absolutely continuous functions γ on $[0, \tau]$ which vanish at τ and whose derivatives $D\gamma$ are square integrable on $[0, \tau]$. H is a Hilbert space under the inner product

$$(1.3) \qquad (\gamma_1, \gamma_2) = \int_0^\tau (D\gamma_1)(s)(D\gamma_2)(s)\, ds.$$

Albeverio and Høegh-Krohn's space $\mathcal{F}(H)$ of Fresnel (or Feynman) integrable functions consists of Fourier transforms of finite Borel measures on H. The spaces $\mathcal{S}$ and $\mathcal{F}(H)$ are isometrically isomorphic as Banach algebras as was discovered by Johnson [**17**]. Using the connection between the spaces $\mathcal{S}$ and $\mathcal{F}(H)$ we are able to use our result about $\mathcal{S}$ and show that functions of the form (1.2), with an appropriate slight modification, are in $\mathcal{F}(H)$ and so are Fresnel integrable. We mention that as an immediate consequence of this and a theorem of Truman [**29**, Theorem 2], Truman's Feynman map exists for this class of functions. As a further consequence, the sequential Feynman integral discussed in Cameron and Storvick's recent memoir [**9**] exists for such functions.

The techniques of the present paper are most closely related to those of the earlier paper [**21**] of Johnson and Skoug, but most of the arguments here are considerably more complicated. It is interesting to note that while the statement of our results involves only the one-parameter Wiener process, the proof involves the two-parameter Wiener process (or Yeh-Wiener process) in a natural way. The most crucial technical step in the paper is the establishment of a stochastic integration by parts formula, Theorem 3.1, involving a mix of Wiener space and Yeh-Wiener space. This result may well be of some independent interest.

The only previous work of which we are aware in the rigorous theory of the Feynman integral which involves functions of the form (1.1) is the paper of Cameron and Storvick [**4**]. In that paper, the integral is interpreted as a bounded linear operator from L_1 to L_∞ rather than as a number.

The techniques of [4] and the conditions on W are very different than in the present paper.

We should mention that our interest in functions of the form (1.1) was stimulated by remarks in a recent expository essay of Nelson [24].

2. Definitions and preliminaries. Let $Q \equiv [0, \tau]^2$ and let $C_2(Q)$ denote Yeh-Wiener space; that is, the space of continuous functions f on Q such that $f(0, t) = f(s, 0) = 0$ for all $t \in [0, \tau]$ and $s \in [0, \tau]$. Let m_1 denote Wiener measure and let m_2 denote Yeh-Wiener measure.

A subset A of $C_1[0, \tau]$ is said to be scale-invariant measurable provided ρA is Wiener measurable for every $\rho > 0$. A scale-invariant measurable set N is said to be scale-invariant null provided $m_1(\rho N) = 0$ for every $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). For a detailed discussion of this topic see [19]. The concept of s-a.e. in Yeh-Wiener space $C_2(Q)$ is defined similarly [10].

Let F be a complex-valued function on $C_1[0, \tau]$ which is s-a.e. defined and scale-invariant measurable and such that the Wiener integral

$$J(\lambda) = \int_{C_1[0,\tau]} F(\lambda^{-1/2}x)\, dm_1(x)$$

exists as a finite number for all $\lambda > 0$. If there exists a function $J^*(\lambda)$ analytic in $C^+ \equiv \{\lambda | \lambda \text{ is complex and } \operatorname{Re}\lambda > 0\}$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the analytic Wiener integral of F over $C_1[0, \tau]$ with parameter λ, and for λ in C^+, we write

$$\int_{C_1[0,\tau]}^{\mathrm{an}\, w_\lambda} F(x)\, dm_1(x) \equiv J^*(\lambda).$$

Let q be a real parameter ($q \neq 0$) and let F be a function whose analytic Wiener integral exists for all λ in C^+. If the following limit exists, we call it the analytic Feynman integral of F over $C_1[0, \tau]$ with parameter q and we write

$$\int_{C_1[0,\tau]}^{\mathrm{an}\, f_q} F(x)\, dm_1(x) \equiv \lim_{\lambda \to -iq} \int_{C_1[0,\tau]}^{\mathrm{an}\, w_\lambda} F(x)\, dm_1(x)$$

where λ approaches $-iq$ through C^+.

The definition of the Banach algebra $\mathscr{S}$ with which we are concerned involves the Paley-Wiener-Zygmund (P.W.Z.) integral [25], a simple type of stochastic integral which we now define.

Let $\{\phi_j\}$ be a complete orthonormal (CON) set of functions of bounded variation on $[0,\tau]$. For g in $L_2[0,\tau]$, let

$$g_n(s) = \sum_{j=1}^{n} (g,\phi_j)\phi_j(s).$$

The P.W.Z. integral is defined by the formula

$$\int_0^\tau g(s)\,\tilde{d}x(s) \equiv \lim_{n\to\infty} \int_0^\tau g_n(s)\,dx(s)$$

for all x in $C_1[0,\tau]$ for which the limit exists. Since, in §3 below we discuss various properties of the P.W.Z. integral for functions of two variables, we won't list the corresponding properties for functions of one variable.

Now let $M = M(L_2[0,\tau])$ be the collection of complex-valued countably additive measures on $\mathscr{B} = \mathscr{B}(L_2[0,\tau])$, the Borel class of $L_2[0,\tau]$. M is a Banach algebra under the total variation norm where convolution is taken as the multiplication. The Banach algebra $\mathscr{S}$ consists of functions F on $C_1[0,\tau]$ expressible in the form

$$F(x) = \int_{L_2[0,\tau]} \exp\left\{ i\int_0^\tau g(s)\,\tilde{d}x(s)\right\} d\sigma(g)$$

for s-a.e. x in $C_1[0,\tau]$, where σ is an element of M. Cameron and Storvick show that the correspondence $\sigma \to F$ is one-to-one [**6**; Theorem 2.1] and carries convolution into pointwise multiplication. Letting $\|F\| \equiv \|\sigma\|$ we have that $\mathscr{S}$ is a Banach algebra of functions on Wiener space. Furthermore the analytic Feynman integral exists for every F in $\mathscr{S}$ [**6**, Theorem 5.1].

Finally, for the convenience of the reader, we state the following two results that are used in §3. The first well-known result, stated in [**21**, p. 283], is a stochastic integration by parts formula for P.W.Z. integrals of functions of one variable.

LEMMA 2.1. *Let* $0 \le a < b \le \tau$. *Then for* $m_1 \times m_1$-*a.e.*$(x,w) \in C_1[0,\tau] \times C_1[0,\tau]$ *we have that*

$$\int_a^b x(s)\,\tilde{d}w(s) = x(b)w(b) - x(a)w(a) - \int_a^b w(s)\,\tilde{d}x(s). \tag{2.1}$$

The second result is due to Skoug [**27**, p. 306].

LEMMA 2.2. *Let* $0 < \beta \le \tau$. *Let A be any subset of* $C_1[0,\tau]$ *and let* $B_A \equiv \{f \in C_2(Q)\colon f(\cdot,\beta) \in A\}$. *Then B_A is Yeh-Wiener measurable if and only if* $[2/\beta]^{1/2}A$ *is Wiener measurable. Furthermore, if either set is* *measurable, we have that*

$$m_2(B_A) = m_1\big([2/\beta]^{1/2}A\big). \tag{2.2}$$

3. Bounded variation on Q; A stochastic integration by parts formula for Wiener $\times$ Yeh-Wiener space. The concept of bounded variation for a function of two variables that we use in this paper was used by Hardy and by Krause [**16**, p. 345]. For the convenience of the reader we briefly review this definition.

Let Δ denote the partition of Q determined by $0 = s_0 < s_1 < \cdots < s_n = \tau$ and $0 = t_0 < t_1 < \cdots < t_m = \tau$. A function $h(s,t)$ is said to be of bounded variation on Q provided the following three conditions hold:

(i) there exists a constant K such that for any partition Δ

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\big|h(s_i,t_j) - h(s_i,t_{j-1}) - h(s_{i-1},t_j) + h(s_{i-1},t_{j-1})\big| \le K, \tag{3.1}$$

(ii) $h(s,t)$ is a function of bounded variation in s for each $t \in [0,\tau]$, and

(iii) $h(s,t)$ is a function of bounded variation in t for each $s \in [0,\tau]$. Hobson then points out that conditions (ii) and (iii) can be relaxed to the requirements that $h(s,t)$ is of bounded variation in s for one fixed value of t and is of bounded variation in t for one fixed value of s.

The Riemann-Stieltjes integral $\int_Q h(s,t)\,df(s,t)$ is then defined in the usual way [16]. Also see [30] for a nice discussion of the n-dimensional Riemann-Stieltjes integral and some of its properties.

The following proposition will be used to establish Corollary 4.2 in §4.

PROPOSITION 3.1. *Let $\delta > 0$ be given. Let h: $Q \to [\delta, \infty)$ be such that the partial derivatives $h_1(s,t)$ and $h_2(s,t)$ are absolutely continuous in s for each t and in t for each s and that the function $h_{12}(s,t)$ is integrable over Q. Then the function $g(s,t) \equiv [h(s,t)]^{1/2}$ is of bounded variation on Q.*

Proof. Let Δ denote the partition of the rectangle Q given $0 = s_0 < s_1 < \cdots < s_n = \tau$ and $0 = t_0 < t_1 < \cdots < t_m = \tau$. Then

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m}&\big|g(s_i,t_j) - g(s_{i-1},t_j) - g(s_i,t_{j-1}) + g(s_{i-1},t_{j-1})\big| \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}\left|\int_{s_{i-1}}^{s_i}\big[g_1(s,t_j) - g_1(s,t_{j-1})\big]\,ds\right| \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}\left|\int_{s_{i-1}}^{s_i}\left[\int_{t_{j-1}}^{t_j} g_{12}(s,t)\,dt\right]ds\right|.\end{aligned}$$

But $g(s,t) \equiv [h(s,t)]^{1/2}$ and so we have that

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\left|\sqrt{h(s_i,t_j)} - \sqrt{h(s_{i-1},t_j)} - \sqrt{h(s_i,t_{j-1})} + \sqrt{h(s_{i-1},t_{j-1})}\right|$$

$$\leq \sum_{i=1}^{n}\sum_{j=1}^{m}\int_{s_{i-1}}^{s_i}\int_{t_{j-1}}^{t_j}\left|\frac{h_{12}(s,t)}{2\sqrt{h(s,t)}} - \frac{h_1(s,t)h_2(s,t)}{4[h(s,t)]^{3/2}}\right| dt\, ds$$

$$= \int_0^T\int_0^T\left|\frac{h_{12}(s,t)}{2\sqrt{h(s,t)}} - \frac{h_1(s,t)h_2(s,t)}{4[h(s,t)]^{3/2}}\right| dt\, ds.$$

Next we note that for all $(s,t) \in Q$,

$$h_1(s,t) = \int_0^t h_{12}(s,v)\, dv + h_1(s,0)$$

and

$$h_2(s,t) = \int_0^s h_{21}(u,t)\, du + h_2(0,t).$$

Hence for all $(s,t) \in Q$ we obtain that

$$|h_1(s,t)| \leq \int_0^T |h_{12}(s,v)|\, dv + |h_1(s,0)| \equiv M(s) \in L_1[0,\tau]$$

and

$$|h_2(s,t)| \leq \int_0^T |h_{21}(u,t)|\, du + |h_2(0,t)| \equiv N(t) \in L_1[0,\tau].$$

Using the above expressions it easily follows that

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\left|\sqrt{h(s_i,t_j)} - \sqrt{h(s_{i-1},t_j)} - \sqrt{h(s_i,t_{j-1})} + \sqrt{h(s_{i-1},t_{j-1})}\right|$$

$$\leq (4\delta)^{-1/2}\int_0^T\int_0^T |h_{12}(s,t)|\, dt\, ds$$
$$+ (16\delta^3)^{-1/2}\int_0^T\int_0^T |h_1(s,t)h_2(s,t)|\, dt\, ds$$
$$\leq (4\delta)^{-1/2}\int_0^T\int_0^T |h_{12}(s,t)|\, dt\, ds$$
$$+ (16\delta^3)^{-1/2}\int_0^T M(s)\, ds \int_0^T N(t)\, dt < \infty.$$

Thus, using equation (3.1), we see that the function $g(s,t) \equiv \sqrt{h(s,t)}$ is of bounded variation on Q.

Next we give the definition of the Paley-Wiener-Zygmund integral (generalized Riemann-Stieltjes integral) for functions of two variables. Let $\{\phi_j\}$ be a complete orthonormal set of functions of bounded variation on

Q (for example the Haar functions on Q). For h in $L_2(Q)$ let

$$h_n(s,t) = \sum_{j=1}^{n} (h, \phi_j)\phi_j(s,t). \tag{3.2}$$

Then the P.W.Z. integral of h with respect to f is defined by

$$\int_Q h(s,t)\, \tilde{d}_2 f(s,t) \equiv \lim_{n\to\infty} \int_Q h_n(s,t)\, df(s,t) \tag{3.3}$$

for all f in $C_2(Q)$ for which the limit exists. Following are some useful facts about the P.W.Z. integral.

(i) For each h in $L_2(Q)$, the P.W.Z. integral $\int_Q h\tilde{d}_2 f$ exists for s-a.e. f in $C_2(Q)$.

(ii) The P.W.Z. integral $\int_Q h\, \tilde{d}_2 f$ is essentially independent of the CON set $\{\phi_j\}$; and thus it is often convenient to let $\{\phi_j\}$ be the Haar functions on Q.

(iii) If h is of bounded variation on Q, the P.W.Z. integral $\int_Q h\, \tilde{d}_2 f$ is s-a.e. equal to the Riemann-Stieltjes integral $\int_Q h\, df$.

(iv) The P.W.Z. integral has the usual linearity properties.

(v) The sequence $\{\int_Q h_n\, df\}$ converges in $L_2(C_2(Q))$ mean to $\int_Q h\, \tilde{d}_2 f$.

For convenience we also state the following well known Yeh-Wiener integration formula.

LEMMA 3.2. *Assume that h is an element of $L_2(Q)$ and that $g(u)$ is a complex-valued Lebesgue measurable function on* $\mathbf{R}$. *Then*

$$\int_{C_2(Q)} g\left(\int_Q h(s,t)\, \tilde{d}_2 f(s,t)\right) dm_2(f)$$
$$= (2\pi)^{-1/2} \int_{-\infty}^{\infty} g(u\|h(\cdot,\cdot)\|_2)\, e^{-u^2/2}\, du$$

where the existence of either integral implies the existence of the other and their equality.

Our next lemma gives a relationship between certain P.W.Z. integrals.

LEMMA 3.3. *For each x in $C_1[0,\tau]$ and s-a.e. f in $C_2(Q)$*

$$\int_Q x(s)\, \tilde{d}_2 f(s,t) = \int_0^\tau x(s)\, \tilde{d}f(s,\tau). \tag{3.4}$$

Proof. We know that for each x in $C_1[0,\tau]$, $\int_Q x(s)\, \tilde{d}_2 f(s,t)$ exists for s-a.e. f in $C_2(Q)$. We also know that $\int_0^\tau x(s)\, \tilde{d}y(s)$ exists for s-a.e. y in $C_1[0,\tau]$. Let $A = \{y \in C_1 | \int_0^\tau x(s)\, \tilde{d}y(s)$ doesn't exist$\}$ and let $B_A = \{f(\cdot,\cdot) \in C_2(Q) | f(\cdot,\tau) \in A\}$. A is a scale-invariant null set and so, by equation (2.2), B_A is also a scale-invariant null set. Thus for each $x \in C_1$, both sides of (3.4) exist for s-a.e. $f \in C_2(Q)$.

Let $\theta_1(s) = \tau^{-1/2}$, $\theta_2(s)$, $\theta_3(s), \ldots$ be a CON set of functions of bounded variation on $[0, \tau]$. Then $\{\theta_i(s)\theta_j(t)\colon i, j = 1, 2, \ldots\}$ is a CON set of functions of bounded variation on Q. Hence for s-a.e. f in $C_2(Q)$ we have that

$$\begin{aligned}
\int_Q x(s)\,\tilde{d}_2 f(s,t) &= \lim_{N\to\infty} \int_Q \left[\sum_{i=1}^{N}\sum_{j=1}^{N} (x, \theta_i\theta_j)\theta_i(s)\theta_j(t)\right] df(s,t) \\
&= \lim_{N\to\infty} \sum_{i=1}^{N}\sum_{j=1}^{N} \left(\int_0^\tau\int_0^\tau x(s)\theta_i(s)\theta_j(t)\,dt\,ds\right) \\
&\quad \cdot \int_Q \theta_i(s)\theta_j(t)\,df(s,t) \\
&= \lim_{N\to\infty} \sum_{i=1}^{N} \left(\int_0^\tau x(s)\theta_i(s)\,ds\right)\int_Q \theta_i(s)\,df(s,t) \\
&= \lim_{N\to\infty} \sum_{i=1}^{N} (x, \theta_i)\int_0^\tau \theta_i(s)\,d[f(s,\tau) - f(s,0)] \\
&= \lim_{N\to\infty} \int_0^\tau \left(\sum_{i=1}^{N} (x, \theta_i)\theta_i(s)\right) df(s,\tau) \\
&= \int_0^\tau x(s)\,\tilde{d}f(s,\tau).
\end{aligned}$$

The following result, Theorem 3.1, is a stochastic integration by parts formula that plays the key role in the proof of our main result, Theorem 4.1. This formula involves a mix of Wiener space and Yeh-Wiener space and thus is a hybrid between the stochastic integration by parts formula (2.1) and a stochastic integration by parts formula recently discovered by the last two authors while working on this paper. This formula involves Yeh-Wiener space in both the integrator and the integrand. This formula seems quite likely to be of interest, but it turned out not to be needed in this paper and so it will not be included here.

THEOREM 3.1. *Assume that h is of bounded variation on Q. Then for $m_1 \times m_2$-a.e.(x, f) in $C_1[0,\tau] \times C_2(Q)$*

$$\int_Q h(s_1, s_2)x(s_1)\,\tilde{d}_2 f(s_1, s_2) \tag{3.5}$$

$$= \int_0^\tau \left(\int_{[s_1,\tau]\times[0,\tau]} h(t_1, t_2)\,df(t_1, t_2)\right) \tilde{d}x(s_1)$$

and

$$\int_Q h(s_1,s_2)x(s_2)\,\tilde{d}_2 f(s_1,s_2) = \int_0^\tau \left(\int_{[0,\tau]\times[s_2,\tau]} h(t_1,t_2)\,df(t_1,t_2) \right) \tilde{d}x(s_2). \tag{3.6}$$

Proof. We will establish equation (3.5) by considering three cases. Equation (3.6) follows similarly since it is really the same formula with the variables interchanged.

Case 1. $h(s_1,s_2) \equiv K$ on Q. In this case we will first note that for each f in $C_2(Q)$ and s-a.e. x in $C_1[0,\tau]$

$$\begin{aligned} \int_0^\tau &\left(\int_{[s,\tau]\times[0,\tau]} h(t_1,t_2)\,df(t_1,t_2) \right) \tilde{d}x(s) \\ &= K\int_0^\tau \left(\int_{[s,\tau]\times[0,\tau]} df(t_1,t_2) \right) \tilde{d}x(s) \\ &\doteq K\int_0^\tau [f(\tau,\tau) - f(s,\tau)]\,\tilde{d}x(s) \\ &= K\left[f(\tau,\tau)x(\tau) - \int_0^\tau f(s,\tau)\,\tilde{d}x(s) \right]. \end{aligned} \tag{3.7}$$

Next using Lemmas 2.1 and 2.2 it follows that for $m_1 \times m_2$-a.e. $(x,f) \in C_1[0,\tau] \times C_2(Q)$

$$\int_0^\tau x(s)\,\tilde{d}f(s,\tau) = f(\tau,\tau)x(\tau) - \int_0^\tau f(s,\tau)\,\tilde{d}x(s). \tag{3.8}$$

Equation (3.5), for Case 1, now easily follows using equations (3.7), (3.8) and (3.4).

Case 2. *$h(s_1,s_2)$ is a step-function of bounded variation on Q*. (For example, h is a Haar function or a finite linear combination of Haar functions.).

Let $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n = \tau$ and $0 = \beta_0 < \beta_1 < \cdots < \beta_m = \tau$ be a partition of Q and assume that $h(s_1,s_2)$ is equal to K_{ij} on the rectangle $Q_{ij} \equiv (\alpha_{i-1},\alpha_i) \times (\beta_{j-1},\beta_j)$. As we proceed it is helpful to note that for each $s \in [0,\tau]$, the integral

$$\int_{[s,\tau]\times[0,\tau]} h(s_1,s_2)\,df(s_1,s_2) \tag{3.9}$$

doesn't depend upon the values of h on the edges of the rectangles Q_{ij} since in forming any Riemann-Stieltjes sum for the integral we can always select points (s_1, s_2) that aren't on the edges of the rectangles Q_{ij}. On the other hand, if we integrate the integral in (3.9) by parts we obtain the formula

$$\begin{aligned}(3.10)\quad \int_{[s,\tau]\times[0,\tau]} h(s_1,s_2)\,df(s_1,s_2) &= f(\tau,\tau)h(\tau,\tau) - f(s,\tau)h(s,\tau)\\ &\quad - \int_0^\tau f(s_1,\tau)\,dh(s_1,\tau) - \int_0^\tau f(\tau,s_2)\,dh(\tau,s_2)\\ &\quad + \int_0^\tau f(s,s_2)\,dh(s,s_2) + \int_{[s,\tau]\times[0,\tau]} f(s_1,s_2)\,dh(s_1,s_2).\end{aligned}$$

While each of the terms on the right hand side of (3.10) may depend upon the values of h on the edges of the rectangles Q_{ij}, the right hand side as a whole doe not.

The rest of the proof of Case 2 is essentially combinatorial. We substitute for $h(t_1, t_2)$ on the right hand side of (3.5), use ideas from the proofs of Lemma 3.3 and Case 1 above, simplify and obtain the left hand side of (3.5). Recall that for each $x \in C_1[0,\tau]$, the left hand side of (3.5) exists for s-a.e. $f \in C_2(Q)$ while for each $f \in C_2(Q)$ the right hand side of (3.5) exists for s-a.e. $x \in C_1[0,\tau]$. Thus for $m_1 \times m_2$-a.e.$(x, f) \in C_1[0,\tau] \times C_2(Q)$

$$\begin{aligned}&\int_0^\tau \left(\int_{[s_1,\tau]\times[0,\tau]} h(t_1,t_2)\,df(t_1,t_2) \right) \tilde{d}x(s_1)\\ &= \sum_{i=1}^n \int_{\alpha_{i-1}}^{\alpha_i} \left(\sum_{j=1}^m \int_{[s_1,\tau]\times[\beta_{j-1},\beta_j]} h(t_1,t_2)\,df(t_1,t_2) \right) \tilde{d}x(s_1)\\ &= \sum_{i=1}^n \sum_{j=1}^m \int_{\alpha_{i-1}}^{\alpha_i} \Big\{ K_{ij}\big[f(\alpha_i,\beta_j) - f(s_1,\beta_j) - f(\alpha_i,\beta_{j-1})\\ &\qquad\qquad + f(s_1,\beta_{j-1})\big]\\ &\qquad + \sum_{l=i+1}^n K_{lj}\big[f(\alpha_l,\beta_j) - f(\alpha_{l-1},\beta_j) - f(\alpha_l,\beta_{j-1})\\ &\qquad\qquad + f(\alpha_{l-1},\beta_{j-1})\big]\Big\} \tilde{d}x(s_1)\\ &= \sum_{i=1}^n \sum_{j=1}^m K_{ij} \Big\{ \int_{\alpha_{i-1}}^{\alpha_i} x(s_1)\,\tilde{d}\big[f(s_1,\beta_j) - f(s_1,\beta_{j-1})\big]\\ &\qquad - x(\alpha_{i-1})\big[f(\alpha_i,\beta_j) - f(\alpha_{i-1},\beta_j) - f(\alpha_i,\beta_{j-1})\\ &\qquad\qquad + f(\alpha_{i-1},\beta_{j-1})\big]\Big\}\end{aligned}$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{l=i+1}^{n} K_{lj}\big[f(\alpha_l,\beta_j)-f(\alpha_{l-1},\beta_j)-f(\alpha_l,\beta_{j-1})+f(\alpha_{l-1},\beta_{j-1})\big]\cdot\big[x(\alpha_i)-x(\alpha_{i-1})\big]$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{m} K_{ij}\int_{Q_{ij}} x(s_1)\,\tilde{d}_2 f(s_1,s_2)$$

$$-\sum_{i=1}^{n}\sum_{j=1}^{m} K_{ij}x(\alpha_{i-1})\big[f(\alpha_i,\beta_j)-f(\alpha_{i-1},\beta_j)-f(\alpha_i,\beta_{j-1})+f(\alpha_{i-1},\beta_{j-1})\big]$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{l=i+1}^{n} K_{lj}\big[f(\alpha_l,\beta_j)-f(\alpha_{l-1},\beta_j)-f(\alpha_l,\beta_{j-1})+f(\alpha_{l-1},\beta_{j-1})\big]\cdot\big[x(\alpha_i)-x(\alpha_{i-1})\big]$$

$$=\int_Q h(s_1,s_2)x(s_1)\,\tilde{d}_2 f(s_1,s_2)$$

$$-\sum_{j=1}^{m}\sum_{i=2}^{n} K_{ij}x(\alpha_{i-1})\big[f(\alpha_i,\beta_j)-f(\alpha_{i-1},\beta_j)-f(\alpha_i,\beta_{j-1})+f(\alpha_{i-1},\beta_{j-1})\big]$$

$$+\sum_{j=1}^{m}\sum_{l=2}^{n}\sum_{i=1}^{l-1} K_{lj}\big[f(\alpha_l,\beta_j)-f(\alpha_{l-1},\beta_j)-f(\alpha_l,\beta_{j-1})+f(\alpha_{l-1},\beta_{j-1})\big]\cdot\big[x(\alpha_i)-x(\alpha_{i-1})\big]$$

$$=\int_Q h(s_1,s_2)x(s_1)\,\tilde{d}_2 f(s_1,s_2)$$

since $\sum_{i=1}^{l-1}[x(\alpha_i)-x(\alpha_{i-1})]=x(\alpha_{l-1})$. Thus equation (3.5) is established for step-functions h of bounded variation on Q.

Case 3. *General case*: *h is a function of bounded variation on Q*. In this case, h is certainly in $L_2(Q)$ and so we can find a sequence of stepfunctions $\{h_n\}$ each of bounded variation on Q such that $\|h_n-h\|_2\to 0$ as $n\to\infty$ (For example let $\{\phi_j\}$ be the Haar functions on Q and let h_n be given by (3.2)). Next let

$$I \equiv \int_{C_2(Q)} \int_{C_1[0,\tau]} \left| \int_0^\tau \left(\int_{[s_1,\tau]\times[0,\tau]} h(t_1,t_2)\, df(t_1,t_2) \right) \tilde{d}x(s_1) - \int_Q h(s_1,s_2) x(s_1) \tilde{d}_2 f(s_1,s_2) \right| dm_1(x)\, dm_2(f).$$

To establish equation (3.5) we need only show that $I = 0$. First note, using case 2 above, that for any integer n

$$\begin{aligned}
I &= \int_{C_2(Q)} \int_{C_1[0,\tau]} \left| \int_0^\tau \left(\int_{[s_1,\tau]\times[0,\tau]} [h(t_1,t_2) - h_n(t_1,t_2)]\, df(t_1,t_2) \right) \tilde{d}x(s_1) \right. \\
&\qquad \left. - \int_Q [h(s_1,s_2) - h_n(s_1,s_2)]\, x(s_1)\, \tilde{d}_2 f(s_1,s_2) \right| dm_1(x)\, dm_2(f) \\
&\le \int_{C_2(Q)} \int_{C_1[0,\tau]} \left| \int_0^\tau \left(\int_{[s_1,\tau]\times[0,\tau]} [h(t_1,t_2) - h_n(t_1,t_2)] \right.\right. \\
&\qquad \left.\left. df(t_1,t_2) \right) \tilde{d}x(s_1) \right| dm_1(x)\, dm_2(f) \\
&\quad + \int_{C_2(Q)} \int_{C_1[0,\tau]} \left| \int_Q [h(s_1,s_2) - h_n(s_1,s_2)] \right. \\
&\qquad \left. \cdot\, x(s_1)\, \tilde{d}_2 f(s_1,s_2) \right| dm_1(x)\, dm_2(f).
\end{aligned}$$

Next we obtain bounds for the two terms on the right hand side of the above inequality. First we will work with the second term. Using the Fubini Theorem and Lemma 3.2 we see that

$$\begin{aligned}
&\int_{C_2(Q)} \int_{C_1[0,\tau]} \left| \int_Q [h(s_1,s_2) - h_n(s_1,s_2)] \right. \\
&\qquad \left. \cdot x(s_1)\, \tilde{d}_2 f(s_1,s_2) \right| dm_1(x)\, dm_2(f) \\
&= \int_{C_1[0,\tau]} \int_{C_2(Q)} \left| \int_Q [h(s_1,s_2) - h_n(s_1,s_2)] \right. \\
&\qquad \left. \cdot x(s_1)\, \tilde{d}_2 f(s_1,s_2) \right| dm_2(f)\, dm_1(x) \\
&= (2\pi)^{-1/2} \int_{C_1[0,\tau]} \left\{ \| [h(\cdot,\cdot) - h_n(\cdot,\cdot)] x(\cdot) \|_2 \right. \\
&\qquad \left. \int_{-\infty}^{\infty} |u| \exp\left\{ -\frac{u^2}{2} \right\} du \right\} dm_1(x)
\end{aligned}$$

$$= \left(\frac{2}{\pi}\right)^{1/2} \int_{C_1[0,\tau]} \| [h(\cdot,\cdot) - h_n(\cdot,\cdot)] x(\cdot) \|_2 \, dm_1(x)$$

$$\leq \left(\frac{2}{\pi}\right)^{1/2} \| h - h_n \|_2 \int_{C_1[0,\tau]} \| x \|_\infty \, dm_1(x) \leq \frac{4\sqrt{\tau}}{\pi} \| h - h_n \|_2$$

since $\|x\|_\infty \in L_1(C_1, m_1)$ and

$$\int_{C_1[0,\tau]} \| x \|_\infty \, dm_1(x) \leq 4(2\pi\tau)^{-1/2} \int_0^\infty u \exp\left\{-\frac{u^2}{2\tau}\right\} du = \left(\frac{8\tau}{\pi}\right)^{1/2}.$$

Again, using the Fubini Theorem, a Wiener integration formula corresponding to Lemma 3.2, and Lemma 3.2 we see that

$$\int_{C_2(Q)} \int_{C_1[0,\tau]} \left| \int_0^\tau \left(\int_{[s_1,\tau]\times[0,\tau]} [h(t_1,t_2) - h_n(t_1,t_2)] \, df(t_1,t_2) \right) \tilde{d}x(s_1) \right| \, dm_1(x) \, dm_2(f)$$

$$= (2\pi)^{-1/2} \int_{C_2(Q)} \left\| \int_{[\cdot,\tau]\times[0,\tau]} [h(t_1,t_2) - h_n(t_1,t_2)] \, df(t_1,t_2) \right\|_2 \cdot \int_{-\infty}^\infty |u| e^{-u^2/2} \, du \, dm_2(f)$$

$$= \left(\frac{2}{\pi}\right)^{1/2} \int_{C_2(Q)} \left\{ \int_0^\tau \left(\int_{[s,\tau]\times[0,\tau]} [h(t_1,t_2) - h_n(t_1,t_2)] \, df(t_1,t_2) \right)^2 ds \right\}^{1/2} dm_2(f)$$

$$\leq \left(\frac{2}{\pi}\right)^{1/2} \left\{ \int_{C_2(Q)} \int_0^\tau \left(\int_{[s,\tau]\times[0,\tau]} [h(t_1,t_2) - h_n(t_1,t_2)] \, df(t_1,t_2) \right)^2 ds \, dm_2(f) \right\}^{1/2}$$

$$= \left(\frac{2}{\pi}\right)^{1/2} \left\{ \int_0^\tau (2\pi)^{-1/2} \int_{-\infty}^\infty u^2 \left(\int_s^\tau \int_0^\tau [h(t_1,t_2) - h_n(t_1,t_2)]^2 \, dt_2 \, dt_1 \right) e^{-u^2/2} \, du \, ds \right\}^{1/2}$$

$$= \left(\frac{2}{\pi}\right)^{1/2} \left\{ \int_0^\tau \left(\int_s^\tau \int_0^\tau [h(t_1,t_2) - h_n(t_1,t_2)]^2 \, dt_2 \, dt_1 \right) ds \right\}^{1/2}$$

$$\leq \left(\frac{2}{\pi}\right)^{1/2} \| h - h_n \|_2 \left\{ \int_0^\tau ds \right\}^{1/2} = \left(\frac{2\tau}{\pi}\right)^{1/2} \| h - h_n \|_2.$$

Combining the two estimates above we see that for all n

$$I \le \left(\frac{2\tau}{\pi}\right)^{1/2} \|h - h_n\|_2 + \frac{4\sqrt{\tau}}{\pi} \|h - h_n\|_2,$$

and so letting $n \to \infty$ we obtain $I = 0$ and hence equation (3.5) is established.

4. The main result. In this section we first develop the main result, Theorem 4.1, and then we proceed to establish several corollaries.

THEOREM 4.1. *Assume that for s-a.e. x in $C_1[0, \tau]$*

$$\text{(4.1)} \quad F(x) = \exp\left\{-\int_0^\tau \int_0^\tau \langle A(s_1, s_2)(x(s_1), x(s_2)), (x(s_1), x(s_2))\rangle \, ds_1 \, ds_2\right\}$$

where $\{A(s_1, s_2) = (a_{ij}(s_1, s_2)) | i, j = 1, 2, (s_1, s_2) \in Q\}$ is a commutative family of 2 by 2 real, symmetric, nonnegative definite matrices such that the (nonnegative) eigenvalues $p_1(s_1, s_2)$ and $p_2(s_1, s_2)$ have square roots which are of bounded variation on Q. Then F is in the Banach algebra $\mathcal{S}$ (and so possesses an analytic Feynman integral for all values of the parameter q).

Proof. Let

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

be an orthogonal matrix such that

$$BA(s_1, s_2)B^{-1} = P(s_1, s_2) = \begin{bmatrix} p_1(s_1, s_2) & 0 \\ 0 & p_2(s_1, s_2) \end{bmatrix}$$

throughout Q. Let $\rho > 0$ be given. Then for a.e. x in $C_1[0, \tau]$ we have that

$$\begin{aligned}
\text{(4.2)} \quad F(\rho x) &= \exp\left\{-\rho^2 \int_0^\tau \int_0^\tau \langle A(s_1, s_2)(x(s_1), x(s_2)), \right. \\
&\qquad \left. (x(s_1), x(s_2))\rangle \, ds_1 \, ds_2\right\} \\
&= \exp\left\{-\rho^2 \int_0^\tau \int_0^\tau \langle B^{-1}P(s_1, s_2)B(x(s_1), x(s_2)), \right. \\
&\qquad \left. (x(s_1), x(s_2))\rangle \, ds_1 \, ds_2\right\} \\
&= \exp\left\{-\rho^2 \int_0^\tau \int_0^\tau \langle P(s_1, s_2)B(x(s_1), x(s_2)), \right. \\
&\qquad \left. B(x(s_1), x(s_2))\rangle \, ds_1 \, ds_2\right\}
\end{aligned}$$

$$= \exp\left\{-\rho^2 \int_0^\tau \int_0^\tau \sum_{j=1}^{2} p_j(s_1, s_2) \left[\sum_{k=1}^{2} b_{jk} x(s_k)\right]^2 ds_1\, ds_2\right\}$$

$$= \prod_{j=1}^{2} \exp\left\{-\frac{1}{2} \int_0^\tau \int_0^\tau 2\rho^2 \left[\sum_{k=1}^{2} b_{jk} \sqrt{p_j(s_1, s_2)}\, x(s_k)\right]^2 ds_1\, ds_2\right\}$$

$$= \prod_{j=1}^{2} (2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp\left\{ i\rho\sqrt{2}\, u \cdot \left(\int_0^\tau \int_0^\tau \left[\sum_{k=1}^{2} b_{jk} \sqrt{p_j(s_1, s_2)}\, x(s_k)\right]^2 ds_1\, ds_2 \right)^{1/2} \right\} \cdot \exp\left\{-\frac{u^2}{2}\right\} du$$

where in the last equality above we used the Fourier transform formula

$$(2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp\left\{ iuv - \frac{u^2}{2} \right\} du = \exp\left\{ \frac{-v^2}{2} \right\}$$

with

$$v^2 \equiv \int_0^\tau \int_0^\tau 2\rho^2 \left[\sum_{k=1}^{2} b_{jk} \sqrt{p_j(s_1, s_2)}\, x(s_k)\right]^2 ds_1\, ds_2.$$

Next using Lemma 3.2, Theorem 3.1, linearity, and the equation (4.2) we see that for a.e. x in $C_1[0, \tau]$,

(4.3) $F(\rho x)$

$$= \prod_{j=1}^{2} \int_{C_2(Q)} \exp\left\{ i\rho\sqrt{2} \int_Q \left[\sum_{k=1}^{2} b_{jk} \sqrt{p_j(s_1, s_2)}\, x(s_k)\right] \tilde{d}_2 f_j(s_1, s_2) \right\} dm_2(f_j)$$

$$= \int_{C_2(Q)} \int_{C_2(Q)} \exp\left\{ i\rho\sqrt{2} \int_Q \left[b_{11}\sqrt{p_1(s_1, s_2)}\, x(s_1) + b_{12}\sqrt{p_1(s_1, s_2)}\, x(s_2) \right] \tilde{d}_2 f_1(s_1, s_2) \right\}$$

$$\cdot \exp\left\{ i\rho\sqrt{2} \int_Q \left[b_{21}\sqrt{p_2(s_1, s_2)}\, x(s_1) + b_{22}\sqrt{p_2(s_1, s_2)}\, x(s_2) \right] \tilde{d}_2 f_2(s_1, s_2) \right\} dm_2(f_1)\, dm_2(f_2)$$

(continues)

(continued)

$$= \int_{C_2(Q)} \int_{C_2(Q)} \exp\left\{ i\rho\sqrt{2} \int_0^\tau \left(\int_{[s_1,\tau]\times[0,\tau]} \left[b_{11}\sqrt{p_1(t_1,t_2)}\, df_1(t_1,t_2) + b_{21}\sqrt{p_2(t_1,t_2)}\, df_2(t_1,t_2) \right] \right) \tilde{d}x(s_1) \right\}$$

$$\cdot \exp\left\{ i\rho\sqrt{2} \int_0^\tau \left(\int_{[0,\tau]\times[s_2,\tau]} \left[b_{12}\sqrt{p_1(t_1,t_2)}\, df_1(t_1,t_2) + b_{22}\sqrt{p_2(t_1,t_2)}\, df_2(t_1,t_2) \right] \right) \tilde{d}x(s_2) \right\} dm_2(f_1)\, dm_2(f_2).$$

Now let $T\colon C_2(Q) \times C_2(Q) \to L_2[0,\tau]$ be defined by the formula

$$T(f_1,f_2)(s) \equiv \sqrt{2} \int_{[s,\tau]\times[0,\tau]} \left[b_{11}\sqrt{p_1(t_1,t_2)}\, df_1(t_1,t_2) + b_{21}\sqrt{p_2(t_1,t_2)}\, df_2(t_1,t_2) \right]$$
$$+ \sqrt{2} \int_{[0,\tau]\times[s,\tau]} \left[b_{12}\sqrt{p_1(t_1,t_2)}\, df_1(t_1,t_2) + b_{22}\sqrt{p_2(t_1,t_2)}\, df_2(t_1,t_2) \right].$$

Applying the integration by parts formula [**30**, Theorem 4] to each term in the definition of $T(f_1,f_2)$ and looking at each term in the resulting expression separately, one sees without too much difficulty that $T(f_1,f_2)(\cdot)$ is in $L_2[0,\tau]$ and that T is a continuous linear operator from $C_2(Q) \times C_2(Q)$ to $L_2[0,\tau]$. Furthermore, substituting $T(f_1,f_2)$ into the last expression in equation (4.3) above, we obtain that for a.e. x in $C_1[0,\tau]$

(4.4)
$$F(\rho x) = \int_{C_2(Q)} \int_{C_2(Q)} \exp\left\{ i\rho \int_0^\tau T(f_1,f_2)(s)\, \tilde{d}x(s) \right\} dm_2(f_1)\, dm_2(f_2).$$

Finally, by the change of variables theorem [**15**, p. 163], we see that for a.e. $x \in C_1[0,\tau]$

$$F(\rho x) = \int_{L_2[0,\tau]} \exp\left\{ i\rho \int_0^\tau g(s)\, \tilde{d}x(s) \right\} d\sigma(g)$$

where $\sigma \equiv (m_2 \times m_2) \circ T^{-1}$ is an element of $M = M(L_2[0, \tau])$. Thus

$$F(x) = \int_{L_2[0,\tau]} \exp\left\{ i \int_0^\tau g(s)\, \tilde{d}x(s) \right\} d\sigma(g) \tag{4.5}$$

for s-a.e. $x \in C_1[0, \tau]$ and so F is an element of the Banach algebra $\mathcal{S}$.

Our first corollary establishes a slight generalization of the relatively simple single Wiener variable case of our earlier paper [**21**, p. 281].

COROLLARY 4.1. *Let* $F(x) = \exp\{-\int_0^\tau a(s)x^2(s)\,ds\}$ *for s-a.e.* x *in* $C_1[0, \tau]$ *where* $a(s)$ *is a nonnegative function on* $[0, \tau]$ *such that* $\sqrt{a(s)}$ *is of bounded variation. Then* F *is in* $\mathcal{S}$ *(and so possesses an analytic Feynman integral for all values of the parameter* q*).*

Proof. Let

$$A(s_1, s_2) = \frac{1}{\tau}\begin{bmatrix} a(s_1) & 0 \\ 0 & 0 \end{bmatrix}.$$

The family $\{A(s_1, s_2)\}$ satisfies the hypotheses of Theorem 4.1. Hence G is in $\mathcal{S}$ where

$$\begin{aligned} G(x) &= \exp\left\{ -\int_0^\tau \int_0^\tau \langle A(s_1, s_2)(x(s_1), x(s_2)), (x(s_1), x(s_2))\rangle\, ds_1\, ds_2 \right\} \\ &= \exp\left\{ -\int_0^\tau \int_0^\tau \frac{1}{\tau} a(s_1)x^2(s_1)\, ds_1\, ds_2 \right\} = \exp\left\{ -\int_0^\tau a(s_1)x^2(s_1)\, ds_1 \right\}. \end{aligned}$$

In our next corollary, we put the hypotheses on the functions a_{ij} rather than on the eigenvalues p_1 and p_2. This allows one to see that certain F's of the form (4.1) are in $\mathcal{S}$ without computing the eigenvalues.

COROLLARY 4.2. *Let the matrices* $\{A(s_1, s_2) = (a_{ij}(s_1, s_2)) \mid i, j = 1, 2\}$ *be a commutative family of 2 by 2 real, symmetric, positive definite matrices. Suppose that the functions* a_{ij} *are continuous on* Q *and have first partial derivatives which are absolutely continuous in* s_1 *for each* s_2 *and are absolutely continuous in* s_2 *for each* s_1 *and that the functions* $\partial^2 a_{ij}/\partial s_2 \partial s_1$ *are integrable on* Q. *Then the eigenvalues* $p_1(s_1, s_2)$ *and* $p_2(s_1, s_2)$ *have square roots which are of bounded variation on* Q *and so the conclusions of Theorem* 4.1 *hold.*

Proof. Since

$$BA(s_1, s_2)B^{-1} = \begin{bmatrix} p_1(s_1, s_2) & 0 \\ 0 & p_2(s_1, s_2) \end{bmatrix}$$

we see that

$$p_j(s_1, s_2) = b_{j1}^2 a_{11}(s_1, s_2) + 2b_{j1}b_{j2}a_{12}(s_1, s_2) + b_{j2}^2 a_{22}(s_1, s_2) \quad \text{for } j = 1, 2.$$

Now the p_j's are continuous and positive on Q and so there exists $\delta > 0$ such that $p_j(s_1, s_2) \geq \delta > 0$ on Q for $j = 1, 2$. Also the functions

$$\frac{\partial p_j(s_1, s_2)}{\partial s_1}, \quad \frac{\partial p_j(s_1, s_2)}{\partial s_2} \quad \text{and} \quad \frac{\partial^2 p_j(s_1, s_2)}{\partial s_2 \partial s_1}$$

satisfy the appropriate conditions in Proposition 3.1 and so $[p_j(s_1, s_2\ ]^{1/2}$ is of bounded variation on Q for $j = 1, 2$.

COROLLARY 4.3. *Let*

$$A(s_1, s_2) \equiv A \equiv \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$$

be a real constant, symmetric, and nonnegative definite matrix. Then $p_1(s_1, s_2) = p_1$ *and* $p_2(s_1, s_2) = p_2$ *are simply nonnegative constants and so the conclusions of Theorem* 4.1 *hold.*

EXAMPLE. Let $F(x)$ be given by (4.1) where A is a real, constant, symmetric, and positive definite matrix. In this case, although the calculation is rather lengthy, we can explicitly compute the analytic Feynman integral of $F(x)$. It turns out that for all real $q \neq 0$

$$(4.6) \quad \int_{C_1[0,\tau]}^{\mathrm{an}f_q} F(x)\, dm_1(x) = \left[\frac{\alpha(a_{11} + a_{22})\operatorname{sech}\alpha}{\alpha(a_{11} + 2a_{12} + a_{22}) - 2a_{12}\tanh\alpha}\right]^{1/2}$$

where $\alpha \equiv [2\tau^3 i(a_{11} + a_{22})/q]^{1/2}$. We will simply outline the calculations.

First using (4.1) and the fact that $BAB^{-1} = P$ (and so $p_1 + p_2 = a_{11} + a_{22}$ and $a_{12} = p_1 b_{11} b_{12} + p_2 b_{21} b_{22}$) it follows that for all $\lambda > 0$

$$\int_{C_1[0,\tau]} F(\lambda^{-1/2}x)\, dm_1(x) = \int_{C_1[0,\tau]} \exp\left\{-\frac{\tau(a_{11} + a_{22})}{\lambda}\int_0^\tau x^2(s)\, ds - \frac{2a_{12}}{\lambda}\left(\int_0^\tau x(s)\, ds\right)^2\right\} dm_1(x).$$

Next letting $g(\lambda) \equiv [2\tau(a_{11} + a_{22})/\lambda]^{1/2}$ and applying an integration theorem of Cameron and Martin [5, Theorem 1a, p. 34] to the last expression above, simplifying, and finally using a well-known Wiener integration formula we obtain

$$\int_{C_1[0,\tau]} F(\lambda^{-1/2}x)\,dm_1(x)$$

$$= (\cos[\tau i g(\lambda)])^{-1/2}$$

$$\cdot \int_{C_1[0,\tau]} \exp\left\{-\frac{2a_{12}}{\lambda}\left[\int_0^\tau (\cos[ig(\lambda)(\tau-s)])\right.\right.$$

$$\left.\left.\cdot\left(\int_0^s \sec[ig(\lambda)(\tau-t)]\,dy(t)\right)ds\right]^2\right\} dm_1(y)$$

$$= (\operatorname{sech}[\tau g(\lambda)])^{1/2}$$

$$\cdot \int_{C_1[0,\tau]} \exp\left\{-\frac{a_{12}}{\tau(a_{11}+a_{22})}\left[\int_0^\tau \tanh[g(\lambda)(t-\tau)]\,dy(t)\right]^2\right\} dm_1(y)$$

$$= \left(\frac{\operatorname{sech}[\tau g(\lambda)]}{2\pi}\right)^{1/2}$$

$$\cdot \int_{-\infty}^{\infty} \exp\left\{-\frac{u^2}{2}\left[1+\frac{2a_{12}}{a_{11}+a_{22}}\left(1-\frac{\tanh[\tau g(\lambda)]}{\tau g(\lambda)}\right)\right]\right\} du$$

$$= (\operatorname{sech}[\tau g(\lambda)])^{1/2}\left[1+\frac{2a_{12}}{a_{11}+a_{22}}\left(1-\frac{\tanh[\tau g(\lambda)]}{\tau g(\lambda)}\right)\right]^{-1/2}$$

for all $\lambda > 0$.

Next we will show that the last expression above is an analytic function of λ for λ in $C^+ = \{\lambda | \operatorname{Re}\lambda > 0\}$. First we note that for $\lambda \in C^+$, $|\arg(\tau g(\lambda))| < \pi/4$ where we choose the branch of $g(\lambda)$ that is positive for positive λ. Next we note that the function sech z is analytic (and of course doesn't vanish) in the open domain $\mathbf{C} \setminus D$ where

$$D = \left\{z = x + yi | x = 0 \quad \text{and} \quad y \in \left(-\infty, \frac{-\pi}{2}\right] \cup \left[\frac{\pi}{2}, \infty\right)\right\}.$$

Hence by [**26**, Theorem 3.1 on page 180], $(\operatorname{sech} z)^{1/2}$ has an analytic branch in $\mathbf{C} \setminus D$; we select the branch that is positive for real z. Thus $(\operatorname{sech}[\tau g(\lambda)])^{1/2}$ is certainly analytic in λ for $\lambda \in C^+$.

For $z = x + yi$ with $x > 0$ and $-x < y < x$ one can show that

$$0 < \operatorname{Re}\left(\frac{\tanh z}{z}\right) \le 2.$$

Also $2|a_{12}| < a_{11} + a_{22}$ since the matrix A is positive definite. Hence for all $\lambda \in C^+$

$$0 < 1 + \frac{2a_{12}}{a_{11} + a_{22}}\left[1 - \mathrm{Re}\left(\frac{\tanh[\tau g(\lambda)]}{\tau g(\lambda)}\right)\right]$$
$$= \mathrm{Re}\left(1 + \frac{2a_{12}}{a_{11} + a_{22}}\left[1 - \frac{\tanh[\tau g(\lambda)]}{\tau g(\lambda)}\right]\right) < 2$$

and so

$$\left(1 + \frac{2a_{12}}{a_{11} + a_{22}}\left[1 - \frac{\tanh[\tau g(\lambda)]}{\tau g(\lambda)}\right]\right)^{-1/2}$$

is an analytic function of λ in C^+.

Now by analytically continuing to C^+ and then taking the limit as $\lambda \to -iq$, we obtain equation (4.6).

COROLLARY 4.4. *Let $A(s_1, s_2) \equiv g(s_1, s_2)A$ where the matrix A is as in Corollary 4.3 and $g(s_1, s_2)$ is a nonnegative function whose square root is of bounded variation on Q. Then the conclusions of Theorem 4.1 hold.*

Proof. Clearly $A(s_1, s_2)$ has the required matrix properties. Furthermore

$$p_1(s_1, s_2) = g(s_1, s_2)\left[b_{11}^2 a_{11} + 2b_{11}b_{12}a_{12} + b_{12}^2 a_{22}\right]$$

and

$$p_2(s_1, s_2) = g(s_1, s_2)\left[b_{21}^2 a_{11} + 2b_{21}b_{22}a_{12} + b_{22}^2 a_{22}\right]$$

have square roots which are of bounded variation on Q.

Our next corollary depends upon a result from **[12]** as well as Theorem 4.1 above. It shows that a broad class of functions G belongs to the Banach algebra $\mathcal{S}$.

COROLLARY 4.5. *Let $\{A(s_1, s_2)\}$ be as in Theorem 4.1. Let η be a Borel measure on Q. Let θ: $Q \times \mathbf{R}^2 \to \mathbf{C}$ be such that for all $(s_1, s_2) \in Q$,*

$$\theta(s_1, s_2; u_1, u_2) = \int_{\mathbf{R}^2} \exp\{iu_1 v_1 + iu_2 v_2\}\, d\sigma_{s_1,s_2}(v_1, v_2)$$

where

(i) $\sigma_{s_1,s_2} \in M(\mathbf{R}^2)$,

(ii) *for all $B \in \mathcal{B}(\mathbf{R}^2)$, $\sigma_{s_1,s_2}(B)$ is a Borel measurable function of (s_1, s_2), and*

(iii) $\|\sigma_{s_1,s_2}\| \in L_1(Q, \mathcal{B}(Q), \eta)$.

Then the function

$$G(x) = \exp\left\{-\int_0^T\int_0^T \langle A(s_1,s_2)(x(s_1),x(s_2)),(x(s_1),x(s_2))\rangle\, ds_1\, ds_2 + \int_0^T\int_0^T \theta(s_1,s_2;x(s_1),x(s_2))\, d\eta(s_1,s_2)\right\}$$

belongs to $\mathscr{S}$.

Proof. Let

$$G_1(x) = \exp\left\{\int_0^T\int_0^T \theta(s_1,s_2;x(s_1),x(s_2))\, d\eta(s_1,s_2)\right\}.$$

Then $G(x) = G_1(x)F(x)$ where $F(x)$ is given by equation (4.1). By a result in **[12]**, G_1 belongs to $\mathscr{S}$ and by Theorem 4.1 above, F belongs to $\mathscr{S}$. Since $\mathscr{S}$ is a Banach algebra we have that G is an element of $\mathscr{S}$.

REMARK 4.1. The most general functions shown to be in $\mathscr{S}$ in **[21]** are discussed in Corollary 4 of that paper. It can be shown without too much difficulty that the single Wiener variable case of that earlier result can be obtained as a corollary to Corollary 4.5 above.

Next we wish to establish the Fresnel integrability of certain functions. Recall that we briefly described Albeverio and Høegh-Krohn's space $\mathscr{F}(H)$ of Fresnel integrable functions in §1 above. Using Theorem 4.1 and ideas from **[17]**, especially page 2093, we obtain Corollary 4.6 below. The essential idea is that equation(4.5) holds for each $x \in C_1[0,\tau]$ which is absolutely continuous and whose derivative is in $L_2[0,\tau]$ since for each such x, equation (3.5) holds for s-a.e. f in $C_2(Q)$.

COROLLARY 4.6. *For each* γ *in* H *let*

$$G(\gamma) = \exp\left\{-\int_0^T\int_0^T < A(s_1,s_2)(\gamma(s_1)-\gamma(0),\gamma(s_2)-\gamma(0)), (\gamma(s_1)-\gamma(0),\gamma(s_2)-\gamma(0)) > ds_1\, ds_2\right\} \tag{4.7}$$

where $\{A(s_1,s_2)\}$ *is as in Theorem* 4.1. *Then* G *is in* $\mathscr{F}(H)$; *that is to say,* G *is Fresnel integrable.*

Using Corollary 4.6, a theorem from **[12]**, and the fact that $\mathscr{F}(H)$ is a Banach algebra we obtain our final corollary.

COROLLARY 4.7. *Let* $\{A(s_1,s_2)\}$, η *and* θ *be as in Corollary* 4.5. *For* γ *in* H *let*

$$G_1(\gamma) = \exp\left\{\int_0^T\int_0^T \theta(s_1,s_2;\gamma(s_1)-\gamma(0),\gamma(s_2)-\gamma(0))\, d\eta(s_1,s_2)\right\}.$$

Then the functions $G_1(\gamma)$ *and* $G_1(\gamma)G(\gamma)$, *where* $G(\gamma)$ *is given by equation* (4.7), *belong to the Banach algebra* $\mathcal{F}(H)$.

REMARK 4.2. Using extensions of the techniques developed in this paper we conjecture that n-dimensional versions of Theorem 4.1 and its various corollaries could be established. That is to say, for appropriate hypotheses on the matrix $A(s_1, s_2, \ldots, s_n)$, the function F: $C_1[0, \tau] \to \mathbf{R}$ defined by the formula

$$F(x) \equiv \exp\left\{-\int_0^\tau \cdots (n) \cdots \int_0^\tau \langle A(s_1, \ldots, s_n)(x(s_1), \ldots, x(s_n)), \right.$$
$$\left. (x(s_1), \ldots, x(s_n))\rangle \, ds_1 \cdots ds_n \right\}$$

belongs to $\mathcal{S}$. We would expect to require that the eigenvalues $p_j(s_1, \ldots, s_n)$ have square roots that are of bounded variation on the n-dimensional rectangle $[0, \tau]^n$.

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