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POSITIVE DEFINITE FUNCTIONS AND L^p CONVOLUTION OPERATORS ON AMALGAMS

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Let K_i be a countable collection of compact groups, and assume that $H = \bigcap_i K_i$ is an open subgroup of K_i for every i. In this paper we consider positive definite functions and convolution operators on the amalgamated product $G = *_H K_i$, and we study their properties in relation with the notion of length of reduced words. In particular, if $\sup_i k_i < \infty$, we show that there exist unbounded approximate identities in $A(G)$, that the space of bounded convolution operators on $L_p(G)$ is the dual space of the algebra $A_p(G)$, and, under the additional assumption that H be finite, that there exist unbounded approximate identities in $A(G)$.

1. Introduction. Considerable attention has been devoted, in the recent literature, to positive definite functions on groups acting isometrically on homogeneous trees. The Fourier-Stieltjes algebra of the free group $\mathbf{F}_r$ with r generators, which acts isometrically on the homogeneous tree of degree $2r$, has been studied in detail in [**5, 14, 9, 6, 1, 13**]. Other free products have been considered in [**16, 17, 4**]. The class of groups acting simply transitively on a homogeneous tree has been considered in [**3**]. Every locally compact group G acting isometrically on a homogeneous or semihomogeneous tree T is isomorphic to the amalgamated product $K_1 *_H K_2$, where K_1 and K_2 are the stability subgroups of two contiguous vertices and $H = K_1 \cap K_2$ is the stability subgroup of the corresponding edge [**18**]. The subgroup H is open, and its indices in K_1 and K_2 are the homogeneity degrees of T. In particular, if the homogeneity degrees are finite, G is the amalgam of two compact groups. Some properties of positive definite functions on amalgams of two factors have been studied in [**2, 11**].

In this paper we consider amalgamated products $G = *_H G_i$, where $\{G_i,\ i \in I\}$ is any collection of locally compact groups and H is a common open subgroup. These groups act isometrically on trees with periodical homogeneity degrees (and on "polygonal graphs": see [**16**]). The homogeneity degrees are finite if and only if the factors G_i are compact; they are bounded if and only if the indices k_i of H in G_i are bounded. For groups of this type, we consider several results originally obtained for free groups in [**14, 8, 13**]. Some of our arguments are adapted from these references: we refer to Chapters 2 and 8 of [13] for a detailed account on free groups.

In §2, we prove that the amalgamated product of two-sided H-invariant functions f_i which are positive definite on the subgroups G_i is a positive definite function on G. This result was originally proved in [4] for free products of two factors. As a consequence, the length of G is a continuous negative definite function: this extends a result of [2] (see also [11]). In §3, we restrict attention to trees with bounded homogeneity degrees, that is, to amalgams $G = *_H K_i$, where the indices $k_i = \#(K_i/H)$ are bounded. The result of the previous section yields a family of positive definite functions f_t on G which are radial, decay exponentially at infinity and converge to 1 uniformly on compact sets as $t \searrow 0$. By means of these functions, if H is finite we construct approximate identities for the Fourier algebra $A(G)$ which are bounded in the multiplier norm (Theorem 3). In the terminology of [14], Theorem 3 asserts that the Fourier algebra $A(G)$ has the metric approximation property; we observe that, by the same argument, the reduced C^*-algebra $C^*_\lambda(G)$ has the metric approximation property (see [14, 11]). Theorem 3 is a consequence of the following result (due to U. Haagerup [13] in the special case of free groups; see also [16]): if f is a function supported on words of length n, then the norm of f as a convolution operators on L^2 grows as $O(n)$ (Theorem 2: this result holds without the assumption that H is finite). For the special case of amalgams of two finite groups, Theorems 2 and 3 have been obtained independently in [11].

We then consider functions on G which are multiplicative with respect to the length and, for every i, are constant on $K_i - H$. Using these functions, we prove a result which indicates that $A(G)$ is "much smaller" than $B(G)$ (Theorem 4; see also [9, 1, 4]). Even though Theorem 4 emphasizes the fact that G is nonamenable, nevertheless, in §4, we prove a result which restores some similarity with amenable groups. For every commutative group G, the space Cv_p of bounded convolution operators on $L^p(G)$ is known to be isometrically isomorphic with the algebra A_p [12]. This result has been extended to all amenable groups in [15]. In the context of nonamenable groups a similar result is known for $SL_2(\mathbf{R})$ and some other semisimple Lie groups [7], and for free groups [8, 13]. We show (Theorem 5) that $A_p^* \simeq Cv_p$ for amalgams $G = *_H K_i$ if the indices k_i are bounded. We do not know whether this assumption can be dropped.

Unexplained terminology and notations are as in [10, 13]. We are grateful to M. Bozejko, M. Enomoto and Y. Watatani who communicated to us their results of [4, 5, 11] prior to publication.

2. Amalgamated products and length. Let I be any set of indices and let $\{G_i,\ i \in I\}$ be a family of locally compact groups containing a common open subgroup $H = \bigcap_i G_i$. Then the amalgamated product $G = *_H G_i$, endowed with the strongest topology such that the natural embeddings $G_i \to G$ are continuous, is a locally compact group [**2**, **17**]. Choose and fix a set S_i of representatives of the right H-cosets in G_i. Then every element s of G can be uniquely written as a reduced word $s = s_{i_1} \cdots s_{i_n} h$ with $h \in H$, $s_{i_j} \in S_{i_j} - H$ and $i_j \neq i_{j+1}$ for every j. The number n is called the *length* of s, and is denoted by $|s|$. In this section we prove that the length is a continuous negative definite function on G.

DEFINITION. Let f_i be right H-invariant functions on G_i, $i \in I$, with $f_i|_H = 1$.

The function $f = *_H f_i$, defined by the rule $f(g) = \prod_{m=1}^n f_{i_m}(s_{i_m})$ if $g = s_{i_1} \cdots s_{i_n} h$, is called the amalgamated product of the f_i's over H. □

The result about the length function is a consequence of the following theorem, whose argument is essentially due to M. Bozejko [**4**, Thm. 1].

THEOREM 1. *For $i \in I$, let f_i be a positive definite bi-H-invariant function on G_i with $f_i|_H = 1$. Suppose that the amalgamated product $f = *_H f_i$ is bi-H-invariant. Then f is positive definite on G.*

Proof. First, we claim that the assumption that f is bi-H-invariant is equivalent to the property

$$f(xy) = f(x)f(y) \quad \text{if } |xy| = |x| + |y|. \tag{1}$$

Indeed, we show that (1) holds if f is bi-H-invariant (the converse, not needed here, is clear; see also the definition and remarks following Thm. 3). Write, as before, $x = s_{i_1} \cdots s_{i_n} h$, and set $x_0 = s_{i_1} \cdots s_{i_n}$. Write also $hy = s_{j_1} \cdots s_{j_n} h_0$, and observe that $s_{j_1} \neq s_{i_n}^{-1} \bmod H$, since $|xy| = |x| + |y|$. Hence, by the definition of f, it follows $f(xy) = f(x_0)f(hy)$. On the other hand, since f is bi-H-invariant, $f(x_0) = f(x_0 h) = f(x)$, and $f(hy) = f(y)$. Thus $f(xy) = f(x_0)f(y) = f(x)f(y)$, and the claim is proved. For $k = 1, \ldots, n$, choose c_k in C, x_k in G. To prove that f is positive definite, we must prove that, for each such choice,

$$\sum_{m,k} c_m \bar{c}_k f(x_k^{-1} x_m) \geq 0. \tag{2}$$

The left hand side in this inequality is a pre-norm for the function $h = \sum c_k \delta_k$; we denote it by $\|h\|_f$. Observe that $\sum c_k f(x_k) = \langle h, \delta_e \rangle_f$ and

$\|\delta_e\|_f = f(e) = 1$. Thus the Cauchy-Schwarz inequality shows that (2) is equivalent to the apparently stronger inequality

$$\sum_{m,k} c_m \bar{c}_k f\left(x_k^{-1} x_m\right) \geq \left|\sum_k c_k f(x_k)\right|^2. \tag{3}$$

For $i \in I$, denote by s_{i_j} the representatives of the cosets in G_i/H, and let $S = \{s_{i_j}\}$. Define a function i: $G \to S$ by $i(x) = i(s_{i_1 j_1} \cdots s_{i_n j_n} h) = s_{i_1 j_1}$, and let $\tilde{x} = s_{i_2 j_2} \cdots s_{i_n j_n} h$. For some $j \in I$, say $j = 1$, the subset A of $U \equiv \{x_1, \ldots, x_n\}$ given by $A = U \cap i^{-1}(G_j/H)$ is non-empty. We now let $B = U - A$, and proceed by induction on $n = \#U$. We first show that (2) holds in the case $U = A$. Let $A_r = A \cap i^{-1}(s_{1r})$. In order to compute the left hand side of (2), it is enough to assume that $A_r \neq A$ for every r: if this is not so for some r, it suffices to replace $U = A$ by $s_{1r}^{-1} \cdot A$. Now (1) and the definition of f yield

$$\sum_{m,k} c_m \bar{c}_k f\left(x_k^{-1} x_m\right) = \sum_r \sum_{A_r \times A_r} c_m \bar{c}_k f\left(\tilde{x}_k^{-1} \tilde{x}_m\right)$$
$$+ \sum_{r \neq j} \sum_{A_r \times A_j} c_m \bar{c}_k f\left(\tilde{x}_k^{-1}\right) f(\tilde{x}_m) f_1\left(i(x_k)^{-1} i(x_m)\right).$$

Set $b_r = \sum_{x_k \in A_r} c_k f(\tilde{x}_k)$. As $\#A_r < \#U$, by the induction hypothesis the right hand side of this inequality is greater than or equal to $\sum_r |b_r|^2 + \sum_{r \neq j} b_r \bar{b}_j f_1(s_{1j}^{-1} s_{1r})$, which is non-negative because f_1 is positive definite on G_1. Thus (2), and therefore (3), hold in this case.

The general case $U = A \cup B$ is handled by the same token. Indeed, we have just proved that (3) holds for the subset A of U, and, as B is properly contained in U, we can assume, by induction on $\#U$, that (3) holds for B. Moreover, if $x \in A$ and $y \in B$, then $|xy| = |x| + |y|$, and we can apply (1). Now we decompose the sum in the left hand side of (3) into three parts, corresponding to the conditions that x_k and x_m both belong to A, or both to B, or one to A and the other to B. By the induction hypothesis, it follows that

$$\sum_{m,k} c_m \bar{c}_k f\left(x_k^{-1} x_m\right) \geq \left|\sum_A c_k f(x_k)\right|^2 + \left|\sum_B c_m f(x_m)\right|^2$$
$$+ 2\,\mathrm{Re} \sum_{A \times B} c_m \bar{c}_k \bar{f}(x_k) f(x_m) \geq \left|\sum_U c_k f(x_k)\right|^2. \quad \square$$

REMARK 1. As a consequence of Theorem 1, for $t \geq 0$, the function $x \to \exp(-t|x|)$ is a (continuous) positive definite function on G. This result is due to U. Haagerup **[14]** for the special case of free groups (see also **[1]**, **[11]**).

3. Convolution operators and positive definite functions on amalgams. From now on, we assume that $G = *_H K_i$, where the K_i, $i \in I$, are a family of compact groups, and the indices k_i of $H = \bigcap_i K_i$ in K_i are bounded. We shall refer to the groups K_i as the *factors* of G. We normalize the Haar measure m of G so that $m(H) = 1$.

Following [16], we say that two words $x = s_{i_1} \cdots s_{i_n} h$, $y = s_{j_1} \cdots s_{j_m} h'$ give rise to a *simple cancellation* if $|xy| = |x| + |y| - 2$, that is, if $s_{i_n} h s_{j_1} \in H$. In order that a cancellation occurs, s_{i_n} and s_{j_1} must belong to the same set S_i, i.e., to the same factor K_i, and the right H-cosets determined by hs_{j_1} and $s_{i_n}^{-1}$ must coincide.

We say that x and y give rise to a *reduction* if $|xy| = |x| + |y| - 1$, that is, if s_{i_n} and s_{j_1} belong to the same factor K_i but do not give rise to a cancellation.

Denote by W_n the set of words of length n, and by $B_{n,m}(x)$ the set $\{(y, z) \in W_n \times W_m\colon x = yz\}$. Obviously $B_{n,m}(x) = \varnothing$ unless $|n - m| \leq |x| \leq n + m$. In particular, if $\operatorname{supp} f \subset W_n$, $\operatorname{supp} g \subset W_m$, then $f * g(x) = 0$ unless $|n - m| \leq |x| \leq n + m$.

LEMMA 1. *Let $p = n + m - t$, $0 \leq t \leq n + m$, and $|x| = p$; write $x = s_{i_1} \cdots s_{i_p} h$ with $h \in H$. Let $q = [(t + 1)/2]$, $q' = [(t - 1)/2]$, $y_0 = s_{i_1} \cdots s_{i_{n-q}}$, $z_0 = s_{i_{n-q'+1}} \cdots s_{i_p} h$. Then:*

(i) *if t is even, $t = 2j$, then $y_0 = s_{i_1} \cdots s_{i_{n-j}}$, $z_0 = s_{i_{n+j+1}} \cdots s_{i_p} h$, and $B_{n,m}(x) = \{(y, z) \in W_n \times W_m\colon$ there exist $w \in W_j$ such that $y = y_0 w$, $z = w^{-1} z_0\}$.*

(ii) *if t is odd, $t = 2j + 1$, then $y_0 = s_{i_1} \cdots s_{i_{n-j-1}}$, $z_0 = s_{i_{n+j+1}} \cdots s_{i_p} h$ and $B_{n,m}(x) = \{(y, z) \in W_n \times W_m\colon$ there exist $p_1, p_2 \in W_1$, $w \in W_j$ such that $y = y_0 p_1 w$, $z = w^{-1} p_2 z_0$ and $p_1 p_2 = s_{i_n}\}$.*

(iii) *If $u \in K_\iota - H$, and π_2 denotes the projection of $G \times G$ onto its second factor, then $\int_{\pi_2(B_{1,1}(u))} dm = k_\iota - 2$.*

Proof. Let $t = 2j$. Then $(y_0, z_0) \in B_{n-j,m-j}(x)$. By uniqueness of the reduced word expression in G [18], for every $(y, z) \in B_{n,m}(x)$ there exists $w \in W_j$ such that $y = y_0 w$. Then $y_0 z_0 = x = y_0 w z$, hence $z = w^{-1} z_0$. To prove (ii), let $t = 2j + 1$: then $x = y_0 s_{i_n} z_0$. If $(y, z) \in B_{n,m}(x)$, then, again, $y = y_0 u$, for some $u \in W_{j+1}$. Then $y_0 s_{i_n} z_0 = x = yz = y_0 u z$ and $z = u^{-1} s_{i_n} z_0$. As $|z| = m$, $|z_0| = m - j - 1$ and $|u| = j + 1$, a reduction occurs in the product $u^{-1} s_{i_n} z_0$. By definition of z_0, no reduction occurs in the product $s_{i_n} z_0$. Hence a reduction occurs in the product $u^{-1} s_{i_n}$. If $s_{i_n} \in K_{i_n}$ and $u^{-1} = s'_{r_1} \cdots s'_{r_{j+1}}$ this means that $e \neq s'_{r_{j+1}} s_{i_n} \in K_{i_n}$. Let $p_1 = s'^{-1}_{r_{j+1}}$, $p_2 = s'_{r_{j+1}} s_{i_n}$, $w = u^{-1} p_1$. Then $|w| = j$, $y = y_0 p_1 w$, $z = w^{-1} p_2 z_0$.

To prove (iii), pick any representative $s \in S_i$. Then there is exactly one representative $s' \in S_i$ such that $u = ss' \bmod H$. Since u is nontrivial, s' is nontrivial if s is trivial, i.e., if $s \in H$. Thus there are exactly $k_i - 2$ choices of s such that both s and s' are nontrivial and $u = ss'h$ for some $h \in H$. By part (ii),

$$\int \int_{\pi_2(B_{1,1}(u))} dm = (k_i - 2)m(H) = k_i - 2. \qquad \square$$

Denote by χ_n the characteristic function of W_n; in particular, $\chi_0 = \chi_H$. We shall denote by H_l the expectation defined by left average over H on $L^1(G) \cap L^\infty(G)$: $H_l f(x) = \int_H f(hx)\,dh$. Similarly, the right average is denoted by H_r. Observe that H_l and H_r have norm 1 on $L^p(G)$, $1 \le p \le \infty$.

Let $k = \max_i k_i = \max_i \#(K_i/H) < \infty$.

LEMMA 2. *Let f, g be square-integrable functions with support in W_n, W_m respectively. Then*:

(i) $\|(f * g)\chi_p\|_2 \le \|f\|_2\|g\|_2$ *if* $p = n + m - 2j$, *with* $0 \le j \le \min(n, m)$

(ii) $\|(f * g)\chi_p\|_2 \le \sqrt{k-2}\ \|f\|_2\|g\|_2$ *if* $p = n + m - 2j - 1$ *with* $0 \le j \le \min(n, m) - 1$.

Proof. The argument follows the lines of [**14**, Lemma 1.3], [**16**, Lemma 2]. By density, we can assume $g \in L^1 \cap L^\infty$. Let us first handle the case $p = n + m$.

Let us write $B(x) = B_{n,m}(x)$. It follows from Lemma 1.i that $m(\pi_2(B(x))) = m(H) = 1$. By the remarks preceding the statement and by Lemma 1.i, one has

$$\begin{aligned}\|(f * g)\chi_p\|_2^2 &= \int_{W_p} dx \left| \int_{\pi_2(B(x))} f(xz)g(z^{-1})\,dm(z)\right|^2 \\ &= \sum_{|s_{i_1}\cdots s_{i_n} s_{i_{n+1}}\cdots s_{i_p}| = p} \left|H_r f(s_{i_1}\cdots s_{i_n})\right|^2 \left|H_l g(s_{i_{n+1}}\cdots s_{i_p})\right|^2 \\ &\le \|H_r f\|_2^2 \|H_l g\|_2^2 \le \|f\|_2^2\|g\|_2^2.\end{aligned}$$

To complete the proof of part (i), denote by λ_y the left translation by y^{-1}, and by ρ_z the right translation by z. Let $p = n + m - 2j$, and define two auxiliary functions f', g', with support in W_{n-j}, W_{m-j} respectively, by

$f'(y) = \|\lambda_{y^{-1}} f\|_{L^2(W_j, m)}$ for $|y| = n - j$, and $g'(z) = \|\rho_z f\|_{L^2(W_j, m)}$ for $|z| = m - j$. It follows again from Lemma 1.ii that $\|f'\|_2 = \|f\|_2$, $\|g'\|_2 = \|g\|_2$. If $|x| = p = n + m - 2j$ and y_0, z_0 are as in Lemma 1.i., one has

$$\begin{aligned} |f * g(x)| &= \left| \int_{\pi_2(B(x))} f(xz) g(z^{-1})\, dm(z) \right| \\ &= f'(y_0) g'(z_0) = f' * g'(x), \end{aligned}$$

by Lemma 1.i. This yields (i).

Let now $p = n + m - 1$, and $|x| = p$, $x = s_{i_1} \cdots s_{i_p} h$.By Lemma 1.iii, $m(\pi_2(B_{1,1}(s_{i_n}))) = k_{i_n} - 2 \leq k - 2$. Write $y = xz^{-1}$. Then Lemma 1.ii, yields

$$\begin{aligned} \|(f * g)\chi_p\|_2^2 &= \int_{W_p} dx \left| \int_{\pi_2(B_{n,m}(x))} f(y) g(z)\, dm(z) \right|^2 \\ &= \int_{W_p} dx \left| \int\int_{\pi_2(B_{1,1}(s_{i_n}))} \int_H f(y_0 p_1 h) g(h^{-1} p_2 z_0)\, dh\, dm(p_2) \right|^2 \\ &\leq \|H_r f\|_2^2 \|H_l g\|_2^2 m\big(\pi_2\big(B_{1,1}(s_{i_n})\big)\big) \leq (k-2)\|f\|_2 \|g\|_2. \end{aligned}$$

Finally, to complete the proof of (ii), we reduce to the case $p = n + m - 1$ by using the auxiliary functions f', g' as in the proof of (i). □

As in [14, 16], the previous lemma provides estimates for the norm of a function supported on W_n as a convolution operator on $L^2(G)$. Indeed, the argument of [14, Lemma 2] now yields:

THEOREM 2. *Let f be a function on $G = *_H K_i$, with support in W_n, and let $k = \max_i k_i < \infty$. Then there exists a constant $C(n, k) = O(n)$ such that the norm of f as a left convolution operator on L^2 satisfies the inequality $\|f\|_{C^*_\lambda} \leq C(n, k)\|f\|_2$.*

Denote by $\|f\|_{Cv_p}$ the norm of a function f as a left convolution operator on $L^p(G)$ (the norm of convolution operators on L^2 will be denoted by $\|\ \|_{C^*_\lambda}$). By Theorem 2, $\|f\|_{C^*_\lambda} \leq C \cdot (n+1)\|f\|_2$ for some constant C independent of f. Moreover, $\|f\|_{Cv_1} \leq \|f\|_1$. Hence bilinear interpolation yields:

COROLLARY 1. *There exists a constant C such that, for every function f with support in W_n, $\|f\|_{Cv_p} \leq C \cdot (n+1)^{1/q}\|f\|_p$, where $1 \leq p \leq 2$ and $1/q = 1 - 1/p$.*

If X, Y are normed spaces of functions on G, we denote by $\mathscr{M}(X,Y)$ the space of pointwise multipliers from X to Y, endowed with the multiplier norm $\|u\|_{\mathscr{M}(X,Y)} = \sup\{\|uv\|_Y\colon \|v\|_X \leq 1\}$. Recall that χ_n denotes the characteristic function of W_n.

COROLLARY 2. $\|\chi_n\|_{\mathscr{M}(A,L^2)} \leq C(n,k)$.

Proof. It is enough to observe that, by the duality between $A(G)$ and C_λ^* **[10]**,

$$\begin{aligned}\|u\chi_n\|_2 &= \sup\left\{\left|\int uv\,dm\right|\colon \|v\|_2 \leq 1, \operatorname{supp} v \subset W_n\right\}\\ &\leq C(n,k)\|u\|_A.\end{aligned}$$

□

Denote by B_λ the closed ideal of $B(G)$ consisting of the functions which can be uniformly approximated on compact sets by bounded sequences in $A(G)$. We can now characterize B_λ as follows:

PROPOSITION 1. *Let X be the largest closed subspace of $B(G)$ such that $\|\chi_n\|_{\mathscr{M}(X,L^2)} = O(n)$. Then $X = B_\lambda$.*

Proof. If $u \in B_\lambda$, there exists a sequence $u_j \in A$, such that $\|u_j\|_A \leq C$ and $u_j(x) \to u(x)$ on compact subsets of G. Then $\|u_j\|_\infty \leq C$, and Corollary 2 yields

$$\begin{aligned}\|u\chi_n\|_2 &= \sup\{\|u\chi_n\chi_K\|_2\colon K \subset G,\ K \text{ compact}\}\\ &= \sup_K \lim_j \|u_j\chi_n\chi_K\|_2 \leq C\cdot C(n,k) = O(n).\end{aligned}$$

Conversely, let $u \in B$, and suppose that $\|u\chi_n\|_2 = O(n)$. Then

$$\left\|e^{-t|x|}u(x)\right\|_2^2 = \sum_{n=0}^{\infty}\left\|e^{-t|x|}u\chi_n\right\|_2^2 = \sum_{n=0}^{\infty} e^{-tn}\|u\chi_n\|_2 < \infty.$$

Therefore $e^{-t|x|}u(x) \in L^2$. Moreover, $e^{-t|x|}u \in B$, by Theorem 1. Hence $e^{-t|x|}u \in A$, and $\|e^{-t|x|}u\|_A \leq \|e^{-t|x|}\|_B\|u\|_B = \|u\|_B$, by Theorem 1 again.

Finally, $\lim_{t\to 0} e^{-t|x|}u(x) = u(x)$ for each x. Thus $u \in B_\lambda$. □

Denote by $\mathscr{M}(A)$ the space of pointwise multipliers of $A(G)$. It is obvious that $B(G) \subset \mathscr{M}A$, with norm-decreasing inclusion. Suppose H is finite (i.e., G is discrete), so that $L^2(G) \subset A(G)$. Then we show that $\mathscr{M}A$ contains B properly.

PROPOSITION 2. *Let* $M = \sum_{n=0}^{\infty} C(n,k)^2 \|u\chi_n\|_\infty^2 < \infty$. *If* H *is finite, then* $u \in \mathcal{M}A$ *and* $\|u\|_{\mathcal{M}A} \le \sqrt{M}$.

Proof. For $h \in A(G)$, Corollary 2 yields $\|uh\|_2^2 \le M\|h\|_A^2$. Then $\|uh\|_A \le \|uh\|_2 \le \sqrt{M}\|h\|_A$. □

THEOREM 3. *If* H *is finite, then, for every* $\varepsilon > 0$, *there exists a sequence* $u_n \in A(G)$ *such that* $\lim_n \|u_n v - v\|_A = 0$ *for every* $v \in A(G)$, *and* $\|u_n\|_{\mathcal{M}A} \le 1 + \varepsilon$.

Proof. As $C(n,k) = O(n)$, for every integer j there exists a smallest integer n_j such that $\sum_{n=n_j}^{\infty} e^{-2n/j} C(n,k)^2 < \varepsilon^2$. Let $u_j(x) = e^{-|x|/j}$ if $|x| < n_j$, $u_j(x) = 0$ otherwise. Then

$$\begin{aligned} \|u_j\|_{\mathcal{M}A} &\le \left\| u_j - e^{-|x|/j} \right\|_{\mathcal{M}A} + \|e^{-|x|/j}\|_{\mathcal{M}A} \\ &\le 1 + \left(\sum_{n=n_j}^{\infty} e^{-2n/j} C(n;k)^2 \right)^{1/2} \le 1 + \varepsilon, \end{aligned}$$

by Theorem 1 and Proposition 2. Moreover, $\lim_j u_j = 1$ uniformly on compact sets. Since all norms are equivalent on finite-dimensional vector spaces, $\|u_j v - v\|_A \underset{j \to \infty}{\to} 0$ for every compactly supported $v \in A$. As the norms $\|u_j\|_{\mathcal{M}A}$ are uniformly bounded, the same is true for every $v \in A$. □

In the remainder, we shall not need to restrict attention to the case of G discrete. We conclude this section by showing that the Fourier-Stieltjes algebra of G is much larger than its ideal B_λ. For this, we introduce a family of positive definite functions as follows:

DEFINITION. A function u on $G = *_H K_i$ is *multiplicative with respect to the length* if $0 < u \le 1$, $u(xy) = u(x)u(y)$ if $|xy| = |x| + |y|$ and, for every $i \in I$, $u_i = u|_{K_i - H}$ is a constant (depending on i). □

It is obvious that every such function u is bi-H-invariant (for the converse, see the proof of Thm.1). It is also immediate that, for every i, u_i is positive definite on K_i. Indeed, denote by c_i the constant value attained by u_i on $K_i - H$. Then $u_i|_{K_i} = c_i\chi_{K_i} + (1 - c_i)\chi_H$, because $u_i \equiv 1$ on H. Thus $u_i|_{K_i}$ is the sum of two positive definite functions. The multiplicative property yields $u = *_H u_i$. Therefore, by Theorem 1, u is positive definite on G.

With notations as in **[10, 13]**, let now P be the central projection, in the von Neumann algebra of the universal representation of G, defined by

$PB(G) = B_\lambda$. If u is a function in $B(G)$, denote by u_λ its projection on B_λ: $u_\lambda = Pu$. We say that $u \in B_\lambda^\perp$ if $u_\lambda = 0$.

THEOREM 4. *Let u be a multiplicative function with respect to the length. If* $\|u\chi_1\|_2 < 1$, *then* $u \in A(G)$. *If* $\|u\chi_1\|_2 = \infty$, *then* $u \in B_\lambda^\perp$.

Proof. Observe that, by the multiplicative property,

$$\|u\chi_n\|_2^2 = \int_{W_n} |u|^2 \le \left(\int_{W_1} |u|^2\right)^n = \|u\chi_n\|_2^{2n}.$$

Thus, if $\|u\chi_1\|_2 < 1$, then

$$\|u\|_2^2 = \sum_{n=0}^{\infty} \|u\chi_n\|_2^2 \le \sum_{n=0}^{\infty} \|u\chi_1\|_2^{2n} < \infty.$$

Hence $u \in B \cap L^2 \subset A$. On the other hand, if $\|u\chi_1\|_2 = \infty$, then the set of indices I is infinite (because u is constant on the compact factors K_i). Let us denote by u_i the constant $u|_{K_i}$. Then there exists a function v on I, and a partition of I into disjoint subsets I_n such that: (i) $\sum_{i\in I_n} u_i v_i = 1$, (ii) $\sum_{i\in I} |v_i|^2 < \infty$. For each $i \in I$, choose a nontrivial representative s_i in S_i, and let $E = \bigcup_I s_i H$, $E_n = \bigcup_{I_n} s_i H$.

The function $f = \sum_{i\in I_n} v_i \chi_{s_i H}$ is supported in $E_n \subset W_1$. Therefore, by Theorem 2, there exists a constant C such that the norm of f_n as a left convolution operator on L^2 satisfies the inequality

$$\|f_n\|_{C_\lambda^*} \le C\|f_n\|_2 < C\left(\sum_{i\in I} |v_i|^2\right)^{1/2} \to 0.$$

If g is a finitely supported function on G, define

$$\|g\|_u^2 = \int_G g^* * g(x) u(x)\, dx.$$

We claim that

$$\lim_n \|f_n - \delta_e\|_u = 0.$$

Indeed,

$$\|f_n - \delta_e\|_u^2 = 1 + \iint u(y^{-1}x) f_n(x)\overline{f_n}(y)\, dx\, dy$$

$$- \int u(y^{-1})\overline{f_n}(y)\, dy - \int u(x) f_n(x)\, dx$$

For every $x, y \in E$, either x and y belong to the same right H-coset or $|x^{-1}y| = |x| + |y|$ and $u(x^{-1}y) = u(x)u(y)$. Thus, by (i),

$$\|f_n - \delta_e\|_u^2 = -1 + \int |f_n|^2 - \int u^2|f_n|^2$$

$$+ \int\int u(x)u(y)f_n(x)\overline{f_n(y)}\,dx\,dy$$

$$= \int_E (1 - u^2(x))|f_n(x)|^2\,dx.$$

The right hand side tends to zero, because of (ii), and the claim is proved. Denote by $\lambda(f)$ the left convolution operator on L^2 determined by f. Let F be the positive functional on $C^*(G)$ associated with the positive definite function u, and define F_λ on $C^*_\lambda(G)$ by $F_\lambda = F \circ P$. By the Cauchy-Schwarz inequality,

$$F_\lambda(\mathbb{1} - \lambda(f_n)) \leq F(P)F\big((\mathbb{1} - \lambda(f_n))^*(\mathbb{1} - \lambda(f_n))\big)$$

$$= F(P)\|f_n - \delta_e\|_u^2,$$

which converges to zero as $n \to \infty$. Thus $F(P\lambda(f_n)) = F_\lambda(\lambda(f_n)) \to F_\lambda(\mathbb{1}) = F(P)$. On the other hand, as $u_\lambda(x) = Pu(x) = F_\lambda(\delta_x)$, one has

$$|F_\lambda(\lambda(f_n))| = \left|\int u_\lambda f_n\,dx\right| \leq \|u_\lambda\|_B\|f_n\|_{C^*_\lambda} \to 0.$$

It follows that $F(P) = 0$. Since $F(P) = u_\lambda(e)$, this implies $u_\lambda = 0$. $\square$

4. Convolution operators on $L^p(G)$ and A_p-algebras. In this section, we characterize the predual of the Banach space Cv_p of left translation invariant operators on $L^p(G)$, $G = *_H K_i$, with $\sup_i \#(K_i/H) < \infty$.

DEFINITION. For $1 < p < \infty$, let $1/q = 1 - 1/p$, and $A_p = \{h = \sum_{i=1}^\infty f_i * g_i : f_i \in L^p(G),\ g_i \in L^q(G),\ \sum_{i=1}^\infty \|f_i\|_p\|g_i\|_q < \infty\}$. We endow A_p with the norm $\|h\|_{A_p} = \inf\{\sum_{i=1}^\infty \|f_i\|_p\|g_i\|_q : h = \sum_{i=1}^\infty f_i * g_i\}$.

We now show that, if $G = *_H K_i$, with $\#(K_i/H)$ bounded, then $A_p^* = Cv_p$.

LEMMA 3. *For $1 < p < \infty$, every bounded operator T on $L^p(G)$ which commutes with right translations can be approximated, in the weak operator topology, by left convolution operators by functions with compact support.*

Proof. Denote by v_α an approximate identity in $L^1(G)$. As T commutes with right translations, for every $u \in L^p(G)$ one has $\lim_\alpha T(v_\alpha * u) = \lim_\alpha T\cdot v_\alpha * u = Tu$. Thus we can assume that T is a left convolution operator by a function $f \in L^p(G)$, $T = \lambda(f)$.

We first handle the case $1 < p \leq 2$. For $t > 0$, let $f_t(x) = \exp(-t|x|)f(x)$. Now $\lambda(f_t) \underset{t\to 0}{\to} \lambda(f)$ in the strong operator topololy, since $B(G) \subset B_p(G) \subset \mathcal{M}(Cv_p)$ **[15]**. Therefore, it suffices to show that there exists a sequence of functions h_n with compact support such that $\lambda(h_n) \underset{n}{\to} \lambda(f_t)$ in norm.

We construct this sequence as follows: $h_n = \sum_{m=0}^n f_t\chi_m$. Indeed, by Corollary 1,

$$\|\lambda(h_n) - \lambda(f_t)\|_{Cv_p} \leq \sum_{m=n+1}^{\infty} \|f_t\chi_m\|_{Cv_p}$$

$$\leq C \sum_{m=n+1}^{\infty} (m+1)^{1/q}\|f_t\chi_m\|_p \leq C \sum_{m=n+1}^{\infty} (m+1)^{1/q}e^{-tm}\|f\|_p.$$

This completes the proof if $1 < p \leq 2$. By the same argument, every right convolution operator on L^p can be approximated, in the strong operator topology, by right convolution operators $\rho(h)$, where h has compact support. Let now J be the involution of L^p given by $Ju(x) = u(x^{-1})$. Then, for $p > 1$, the adjoint of $\lambda(f)$ on L^p is the operator $J\rho(f)J$ on L^q. If $p > 2$, the first part of the proof yields a sequence of operators $\rho(h_n)$ on L^q, with h_n of compact support, which approximate $\rho(f)$ in the weak operator topology. The result now follows from the fact that $J\rho(h_n)J$ converges to $J\rho(f)J$ in the weak operator topology. □

THEOREM 5. *Let $G = *_H K_\iota$, with $\#(K_\iota/H)$ bounded. Then the dual space of A_p is isometrically isomorphic with Cv_p.*

Proof. Let $T \in Cv_p$, $h = \sum_{i=1}^\infty f_i * g_i$, $f_i \in L^p(G)$, $g_i \in L^q(G)$, and $\langle T, h\rangle = \sum_{i=1}^\infty Tf_i * g_i(e)$. The inclusion $Cv_p \subset A_p^*$ is immediate if we prove that the map $h \to \langle T, h\rangle$ is well defined on A_p. To prove this, let $\sum_i f_i * g_i(x) = 0$ for every $x \in G$. Then $\sum_i \lambda(h)f_i * g_i(e) = 0$ for every function h with compact support. By Lemma 3, $\sum_i Tf_i * g_i(e) = 0$.

Conversely, let $\Phi \in A_p^*$, and fix f in $L^p(G)$. Then Φ determines a continuous linear functional on $L^q(G)$ by the rule $g \to \Phi(f * g)$. This functional is associated with a function $F \in L^p(G)$. Define an operator T on $L^p(G)$ by $Tf = F$. It is easily seen that T commutes with right translations, that is, $T \in Cv_p$, and $\|T\| = \|\Phi\|$. Thus $A_p^* \simeq Cv_p$. □

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