PACIFIC JOURNAL OF MATHEMATICS
Vol. 123, No. 2, 1986

COHOMOLOGY WITH SUPPORTS

E. SPANIER

In this paper we study cohomology theories on a space X with supports in a family of supports Φ. There is a uniqueness theorem asserting that a homomorphism between two cohomology theories on the space X with the same family of supports Φ which is an isomorphism for every $A \in \Phi$ is an isomorphism for every closed set $A \subset X$.

1. Introduction. By using cohomology with supports in a given family it is possible to pass from cohomology theories on X to cohomology theories on subsets of X with suitably related families of supports. In particular, compactly supported cohomology theories on a locally compact space X correspond to cohomology theories on the one-point compactification of X which vanish at infinity. Similarly, cohomology theories on a locally paracompact space X with relatively paracompact supports correspond to cohomology theories on the one-point paracompactification of X which vanish at infinity.

We also prove a uniqueness theorem for homomorphisms between additive cohomology theories with paracompact supports on finite dimensional space.

The remainder of the paper is divided into four sections. Section 2 contains the definition of a cohomology theory with supports in a family Φ, a uniqueness theorem for two cohomology theories with the same family of supports, and a characterization of cohomology with supports in suitable families in terms of limit properties.

Section 3 is devoted to the construction of cohomology theories on a space X with supports in a given family Φ from an ES theory on X. The definition of an ES theory on X is given and it is shown that given an ES theory on X and a family Φ of supports on X there is another ES theory on X with supports in Φ.

In Section 4 the relation between cohomology theories on X and on open subsets of X is studied. The concept of a cohomology theory on X concentrated on a subset $Y \subset X$ (i.e. which vanishes for every closed subset of X contained in $X - Y$) is introduced. The main result is a bijection between cohomology theories on X concentrated on an open set Y with supports in Φ and cohomology theories on Y with supports in a suitable family $\Phi|Y$.

The particular cases of compact and paracompact supports are studied in §5. Cohomology theories on a locally compact (locally paracompact) space X with compact (relatively paracompact) supports are shown to correspond to cohomology theories on the one-point compactification (paracompactification) X^+ which are concentrated on X. There is also established a uniqueness theorem for addititive cohomology theories with paracompact supports on finite dimensional normal spaces.

2. Supports. We consider cohomology theories on a space X with a given family of closed subsets of X as supports. The uniqueness theorem extends to this case and asserts that if φ: H, $\delta \to H', \delta'$ is a homomorphism between cohomology theories on the same space X with the same family Φ of supports such that for some integer n, φ_A: $H(A) \to H'(A)$ is an n-equivalence for all $A \in \Phi$ then φ_A is an n-equivalence for all closed $A \subset X$.

All topological spaces will be assumed to be Hausdorff spaces. *A cohomology theory* **[7, 8]** H, δ on X consists of:

(i) a contravariant functor H from the category cl(X) of closed subsets of X and inclusion maps to the category of graded abelian groups and homomorphisms of degree 0 such that $H(\varnothing) = 0$, and

(ii) a natural transformation δ: $H(A \cap B) \to H(A \cup B)$ of degree 1 for every two closed sets A, B in X, such that the following are satisfied:

Continuity. For closed $A \subset X$ there is an isomorphism

$$\rho\colon \varinjlim \{H(N) | N \text{ a closed nbhd of } A \text{ in } X\} \approx H(A)$$

where $\rho\{u\} = u | A$ for $u \in H(N)$.

MV exactness. For A, $B \subset X$ there is an exact sequence

$$\dots \xrightarrow{\beta} H^{q-1}(A \cap B) \xrightarrow{\delta} H^q(A \cup B) \xrightarrow{\alpha} H^q(A) \oplus H^q(B)$$

$$\xrightarrow{\beta} H^q(A \cap B) \xrightarrow{\delta} \dots$$

where $\alpha(u) = (u | A, u | B)$ for $u \in H^q(A \cup B)$ and $\beta(u, v) = u | A \cap B - v | A \cap B$ for $u \in H^q(A)$, $v \in H^q(B)$.

A cohomology theory H, δ is said to be *non negative* if $H^q(A) = 0$ for $q < 0$ and all closed $A \subset X$. A cohomology theory H, δ is said to be *additive* if for every discrete[1] family $\{A_j\}$ of closed sets there is an isomorphism

$$H(\cup A_j) \approx \prod H(A_j)$$

sending $u \in H(\cup A_j)$ to $\{u | A_j\}$.

A *family of supports* [1] Φ on X consists of a collection of closed subsets of X such that

(i) $A \in \Phi$, B closed in $A \Rightarrow B \in \Phi$.

(ii) $A, B \in \Phi \Rightarrow A \cup B \in \Phi$.

If Φ also has the property

(iii) $A \in \Phi \Rightarrow$ there is a closed nbhd N of A in X with $N \in \Phi$,

we say Φ is a *nbhd family of supports*.

EXAMPLES (2.1). The collection of all closed subsets of X is a family of supports on X. In case X is a normal space, it is a nbhd family of supports.

(2.2). The collection of all compact subsets of X is a family of supports on X. In case X is locally compact, it is a neighborhood family of supports.

(2.3). The collection of all paracompact subsets of X is a family of supports on X. The collection of all closed subsets of X having closed paracompact nbhds in X is a nbhd family of supports on X.

(2.4). If $Y \subset X$ and Φ is a family of supports on X, then $\Phi | Y = \{A \in \Phi | A \subset Y\}$ is a family of supports on Y and on X. If Y is open in a normal space X and Φ is a nbhd family in X, then $\Phi | Y$ is a nbhd family on Y and on X.

If Φ is a family of supports on X and H, δ is a cohomology theory on X, then H, δ has *supports* in Φ if given $u \in H(A)$ there exist B closed, $C \in \Phi$ with $A = B \cup C$ and $u | B = 0$.

This definition is a generalization of compactly supported cohomology [7, 8]. Note that the definition does not involve the natural transformation δ. Obviously every cohomology theory on X has supports in the family of all closed subsets of X.

If H, δ and H', δ' are cohomology theories on the same space X, a *homomorphism* φ from H, δ to H', δ' is a natural transformation from H to H' commuting up to sign with δ, δ'.

[1]A family $\{A_j\}$ of subsets of a space X is *discrete* if every point of X has a nbhd meeting at most one member of the family.

The following is a generalization of [8, Proposition (2.8)] to arbitrary families of supports.

THEOREM (2.5). *Let φ: $H, \delta \to H', \delta'$ be a homomorphism between two cohomology theories on X with supports in Φ and suppose $n \in \mathbf{Z}$ is such that φ_A: $H(A) \to H'(A)$ is an n-equivalence*[2] *for every $A \in \Phi$. Then φ_A is an n-equivalence for every closed $A \subset X$.*

Proof. The proof parallels that of [**8**, Proposition (2.8)] and will, therefore, be omitted. □

Given a family Φ of supports on X a set $S \subset X$ is a *co-Φ set* if $\overline{X - S} \in \Phi$. The following is a useful criterion for verifying that a contravariant functor on $\mathrm{cl}(X)$ is continuous and has supports Φ.

PROPOSITION (2.6). *Assume H is a contravariant functor from* $\mathrm{cl}(X)$ *to the category of graded abelian groups such that $H(\varnothing) = 0$, Φ is a family of supports on X, and for every closed $A \subset X$ there is an isomorphism*

$$(*) \qquad \rho\colon \varinjlim \{H(N) \mid N \text{ a closed co-}\Phi \text{ nbhd of } A \text{ in } X\} \approx H(A)$$

where $\rho\{u\} = u \mid A$ for $u \in H(N)$. Then H is continuous and has supports in Φ.

Proof. We first show H is continuous. Let ρ': $\varinjlim \{H(N) \mid N$ a closed neighborhood of A in $x\} \to H(A)$ be the map of the continuity property. Then $(*)$ implies that ρ' is surjective. To show it injective assume N is a closed neighborhood of A in X and $u \in H(N)$ is such that $u \mid A = 0$. By $(*)$ there is a closed co-Φ neighborhood $\overline{N}$ of N in X and $\bar{u} \in H(\overline{N})$ such that $u = \bar{u} \mid N$. Then $\bar{u} \mid A = 0$ so, again by $(*)$, there is a closed co-Φ neighborhood M of A in $\overline{N}$ such that $\bar{u} \mid M = 0$. Then $N \cap M$ is a closed neighborhood of A in N and $u \mid N \cap M = \bar{u} \mid N \cap M = 0$ proving that ρ' is injective. Therefore, H is continuous.

To show H has supports in Φ let $u \in H(A)$. By $(*)$ there is a closed co-Φ neighborhood N of A in X and $v \in H(N)$ such that $v \mid A = u$. Since $v \mid \varnothing = 0$ because $H(\varnothing) = 0$, it follows from $(*)$ again that there is a closed co-Φ M in N such that $v \mid M = 0$. Let $B = A \cap M$ and $C = \overline{A - M}$. Then $A = B \cup C$ where B is closed, $C \in \Phi$ and $u \mid B = 0$. □

[2]A homomorphism φ: $G \to G'$ of degree 0 between graded abelian groups is an *n-equivalence* if φ: $G^q \to G'^q$ is an isomorphism for all $q < n$ and a monomorphism for $q = n$.

In the case of nbhd families of supports and for cohomology theories H, δ there is the following converse of Proposition (2.6).

PROPOSITION (2.7). *Assume H, δ is a cohomology theory on X with supports in a nbhd family Φ. Then there is an isomorphism*

$$\rho\colon \varinjlim \{H(N) \mid N \textit{ a closed co-}\Phi \textit{ nbhd of } A \textit{ in } X\} \approx H(A)$$

where $\rho\{u\} = u \mid A$ for $u \in H(N)$.

Proof. (1) Let $u \in H(A)$ and suppose M is a closed nbhd of A and $v \in H(M)$ are such that $v \mid A = u$ (such M, v exist because H is continuous). Since H has supports in Φ, $M = B \cup C$ where B is closed, $C \in \Phi$ and $v \mid B = 0$. Since Φ is a nbhd family there is a closed nbhd C' of C with $C' \in \Phi$. Let $N = M \cup (X - \operatorname{int} C')$. Then N is a closed nbhd of A in X and

$$\overline{X - N} = \overline{X - M} \cap \overline{\operatorname{int} C'} \subset C'$$

so N is a co-Φ set. Since $C \cap [B \cup (X - \operatorname{int} C')] = C \cap B$, there is an exact sequence

$$H(N) \xrightarrow{\alpha} H(C) \oplus H(B \cup (X - \operatorname{int} C')) \xrightarrow{\beta} H(C \cap B).$$

Since $(v \mid C, 0) \in H(C) \oplus H(B \cup (X - \operatorname{int} C'))$ is in $\ker \beta$, there is $w \in H(N)$ such that $w \mid C = v \mid C$ and $w \mid [B \cup (X - \operatorname{int} C')] = 0$. Then $w \mid M$ and v have the same restrictions to C and to B so by exactness of

$$H(C \cap B) \xrightarrow{\delta} H(M) \xrightarrow{\alpha} H(C) \oplus H(B)$$

there is $w' \in H(C \cap B)$ such that $\delta w' = v - w \mid M$. There is a commutative diagram

$$\begin{array}{ccc} H(C \cap [(B \cup (X\text{-}\operatorname{int} C'))]) & = & H(C \cap B) \\ \delta' \downarrow & & \downarrow \delta \\ H(N) & \xrightarrow{\rho} & H(M). \end{array}$$

It follows that $v = \delta w' + w \mid M = (\delta' w' + w) \mid M$. Therefore, $\delta' w' + w \in H(N)$ is such that

$$(\delta' w' + w) \mid A = ((\delta' w' + w) \mid M) \mid A = v \mid A = u.$$

This proves that the map ρ in the Proposition is an epimorphism.

(2) To show ρ is a monomorphism assume $u \in H(N)$ where N is a closed co-Φ nbhd of A in X is such that $u \mid A = 0$. Since H has supports

in Φ, $N = B \cup C$ with B closed, $C \in \Phi$ and $u \mid B = 0$. By continuity of H there is also a closed nbhd M of A in N such that $u \mid M = 0$. There is an exact sequence

$$H(M \cap B) \xrightarrow{\delta} H(M \cup B) \xrightarrow{\alpha} H(M) \oplus H(B)$$

and $u \mid (M \cup B)$ is in $\ker \alpha$ so there is $v \in H(M \cap B)$ with $\delta v = u \mid (M \cup B)$. By (1) above there are a closed co-Φ nbhd L of $M \cap B$ in N and $w \in H(L)$ such that $w \mid M \cap B = v$. Clearly $L = (M \cup L) \cap (B \cup L)$ and there is a commutative diagram

$$\begin{array}{ccccc} H(L) & \xrightarrow{\delta'} & H(M \cup L \cup B) & \xrightarrow{\alpha'} & H(M \cup L) \oplus H(B \cup L) \\ \rho' \downarrow & & \rho \downarrow & & \downarrow \rho'' \\ H(M \cap B) & \xrightarrow{\delta} & H(M \cup B) & \xrightarrow{\alpha} & H(M) \oplus H(B) \end{array}$$

Since $M \cup B$, $M \cup L$ are closed co-Φ nbhds of A in N (because $\overline{X - M \cup B} = \overline{X - M} \cap \overline{X - B} \subset \overline{X - B} = \overline{X - N} \cup \overline{N - B} \in \Phi$ and $\overline{X - M \cup L} \subset \overline{X - L} \in \Phi$), it follows that $D = (M \cup B) \cap (M \cup L)$ is a closed co-Φ nbhd of A in N. Clearly

$$\begin{aligned} u \mid D &= (u \mid (M \cup B)) \mid D = (\delta v) \mid D = (\delta \rho' w) \mid D = (\rho \delta' w) \mid D \\ &= ((\delta' w) \mid (M \cup L)) \mid D = 0 \mid D = 0 \end{aligned}$$

proving that ρ is a monomorphism. □

3. Existence of cohomology with given supports. In this Section we show how to obtain cohomology theories on a normal space X with supports in a given nbhd family of supports Φ from an ES theory on X. We begin by recalling the definition of an ES theory on X and some of its properties. See **[8]** for more details.

As *ES theory* H, δ^* on X consists of:

(i) a contravariant functor H from $\mathrm{cl}(X)^2$ (the category of closed pairs in X and inclusion maps between them) to the category of graded abelian groups, and

(ii) a natural transformation

$$\delta^*: H(B, \varnothing) \to H(A, B)$$

of degree 1 for every closed pair (A, B) in X, such that the following are satisfied:

Continuity. For every closed A in X there is an isomorphism

$$\rho: \varinjlim \{ H(N, \varnothing) \mid N \text{ a closed nbhd of } A \text{ in } X \} \approx H(A, \varnothing)$$

where $\rho\{u\} = u \mid (A, \varnothing)$ for $u \in H(N, \varnothing)$.

Exactness. For every closed pair (A, B) in X the following sequence is exact

$$\dots \xrightarrow{\delta^*} H^q(A,B) \xrightarrow{H(j)} H^q(A,\varnothing) \xrightarrow{H(i)} H^q(B,\varnothing) \xrightarrow{\delta^*} H^{q+1}(A,B) \to \dots$$

where $i\colon (B, \varnothing) \subset (A, \varnothing)$ and $j\colon (A, \varnothing) \subset (A, B)$.

Excision. For closed sets A, B in X there is an isomorphism

$$\rho\colon H(A \cup B, B) \approx H(A, A \cap B)$$

It is standard [2] that if (A, B, C) is a closed triple in X there is a corresponding exact sequence of the triple and that there is a cohomology theory H', δ' on X with $H'(A) = H(A, \varnothing)$ for $A \in \mathrm{cl}(X)$. An *ES* theory has *supports* in a family Φ if the corresponding cohomology theory has supports in Φ.

The following will be useful in constructing *ES* theories with supports in Φ.

LEMMA (3.1). *Let Φ be a nbhd family of supports on a normal space X. If A is closed and N is a co-Φ nbhd of A in X, there is a closed co-Φ nbhd of A contained in the interior of N.*

Proof. By hypothesis A is disjoint from $\overline{X - N} \in \Phi$. Let $M \in \Phi$ be a nbhd of $\overline{X - N}$. Since A and $\overline{X - N}$ are disjoint closed sets in X there exist disjoint closed nbhds A' of A and B' of $\overline{X - N}$. Then $B' \cap M \in \Phi$ is a nbhd of $\overline{X - N}$ disjoint from A'. Therefore, $N' = \overline{X - B' \cap M}$ is a closed nbhd of A contained in X-$\overline{X - N}$ = interior of N and $\overline{X - N'} \subset B' \cap M \in \Phi$ so N' is a co-Φ nbhd of A contained in the interior of N. □

THEOREM (3.2). *Let H, δ^* be an ES theory on a normal space X and let Φ be a nbhd family of supports on X. Then there is an ES theory H_Φ, δ^*_Φ on X with supports in Φ where*

$$H_\Phi(A,B) = \varinjlim \{H(M,N) \mid (M,N) \text{ a closed co-}\Phi \text{ nbhd of } (A,B) \text{ in } X\}.$$

Proof. Note that the intersection of two closed co-Φ nbhds of (A, B) is a closed co-Φ nbhd of (A, B) so the collection of closed co-Φ nbhds of (A, B) is directed downward by inclusion and we can define

$$H_\Phi(A,B) = \varinjlim \{H(M,N) \mid (M,N) \text{ a closed co-}\Phi \text{ nbhd of } (A,B) \text{ in } X\},$$

and H_Φ is a contravariant functor on $\mathrm{cl}(X)^2$.

Consider closed triples (M, N, P) of co-Φ sets such that (M, N) is a nbhd of (A, B). As (M, N, P) vary over the collection of all such triples (which is directed downward by inclusion) note that:

(1) (M, N) varies over all closed co-Φ nbhds of (A, B) in X (to such (M, N) there is the triple (M, N, N)),

(2) (M, P) varies over all closed co-Φ nbhds of $(A, \varnothing)$ (to such (M, P) there is the triple (M, M, P)), and

(3) (N, P) varies over all closed co-Φ nbhds of $(B, \varnothing)$ (to such (N, P) there is the triple (X, N, P)).

Corresponding to such a triple (M, N, P) there is an exact sequence

$$\dots \to H^q(M, N) \to H^q(M, P) \to H^q(N, P) \xrightarrow{\delta^*} H^{q+1}(M, N) \to \dots$$

Taking the direct limit of these exact sequences over all such triples (M, N, P) and using (1), (2), (3) we obtain an exact sequence

$$\dots \to H^q_\Phi(A, B) \to H^q_\Phi(A, \varnothing) \to H^q_\Phi(B, \varnothing) \xrightarrow{\delta^*_\Phi} H^{q+1}_\Phi(A, B) \to \dots$$

This defines the natural transformation δ^*_Φ of degree 1 such that H_Φ, δ^*_Φ satisfy exactness.

We verify excision. Given closed sets A, B in X let (M, N) be a closed co-Φ nbhd of $(A, A \cap B)$ in X and (M', N') a closed co-Φ nbhd of $(A \cup B, B)$ in X. Then A-int $N \cap N'$ and B-int $N \cap N'$ are disjoint closed subsets of M' so there exist disjoint closed nbhds E of A-int $N \cap N'$ and F of B-int $N \cap N'$ in M'. Then $M'' = [E \cup (N \cap N')] \cap M$ is a closed co-Φ nbhd of A contained in $M' \cap M$ and $N'' = (F \cup N) \cap N'$ is a closed co-Φ nbhd of B contained in N' such that $M'' \cup N'' \subset M'$ and $M'' \cap N'' = N \cap N' \cap M \subset N$. Thus, $(M'', N'' \cap M'')$ is a closed co-Φ nbhd of $(A, A \cap B)$ contained in (M, N) and $(M'' \cup N'', N'')$ is a closed co-Φ nbhd of $(A \cup B, B)$ contained in (M', N'). Since H satisfies excision,

$$H(M'' \cup N'', N'') \approx H(M'', M'' \cap N'').$$

Since this isomorphism is valid for a cofinal system of closed co-Φ nbhds of $(A \cup B, B)$ and of $(A, A \cup B)$ on taking direct limits we obtain an isomorphism

$$H_\Phi(A \cup B, B) \approx H_\Phi(A, A \cap B).$$

To complete the proof we show that H_Φ is continuous and has supports in Φ. Since $H_\Phi(\varnothing, \varnothing) = 0$, in view of Proposition (2.6) it suffices to verify that the homomorphism

$$\rho\colon \varinjlim \{ H_\Phi(M, \varnothing) \mid M \text{ a closed co-}\Phi \text{ nbhd of } A\} \to H_\Phi(A, \varnothing)$$

is an isomorphism.

Let $u \in H_\Phi(A, \varnothing)$. By definition of H_Φ there is a closed co-Φ nbhd (M, N) of $(A, \varnothing)$ and $v \in H(M, N)$ such that $u = \{v\}_{(A,\varnothing)}$. By Lemma (3.1) there is a closed co-Φ nbhd M' of A contained in $\operatorname{int} M$. Then v determines $\{v\}_{(M',\varnothing)} \in H_\Phi(M', \varnothing)$ such that

$$\{v\}_{(M',\varnothing)} | (A, \varnothing) = \{v\}_{(A,\varnothing)} = u$$

proving that ρ is an epimorphism.

To show that ρ is a monomorphism let $u \in H_\Phi(M, \varnothing)$ be such that $u|(A, \varnothing) = 0$ where M is a closed co-Φ nbhd of A. By definition of H_Φ there is a closed co-Φ nbhd (M', N') of $(M, \varnothing)$ and $v \in H(M', N')$ such that $u = \{v\}_{(M,\varnothing)}$. Since $0 = u|(A, \varnothing) = \{v\}_{(A,\varnothing)}$ there is a closed co-Φ nbhd (M'', N'') of $(A, \varnothing)$ contained in (M', N') such that $v|(M'', N'') = 0$. Since $M \cap M''$ is a closed co-Φ nbhd of A it follows from Lemma (3.1) that there is a closed co-Φ nbhd P of A contained in $\operatorname{int}(M \cap M'')$. Then $(M \cap M'', M \cap N'')$ is a closed co-Φ nbhd of $(P, \varnothing)$ and

$$u|(P, \varnothing) = \{v\}_{(M,\varnothing)}|(P, \varnothing) = \{v|(M \cap M'', M \cap N'')\}_{(P,\varnothing)} = 0$$

showing that ρ is a monomorphism. □

The following is an interesting alternate description of the functor H_Φ defined in Theorem (3.2).

PROPOSITION (3.3). *Let H, δ^* be an ES theory on a normal space X and Φ a nbhd family of supports on X. Let H_Φ be the contravariant functor on* $\operatorname{cl}(X)^2$ *defined in Theorem* (3.2). *For any* $(A, B) \in \operatorname{cl}(X)^2$ *there is an isomorphism*

$$\rho' \colon H_\Phi(A, B) \approx \varinjlim_{B'} \{H(A, B') \mid B' \text{ closed}, B \subset B' \subset A \text{ and } \overline{A - B'} \in \Phi\}$$

where $\rho'\{v\}_{(A,B)} = \{v|(A, N \cap A)\}'$ *for* $v \in H(M, N)$, (M, N) *a closed co-Φ nbhd of* (A, B).

Proof. In the above and in the proof we use $\{\ \}$ to denote elements of $H_\Phi(A, B)$ and $\{\ \}'$ to denote elements of the direct limit which is the codomain of ρ'. It is clear that ρ' as defined above is a homomorphism.

We show ρ' is an epimorphism. Let $\{v\}'_{(A,B)} \in \varinjlim_{B'}\{H(A, B') \mid B'$ closed, $B \subset B' \subset A, \overline{A - B'} \in \Phi\}$ where $v \in H(A, B')$. Since $\overline{A - B'} \in \Phi$, $X - (\overline{A - B'})$ is a co-Φ nbhd of $\varnothing$ in X. By Lemma (3.1) there is a closed co-Φ nbhd N of $\varnothing$ contained in $\operatorname{int}(X - (\overline{A - B'})) = X - (\overline{A - B'})$. Then A and $B' \cup N$ are closed sets such that

$$A \cap (B' \cup N) = [\overline{A - B'} \cup B'] \cap [B' \cup N] = B'.$$

Therefore, there is an excision isomorphism

$$H(A \cup (B' \cup N), B' \cup N) \approx H(A, B').$$

Let $v' \in H(A \cup N, B' \cup N)$ be such that $v'|(A, B') = v$. By continuity of H there is a closed nbhd (M, M') of $(A \cup N, B' \cup N)$ and $v'' \in H(M, M')$ such that $v''|(A \cup N, B' \cup N) = v'$. Then (M, M') is a closed co-Φ nbhd of (A, B) so $\{v''\}_{(A,B)} \in H_\Phi(A, B)$ and $\rho'\{v''\}_{(A,B)} = \{v''|(A, A \cap M')\}'_{(A,B)} = \{v''|(A, B')\}'_{(A,B)} = \{v\}'_{(A,B)}$. So ρ' is an epimorphism.

To show ρ' is a monomorphism let $u \in H_\Phi(A, B)$ be such that $\rho'(u) = 0$ and let (M, N) be a closed co-Φ nbhd of (A, B) and $v \in H(M, N)$ be such that $u = \{v\}_{(A,B)}$. Then

$$0 = \rho'(u) = \{v|(A, A \cap N)\}'_{(A,B)}$$

so there is closed B', $B \subset B' \subset A \cap N$, $\overline{A - B'} \in \Phi$ such that $v|(A, B') = 0$. Then $N \cap [X - \overline{A - B'}]$ is a co-Φ nbhd of $\varnothing$ in X. By Lemma (3.1) there is a closed co-Φ nbhd N' of $\varnothing$ contained in

$$\operatorname{int}(N \cap [X - \overline{A - B'}]) = (\operatorname{int} N) \cap [X - \overline{A - B'}].$$

Then $A \cup (B' \cup N') = A \cup N'$ and $A \cap (B' \cup N') = B'$ so there is an excision isomorphism

$$H(A \cup N', B' \cup N') \approx H(A, B').$$

Since $v|(A, B') = 0$, it follows that $v|(A \cup N', B' \cup N') = 0$. By continuity of H, there is a closed nbhd (M'', N'') of $(A \cup N', B' \cup N')$ in (M, N) such that $v|(M'', N'') = 0$. Then (M'', N'') is a closed co-Φ nbhd of (A, B) contained in (M, N) and

$$u = \{v\}_{(A,B)} = \{v|(M'', N'')\}_{(A,B)} = 0$$

proving that ρ' is a monomorphism. □

4. Cohomology of open subsets. We consider relations between cohomology theories on a space X and cohomology theories on subsets Y of X.

A *cohomology theory* H, δ on a space X is said to be *concentrated* on a subset $Y \subset X$ if $H(A) = 0$ for all closed $A \subset X - Y$. An *ES* theory is said to be *concentrated* on Y if the corresponding cohomology theory is concentrated on Y.

EXAMPLE (4.1). Let Y be a closed subset of a normal space X and let H, δ be a cohomology theory on X. The restriction of H, δ to Y is a cohomology theory $\overline{H}$, $\bar{\delta}$ on Y and the direct image of $\overline{H}$, $\bar{\delta}$ under the closed continuous map i: $Y \subset X$ is a cohomology theory H', δ' with $H'(A) = H(A \cap Y)$. Clearly H' is concentrated on Y.

The following shows how to obtain cohomology theories concentrated on an open subset $Y \subset X$ given an ES theory on X.

PROPOSITION (4.2). *Let H, δ^* be an ES theory on X and let Y be an open subset of X. There is an ES theory H', δ' concentrated on Y with $H'(A,B) = H(A \cup (X - Y), B \cup (X - Y))$ for closed (A, B) in X.*

Proof. H' as defined in the statement of the Proposition is clearly a contravariant functor on $\mathrm{cl}(X)^2$. The exact cohomology sequence of the triple $(A \cup (X - Y), B \cup (X - Y), X - Y)$ in H, δ^* becomes the exact cohomology sequence of the pair (A, B) in H', δ' (this defines the natural transformation

$$\delta'\colon H'(B, \varnothing) \to H'(A, B)$$

of degree 1 such that H', δ' satisfy exactness). Excision for H', δ' follows from excision for H, δ^*. To verify continuity for H', δ' note that

$$H'(A, \varnothing) = H(A \cup (X - Y), X - Y) \approx H(A, A \cap (X - Y)).$$

As N varies over closed nbhds of A in X, $N \cap (X - Y)$ varies over closed nbhds of $A \cap (X - Y)$ in $X - Y$. It follows from continuity of H that

$$\rho\colon \varinjlim \{ H(N, N \cap (X - Y)) \mid N \text{ a closed nbhd of } A \text{ in } X\}$$
$$\approx H(A, A \cap (X - Y)).$$

This implies that

$$\rho\colon \varinjlim \{ H'(N, \varnothing) \mid N \text{ a closed nbhd of } A \text{ in } X\} \approx H'(A, \varnothing)$$

and so H' satisfies continuity.

Thus, H', δ' is an ES theory on X. It is concentrated on Y for if $A \subset X - Y$ then

$$H'(A, \varnothing) = H(A \cup (X - Y), X - Y) = H(X - Y, X - Y) = 0. \qquad \square$$

LEMMA (4.3). *If H, δ is a cohomology theory on X concentrated on an open set $Y \subset X$, then for every closed $A \subset X$, there is an isomorphism*

$$\rho'\colon H(A \cup (X - Y)) \approx H(A).$$

Proof. This is immediate from exactness of

$$0 = H(A \cap (X - Y)) \xrightarrow{\delta} H(A \cup (X - Y)) \xrightarrow{\alpha} H(A) \oplus H(X - Y) \to 0$$

and the fact that $H(X - Y) = 0$. □

The following relates cohomology concentrated on an open subset and cohomology having supports in a nbhd family.

PROPOSITION (4.4). *Let H, δ be a cohomology theory on X, Φ be a nbhd family of supports on X, and Y an open subset of X. Then H, δ has supports in Φ and is concentrated on Y if and only if H, δ has supports in $\Phi \mid Y$.*

Proof. If H, δ has supports in $\Phi|$Y, it clearly has supports in the larger family Φ. We show it is concentrated on Y. Assume $A \subset X - Y$ and $u \in H(A)$. Since H has supports in $\Phi \mid Y$, $A = B \cup C$ where B is closed, $C \in \Phi \mid Y$ and $u \mid B = 0$. Since $C \subset A \cap Y = \varnothing$, $B = A$ so $u = u \mid B = 0$. Therefore, $H(A) = 0$ so H is concentrated on Y.

Conversely, assume H has supports in Φ and is concentrated on Y. Let $u \in H(A)$ where A is closed in X. By Lemma (4.3) there is $u' \in H((A \cup (X - Y)))$ such that $u' \mid A = u$. Since Φ is a nbhd family, it follows from Proposition (2.7) that there is a closed co-Φ nbhd N of $A \cup (X - Y)$ in X and an element $v \in H(N)$ such that $v \mid [A \cup (X - Y)] = u'$. Since H is concentrated on Y, $v \mid (X - Y) = 0$. Again, by Proposition (2.7), there is a closed co-Φ nbhd M of $X - Y$ contained in N such that $v \mid M = 0$. Since M is a nbhd of $X - Y$, $\overline{X - M} \subset Y$ so $\overline{X - M} \in \Phi \mid Y$. Then $A = (A \cap M) \cup (\overline{A - M})$ where $A \cap B$ is closed,

$$\overline{A - M} = A \cap \overline{X - M} \in \Phi \mid Y$$

and

$$u \mid (A \cap M) = (v \mid A) \mid (A \cap M) = (v \mid M) \mid (A \cap M) = 0.$$

Hence, H has supports in $\Phi \mid Y$. □

The next result asserts, for Y open in a normal space X and Φ a nbhd family of supports on X, that cohomology theories on X with supports in $\Phi \mid Y$ are essentially the same as cohomology theories on Y with supports in $\Phi \mid Y$.

THEOREM (4.5). *Given Y open in a normal space X and given Φ a nbhd family of supports on X, there is a bijection between cohomology theories H, δ on X with supports in $\Phi \mid Y$ and cohomology theories H', δ' on Y with supports $\Phi \mid Y$ such that, for A closed in Y, $H'(A) = H(A \cup (X - Y))$.*

Proof. Given H, δ on X for A closed in Y define $H'(A) = H(A \cup (X - Y))$, and for closed A, B in Y define δ': $H'(A \cap B) \to H'(A \cup B)$ to equal

$$\delta\colon H([A \cup (X - Y)] \cap [B \cup (X - Y)]) \\ \to H([A \cup (X - Y)] \cup [B \cup (X - Y)]).$$

Then δ' is a natural transformation of degree 1 such that H', δ' satisfy MV exactness. By Proposition (4.4), H is concentrated on Y so that

$$H'(\varnothing) = H(X - Y) = 0.$$

By Proposition (2.7) there is an isomorphism

$$\rho\colon \varinjlim \{ H(N) \mid N \text{ a closed co-}\Phi \mid Y \text{ nbhd of } A \cup (X - Y) \text{ in } X \} \\ \approx H(A \cup (X - Y)).$$

It is clear that N is a closed co-$\Phi \mid Y$ nbhd of $A \cup (X - Y)$ if and only if $N = (N \cap Y) \cup (X - Y)$ where $M = N \cap Y$ is a closed co-$\Phi \mid Y$ nbhd of A in Y. Therefore, there is an isomorphism

$$\rho\colon \varinjlim \{ H'(M) \mid M \text{ a closed co-}\Phi \mid Y \text{ nbhd of } A \text{ in } Y \} \approx H'(A).$$

By Proposition (2.6) H' is continuous on Y and has supports in $\Phi \mid Y$. Therefore, H', δ' is a cohomology theory on Y with supports in $\Phi \mid Y$.

Conversely, let H', δ' be a cohomology theory on Y with supports in $\Phi \mid Y$. Define a contravariant functor H on cl(X) by $H(A) = H'(A \cap Y)$ for A closed in X. Also, for A, B closed in X define δ: $H(A \cap B) \to H(A \cup B)$ to equal

$$\delta'\colon H'((A \cap Y) \cap (B \cap Y)) \to H'((A \cap Y) \cup (B \cap Y)).$$

Then δ is a natural transformation of degree 1 such that H, δ satisfy MV exactness. By definition, if $A \subset X - Y$, $H(A) = H'(A \cap Y) = H'(\varnothing) = 0$.

By Proposition (2.7) there is an isomorphism

$$\rho\colon \varinjlim \{ H'(M) \mid M \text{ a closed co-}\Phi \mid Y \text{ nbhd of } A \cap Y \text{ in } Y \} \\ \approx H'(A \cap Y).$$

It is clear that M is a closed co-$\Phi \mid Y$ nbhd of $A \cap Y$ in Y if and only if $N = M \cup (X - Y)$ is a closed co-$\Phi \mid Y$ nbhd of A in X. Therefore, there is an isomorphism

$$\rho\colon \varinjlim \{ H(N) \mid N \text{ a closed co-}\Phi \mid Y \text{ nbhd of } A \text{ in } X \} \approx H(A).$$

By Proposition (2.6), H is continuous and has supports in $\Phi \mid Y$. Therefore, H, δ is a cohomology theory on X with supports in $\Phi \mid Y$.

Given H, δ on X let H', δ' be the corresponding cohomology theory on Y and $\bar{H}, \bar{\delta}$ the cohomology theory on X corresponding to H', δ'. Then for closed $A \subset X$,

$$\bar{H}(A) = H'(A \cap Y) = H((A \cap Y) \cup (X - Y)) = H(A \cup (X - Y)).$$

Since H is concentrated on Y by Proposition (4.4), Lemma (4.3) implies that $H(A \cup (X - Y)) \approx H(A)$. Thus, $\bar{H} \approx H$ and similarly $\bar{\delta}$ corresponds to δ.

Given H', δ' on Y let H, δ be the corresponding cohomology theory on X and H'', δ'' the cohomology theory on Y defined by H, δ. Then, for A closed in Y,

$$H''(A) = H(A \cup (X - Y)) = H'([A \cup (X - Y)] \cap Y) = H'(A)$$

and similarly δ'' corresponds to δ. Thus, the passage from H, δ on X to H', δ' on Y is a bijection of cohomology theories with supports in $\Phi \mid Y$. □

Combining the last two results we obtain:

COROLLARY (4.6). *If Y is an open subset of a normal space X and Φ is a nbhd family on X there is a bijection between cohomology theories H, δ on X with supports in Φ concentrated on Y and cohomology theories H', δ' on Y with supports in $\Phi \mid Y$.* □

5. Compact and paracompact supports. We consider the special cases of the nbhd family of compact supports in a locally compact space and the nbhd family of relatively paracompact supports in a locally paracompact space. We also consider the uniqueness theorem for compactly supported and paracompactly supported cohomology.

Given a topological space X let Φ_c be the nbhd family of all closed subsets of X having a compact nbhd in X and let $X_{lc} = \bigcup\{A \in \Phi_c\}$. Then X_{lc} is the union of all locally compact open subsets of X so is the largest open subset of X which is locally compact. Clearly $\Phi_c \mid X_{lc} = \Phi_c$ and Φ_c is exactly the family of all compact subsets in X_{lc}. It follows from Theorem (4.5) that cohomology theories on X with supports in Φ_c correspond bijectively to compactly supported cohomology theories on the locally compact space X_{lc}. Thus, the study of cohomology theories with supports in Φ_c is reduced to the study of compactly supported cohomology theories on locally compact spaces.

Our next result implies that the compactly supported cohomology theories on a locally compact space correspond to cohomology theories on its one-point compactification which are concentrated on the space.

THEOREM (5.1). *Let X be an open subset of a compact space Z. There is a bijection between compactly supported cohomology theories on X and cohomology theories on Z which are concentrated on X.*

Proof. If Φ is the nbhd family of all closed (or, equivalently, compact) subsets of Z, then every cohomology theory on Z has supports in Φ. Clearly $\Phi \mid X = \Phi_c$ the nbhd family of all compact subsets of X. The theorem follows from Corollary (4.6). □

We consider similar definitions for the paracompact rather than the compact case. Given a space X let Φ_p be the family of all closed subsets of X having a paracompact nbhd in X. Clearly X_p is a nbhd family of supports and if $X_{lp} = \bigcup\{A \in \Phi_p\}$, then X_{lp} is the largest open subset of X which is locally paracompact. Obviously $\Phi_p \mid X_{lp} = \Phi_p$, but in this case Φ_p is *not* the family of all paracompact subsets of X_{lp} but is the family of all closed subsets of X_{lp} having paracompact nbhds in X_{lp}. It can be shown that this family is identical to the family of all relatively paracompact subsets of X_{lp} (a subset A of a space X is *relatively paracompact* if given a collection $\mathscr{U}$ of open subsets of X covering A there is a collection $\mathscr{V}$ of open subsets of X covering A which refines $\mathscr{U}$ and is locally finite in X). In case X is paracompact the family $\Phi_p =$ the family of all closed sets.

It follows from Theorem (4.5) that cohomology theories on X with supports in Φ_p correspond bijectively to cohomology theories on the locally paracompact space X_{lp} with relatively paracompact supports. The following implies that cohomology theories with relatively paracompact supports on a locally paracompact space correspond to cohomology theories on its one-point paracompactification which are concentrated on the space.

THEOREM (5.2). *Let X be an open subset of a paracompact space Z. There is a bijection between cohomology theories on X with relatively paracompact supports and cohomology theories on Z which are concentrated on X.*

Proof. Analogously to Theorem (5.1) this follows from Corollary (4.6). □

Let φ: H, $\delta \to H'$, δ' be a homomorphism between compactly supported cohomology theories on the same space X such that for some $n \in \mathbf{Z}$, φ_x: $H(x) \to H'(x)$ is an n-equivalence for all $x \in X$. We would like to deduce that φ_A: $H(A) \to H'(A)$ is an n-equivalence for all closed $A \subset X$. In case H, H' are nonnegative if follows from [**7**, Theorem 3.1] and in case X is a finite dimensional separable metric case it follows from [**8**, Corollary (4.3)].

Now suppose φ: H, $\delta \to H'$, δ' is a homomorphism between additive paracompactly supported cohomology theories on X such that for some $n \in \mathbf{Z}$, φ_x: $H(x) \to H'(x)$ is an n-equivalence for all $x \in X$. We would like to deduce that φ_A: $H(A) \to H'(A)$ is an n-equivalence for all closed $A \subset X$. In case H, H' are nonnegative it follows from [**7**, Theorem 4.1] and Theorem (2.5). In [**8**, Corollary (4.7)] it was shown to follow if X is a locally compact finite dimensional separable metric space. Below we show that the hypothesis of local compactness is unnecessary. First we prove a result about finite dimensional spaces. We use the definition of dimension denoted Ind in [**5**]. Thus, X has dimension -1 if $X = \varnothing$, and for $m \geq 1$, X has dimension $\leq m$ if every pair of disjoint closed subsets of X can be separated by a closed set of dimension $\leq m - 1$.

LEMMA (5.3). *Let A, B be closed subsets of an m-dimensional space X such that $A \cup B = \operatorname{int}_{A \cup B} A \cup \operatorname{int}_{A \cup B} B$. Then there exist closed sets A', B' of X with $A' \subset A$, $B' \subset B$, $A \cup B = A' \cup B'$ and* $\dim A' \cap B' < m$.

Proof. Let $C = A \cup B\text{-}\operatorname{int}_{A \cup B} B$ and $D = A \cup B\text{-}\operatorname{int}_{A \cup B} A$. Then C, D are disjoint closed subsets of $A \cup B$. Since $\dim(A \cup B) \leq m$, there exists a closed subset $E \subset A \cup B$ with $\dim E < m$ which separates C, D. Therefore, $A \cup B - E = C' \cup D'$ where C', D' are each open in $A \cup B - E$ (so open in $A \cup B$), $C' \cap D' = \varnothing$, and $C \subset C'$, $D \subset D'$. Let $A' = C' \cup E$, $B' = D' \cup E$. Then A', B' are closed subsets of $A \cup B$ (so closed in X), $A' \subset A \cup B - D = \operatorname{int}_{A \cup B} A \subset A$, and similarly $B' \subset B$, $A' \cup B' = A \cup B$ and

$$\dim A' \cap B' = \dim[(C' \cup E) \cap (D' \cup E)] = \dim E < m. \qquad \square$$

THEOREM (5.4). *Let φ: H, $\delta \to H'$, δ' be a homomorphism between additive cohomology theories on a finite dimensional normal space X both having paracompact supports and suppose there is $n \in \mathbf{Z}$ such that φ_x: $H(x) \to H'(x)$ is an n-equivalence for all $x \in X$. Then φ_A: $H(A) \to H'(A)$ is an n-equivalence for all closed $A \subset X$.*

Proof. By Theorem (2.5) it suffices to prove the result in case A is a paracompact subset of X. We do this by induction on $\dim A$. If $\dim A = 0$, $A = \varnothing$ and φ_A is an isomorphism. Assume the result valid for all paracompact subsets of dimension $< m$ where $m \geq 1$ and let A be a paracompact subset such that $\dim A = m$.

First, we show φ_A: $H^q(A) \to H'^q(A)$ is an epimorphism for $q < n$. Let $u \in H'^q(A)$ be fixed and let $\mathscr{C}$ be the collection of all closed subsets $B \subset A$ such that $u \mid B \in \operatorname{im} \varphi_B$. The hypothesis on φ_x and continuity of H, H' imply that every point of A has a closed nbhd in A which is an element of $\mathscr{C}$. From the definition of $\mathscr{C}$ it is clear that $B' \subset B$, $B \in \mathscr{C}$ imply $B' \in \mathscr{C}$. The additivity of H, H' imply that the union of a discrete family of elements of $\mathscr{C}$ is an element of $\mathscr{C}$.

We prove that B, $B' \in \mathscr{C}$ and

$$B \cup B' = \operatorname*{int}_{B \cup B'} B \cup \operatorname*{int}_{B \cup B'} B'$$

imply $B \cup B' \in \mathscr{C}$. By Lemma (5.3) there exist closed C, C' such that $C \subset B$, $C' \subset B'$, $B \cup B' = C \cup C'$ and $\dim(C \cap C') < m$. The following diagram has exact rows and commutes up to sign

$$\begin{array}{ccccccc}
H^{q-1}(C \cap C') & \xrightarrow{\delta} & H^q(C \cup C') & \xrightarrow{\alpha} & H^q(C) \oplus H^q(C') & \xrightarrow{\beta} & H^q(C \cap C') \\
\varphi \downarrow \approx & & \varphi \downarrow & & \downarrow \varphi & & \approx \downarrow \varphi \\
H'^{q-1}(C \cap C') & \xrightarrow{\delta'} & H'^q(C \cup C') & \xrightarrow{\alpha'} & H'^q(C) \oplus H'^q(C') & \xrightarrow{\beta'} & H'^q(C \cap C')
\end{array}$$

The two vertical maps on the ends are isomorphism because $\dim(C \cap C') < m$, $q < n$, and the inductive hypothesis. It follows [**7**, part 2) of Lemma 2.19] that $\alpha'^{-1}(\operatorname{im} \varphi) \subset \operatorname{im} \varphi$. Since C, $C' \in \mathscr{C}$, $(u \mid C, u \mid C') \in \operatorname{im} \varphi$. Therefore, $u \mid (C \cup C') \in \operatorname{im} \varphi$. Since $C \cup C' = B \cup B'$, $B \cup B' \in \mathscr{C}$. By [**4**, Theorem 5.5] $A \in \mathscr{C}$ so $u \in \operatorname{im} \varphi_A$.

Next, we show φ_A: $H^q(A) \to H'^q(A)$ is a monomorphism for $q \leq n$. Let $u \in \ker \varphi_A$ be fixed and let $\mathscr{C}$ be the collection of all closed subsets $B \subset A$ such that $u \mid B = 0$. Again every point of A has a closed nbhd in A which is an element of $\mathscr{C}$, every closed subset of an element of $\mathscr{C}$ is an element of $\mathscr{C}$, and the union of a discrete family of elements of $\mathscr{C}$ is an element of $\mathscr{C}$. Using Lemma (5.5) as above, it is not hard to show that B, $B' \in \mathscr{C}$ and

$$B \cup B' = \operatorname*{int}_{B \cup B'} B \cup \operatorname*{int}_{B \cup B'} B'$$

imply $B \cup B' \in \mathscr{C}$. It follows again that $A \in \mathscr{C}$. $\square$

REFERENCES

[1] G. Bredon, *Sheaf Theory*, Mc Graw-Hill, New York, N. Y. 1967.

[2] S. Eilenberg and N. E. Steenrod, *Foundations of Algebraic Topology*, Princeton University Press, Princeton, N. J., 1952.

[3] J. D. Lawson, *Comparison of taut cohomologies*, Aeq. Math., **2** (1973), 201–209.

[4] E. Michael, *Local properties of topological spaces*, Duke Math. J., **21** (1954), 163–171.

[5] K. Nagami, *Dimension Theory*, Academic Press, New York, N. Y. 1970.

[6] E. Spanier, *Algebraic Topology*, Springer Verlag, New York, N. Y. 1982.

[7] ——, *Cohomology isomorphisms*, Contemporary Math., **12** (1982), 315–329.

[8] ——, *Cohomology theories on spaces*, to appear.

Received October 22, 1984. The author gratefully acknowledges hospitality by the Institut des Hautes Etudes Scientifiques in Bures-sur-Yvette and partial support by the Max-Planck Institut für Mathematik in Bonn during the preparation of this year.

UNIVERSITY OF CALIFORNIA
BERKELEY, CA 94720