PACIFIC JOURNAL OF MATHEMATICS
Vol. 124, No. 1, 1986

MINIMAL SURFACES AND HEEGAARD SPLITTINGS OF THE THREE-TORUS

CHARLES FROHMAN

An unknotting criterion for proper arcs in $F \times I$ is proved. This is then applied to show the topological uniqueness of least genus one-sided Heegaard splittings of the three-torus and of genus three two-sided minimal surfaces in a flat three-torus.

Lawson has proved that if $F_1, F_2 \subseteq S^3$ are embedded minimal surfaces having the same genus in metrics having positive Ricci curvature, then there exists a homeomorphism $h\colon S^3 \to S^3$ with $h(F_1) = F_2$. Meeks has proved that up to homeomorphism there is at most one minimal surface F of genus g in a flat, convex three-ball B having a given Jordan curve $\gamma \subseteq \partial B$ as boundary. The outlines of their respective arguments are the same. First it is shown that the minimal surface of interest is a Heegaard splitting, then the topological uniqueness of Heegaard splittings of the three-sphere is used to derive the result. In this paper we prove that up to homeomorphism there is only one surface of genus 3 in the three-torus that can be a minimal surface in a flat metric on the three-torus. The outline of the proof is similar to the proofs of Lawson and Meeks described above, except that because so little is known about the Heegaard splittings of the three-torus, the topological part of our argument is more involved than theirs.

The notion of a one-sided Heegaard splitting was introduced by Rubinstein in [**R**]. In this paper we prove that all least genus, one-sided Heegaard splittings of the three-torus are topologically equivalent. This result and our result concerning genus three minimal surfaces are both proved using the following unknotting lemma. We say a proper arc in $F \times I$ is unknotted if it is isotopic to $* \times I$.

LEMMA 1.1. *Let F be a closed surface of positive genus, and k a proper arc in $F \times I$. The arc k is unknotted if and only if $^-(F \times I - N(k))$ is a handlebody.*

This lemma generalizes Papakyriakopoulos' unknotting lemma for circles in the three-sphere. In the early 1970's, E. Brown [**B**] and C.

Feustal, **[Fe]**, developed results that are similar to Lemma 1.1. Although it is enticing to try to prove our result directly from **[Fe]**, it is not clear that there is any savings in doing so.

We will assume that all spaces have a fixed PL structure and that all maps are PL. Furthermore we assume that the reader is familiar with the theory of incompressible surfaces and Haken manifolds as is described in [**W2**, **W3**]. A *Heegaard splitting* of a closed three-manifold is a surface $F \subseteq M$ such that $M - F$ is the union of two open handlebodies. A *one-sided Heegaard splitting* is a surface $K \subseteq M$ such that $M - K$ is an open handlebody. We say that two (one-sided) Heegaard splittings F and F' of M and M' are topologically equivalent if there exists a homeomorphism $h\colon M \to M'$ such that $h(F) = F'$. We say that two subsets X, X' of M are *isotopic* if there exists an isotopy $h\colon M \times I \to M$ such that $h_0 = \mathrm{Id}$ and $h_1(X) = X'$. We say that a (one-sided) Heegaard splitting F has *least genus* if the genus of F is minimal among all (one-sided) Heegaard splittings of M.

1. The unknotting lemma. The purpose of this section is to prove the following lemma.

LEMMA 1.1. *Let F be a closed surface of positive genus, and k a proper arc in $F \times I$. The arc k is unknotted if and only if $^-(F \times I - N(k))$ is a handlebody.*

An easy corollary of this is:

COROLLARY 1.2. *Let F be a closed surface of positive genus, and k a proper arc in $F \times I$. The arc k is unknotted if and only if $\pi_1(F \times I - N(k), *)$ is free.*

□

REMARK. When F is homeomorphic to S^2 the lemma is false. For F homeomorphic to $S^1 \times S^1$ we have an algebraic proof based on the fact that one-relator presentations of $\pi_1(S^1 \times S^1, *)$ are "standard". The idea for the proof we give here is due to Marty Scharlemann.

Proof. First assume that k is a proper arc in $F \times I$ that is isotopic to $* \times I$. We then know that $^-(F \times I - N(k))$ is homeomorphic to $^-((F - N(*)) \times I)$ where $N(*)$ is a regular neighborhood of $*$ in F. Let $p\colon\ ^-(F - N(*)) \times I \to {}^-(F - N(*))$ be the natural projection.

Choose a family of nontrivial arcs $\{c_i\}$ on $^-(F - N(*))$ that are pairwise disjoint and cut $^-(F - N(*))$ into a disk. Notice that $\{p^{-1}(c_i)\}$ is a system of meridian disks cutting $^-((F - N(*)) \times I)$ into a ball. Therefore $^-(F \times I - N(k))$ is a handlebody.

Now assume that $^-(F \times I - N(k))$ is a handlebody. Let M be a meridian disk for $^-(F \times I - N(k))$ such that ∂M is transverse to $\operatorname{fr} N(k)$ in $\partial^-(F \times I - N(k))$ and the number of components of $\partial M \cap \operatorname{fr} N(k)$ is minimal. Since $F \times \{i\} \subseteq F \times I$, $i \in \{0, 1\}$, is incompressible we see that $^-(F \times \{i\} - N(k))$ is an incompressible surface in $^-(F \times I - N(k))$. Thus the number of components of $\partial M \cap \operatorname{fr} N(k)$ is at least 2. The regular neighborhood $N(k)$ supports a product structure $k \times D^2$ where k is the set $k \times \{0\}$. Since $\partial M \cap \operatorname{fr} N(k)$ has the minimum number of components we may assume that in our coordinate system $\partial M \cap \operatorname{fr} N(k)$ $= k \times \{d_i\}_{i=1}^n$, where $d_i \in \partial D^2$. Viewing D^2 as the cone of ∂D^2 on 0 we construct $\overline{M}$ by letting

$$\overline{M} = M \cup \{(\rho d_i, t) \in N(k) | \rho, t \in [0, 1]\}.$$

We can view $\overline{M}$ as a disk $\overline{D}$ with segments of its boundary identified. In specific we may partition the boundary into closed segments $\{b_i\}_{i=1}^n$ and open segments $\{a_i\}_{i=1}^n$ such that the segments b_i are identified to one another (and mapped homeomorphically onto k) and the segments a_i are disjoint from one another and lie in $F \times \{0, 1\}$. See Figure 1 for a schematic description. Throughout this argument when we want to view a

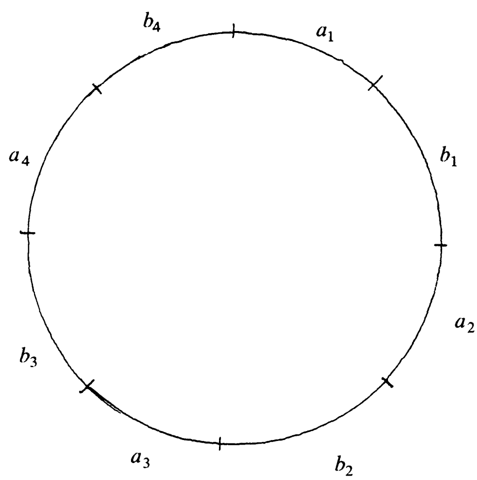

FIGURE 1

subset of $\overline{M}$ as being in $F \times I$ we will refer to it as contained in $\overline{M}$. Because of the inductive nature of our argument we need to view subsets of $\overline{M}$ as subsets of $\overline{D}$, in these cases we will refer to them as subsets of $\overline{D}$.

Let p: $F \times I \to F$ be the canonical projection. A subset S of $F \times I$ is vertical if $p^{-1}(p(S)) = S$. Let $A, V_1, \dots, V_m$ be a system of vertical surfaces in $F \times I$ such that A is a proper annulus, $\partial V_i \subseteq F \times \{0,1\} \cup A$ and $^{-}(F \times I - N(A \cup \bigcup_i V_i))$ is a three-ball. Furthermore we may assume that $\partial k \cap (A \cup \bigcup_i V_i) = \varnothing$, $\overline{M}$ is transverse to A, $\overline{D} \cap A$ consists of only arcs that are proper in $\overline{D}$, and finally the number of components of $\overline{D} \cap A$ is minimal among all compressing disks $\overline{D}$.

If there exists an arc χ in $\overline{D} \cap A$ or $\overline{D} \cap V_i$ such that the endpoints of χ lie in distinct intervals a_i and a_j of $\partial\overline{D}$ with a_i and a_j being adjacent to an interval b_l, then the disk $D' \subseteq \overline{D}$ cut out by χ and containing b_l can be used to isotope k into A or V_i, (see Figure 2). This is enough to show that k is unknotted. Consequently we will call an arc χ as described above an *excellent arc*. The proof of Lemma 1.1 consists of a step by step search for an excellent arc in $\overline{D} \cap A$ or $\overline{D} \cap V_i$ for some i. If there is no excellent arc in $\overline{D} \cap (A \cap V_1 \cdots \cup V_t)$ then we will isotope $\overline{D}$ so that it is in a "nice" position and continue our search in $\overline{D} \cap V_{i+1}$. If we get all the way through $\overline{D} \cap (A \cup \bigcup_i V_t)$ without finding an excellent arc then we derive a contradiction.

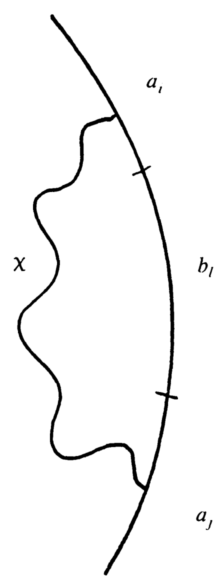

FIGURE 2

Step 1. Assume that there are no excellent arcs in $\overline{D} \cap A$. We will isotope $\overline{D}$ so that $\overline{D} \cap A$ consists only of vertical arcs missing $\bigcup V_i$. We distinguish five different types of arcs in $\overline{D} \cap A$. The first and second types have their endpoints in $\bigcup b_i$. An arc of the third type has its endpoints in some a_i. An arc of the fourth type has one endpoint in some a_i and the other in some b_j. Finally the fifth type of arc has its endpoints in distinct intervals a_i and a_j. If χ is an arc in $\overline{D} \cap A$ of the first four types that is outermost on $\overline{D}$ then we know how to isotope $\overline{M}$ so that $\overline{D} \cap A$ is simplified. If there are no outermost arcs χ of types one through four, then every arc in $\overline{D} \cap A$ is of type five. We show that if this is the case, then $\overline{M}$ can be isotoped so that $\overline{M} \cap A$ is vertical and misses $\bigcup V_i$. This will complete step one of the proof.

Assume that χ is an arc in $\overline{D} \cap A$ that is outermost on $\overline{D}$, and of the first four types. There are four cases to consider.

Case 1. The endpoints of χ lie in some b_i. Let D' be the subdisk of $\overline{D}$ cut out by χ that intersects A only in χ. Since no pairs of distinct points of b_i are identified to one another in forming $\overline{M}$, the disk D' is embedded in $F \times I$. Use this disk as a guide to isotope $\overline{M}$ so that there are two fewer points of intersection in $k \cap A$. Then isotope $\overline{M}$ relative to k to remove any simple closed curves that may have been created in $A \cap \overline{D}$.

Case 2. The endpoints of χ lie in distinct intervals b_i and b_j. Since if $b_i \cap A = \varnothing$ for any i then $b_i \cap A = \varnothing$ for all i, we may assume that the intervals b_i and b_j are adjacent on $\partial\overline{D}$, with some a_k between them.

Since χ is outermost the endpoints of χ must be identified in $\overline{M}$. There are two situations. The first is when χ forms a trivial simple closed curve on A, and the second when χ forms a nontrivial simple closed curve on A. For illustrations of these situations see Figure 3.

Situation a. In this case χ bounds a disk D'' on A and $D'' \cup D'$ form a disk in $F \times I$ having $\bar{a}_k$ as its boundary. Since $\partial(F \times I)$ is incompressible the simple closed curve $\bar{a}_k$ bounds a disk on $\partial(F \times I)$. We may use this disk to isotope M so that $\partial M \cap \operatorname{fr} N(k)$ has fewer components. Hence situation a cannot occur.

Situation b. Since the fundamental group of $F \times I$ is carried by either of its boundary components, we have that $\bar{a}_k$ and one of the boundary components B of A bound an annulus on $\partial(F \times I)$. Furthermore D' is identified to itself along $D' \cap b_i$ and $D' \cap b_j$ to form an annulus, and B

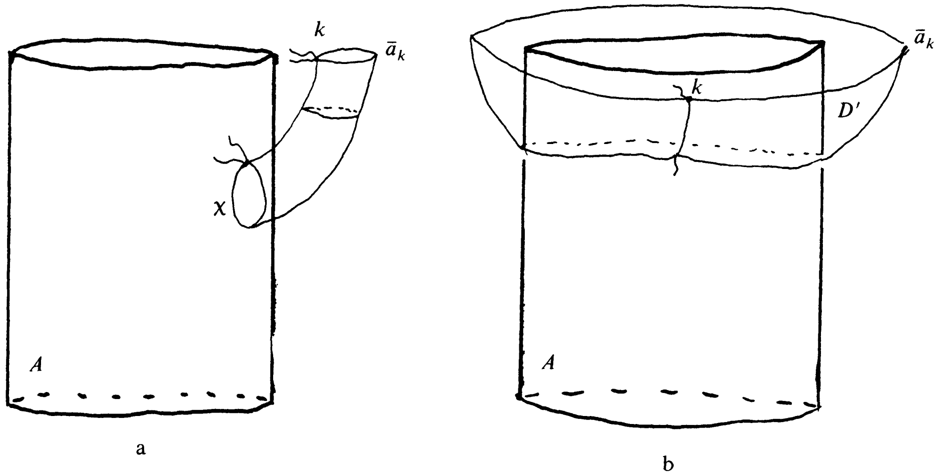

FIGURE 3

and χ bound an annulus on A. These three annulli form the boundary of a solid torus in $F \times I$ that can be used as a guide to isotope $\overline{M}$ so as to reduce the cardinality of $k \cap A$.

Case 3. Suppose χ has both its endpoints in the same a_i. Since $\partial(F \times I)$ is incompressible some arc of a_i along with an arc of ∂A bound a disk in $\partial(F \times I)$, this along with the disks on A and $\overline{D}$ split off by χ bound a ball. Use this as a guide to isotope $\overline{M}$ so as to reduce the number of components of $\overline{D} \cap A$.

Case 4. One endpoint of χ lies in an interval a_i, and the other endpoint of χ lies in an interval b_j. Since χ is outermost the arcs a_i and b_j must be adjacent on $\partial\overline{D}$. Hence the outermost disk D' cut out by χ has boundary consisting of χ, a segment of b_j and a segment of a_i. Use D' as a guide to isotope $\overline{M}$ so as to reduce the cardinality of $k \cap A$. Then isotope away any simple closed curves in $\overline{D} \cap A$ that may have been created.

Because of the constructions above we may assume that if χ is an outermost arc in $\overline{D} \cap A$, then the endpoints of χ lie in distinct intervals a_i and a_j. Let $D' \subseteq \overline{D}$ be an outermost disk cut out by χ. Since the intervals a_i and b_l alternate on $\partial\overline{D}$ there must be some interval b_l in $\partial D'$. But this means that $b_l \cap A = \varnothing$. Since the intervals $\bigcup b_i$ are all identified to one another in $\overline{M}$ we have that there are no arcs in $\overline{D} \cap A$ having an endpoint in $\bigcup b_i$. If there exists an arc in $\overline{D} \cap A$ both of whose endpoints in an interval a_i, then there exists an outermost such arc. Hence our assumption that all the outermost arcs in $\overline{D} \cap A$ are of type five implies that all the arcs in $\overline{D} \cap A$ are of the fifth type. Without looking further we can distinguish two types of such arcs. An arc of type i has both its endpoints in the same component of $\partial(F \times I)$. An arc of type ii has its endpoints in distinct components of $\partial(F \times I)$.

Type i. Let D' be the disk on A that is cut out by χ. By passing to a different χ if necessary we may assume that $\operatorname{int} D' \cap \overline{M} = \varnothing$. Replace one of the components of $\overline{D} - \chi$ by a pushoff of D', call the new singular disk $\overline{M}'$. Notice that the number of components of $\overline{M}' \cap \operatorname{fr} N(k)$ is smaller than the number of components of $\overline{M} \cap \operatorname{fr} N(k)$. This contradicts our choice of $\overline{M}$.

Type ii. We may thus assume that all χ are of type ii. Isotope $\overline{M}$ so that χ is vertical in A and $\chi \cap (\bigcup V_i) = \varnothing$ without increasing the number of components of $\overline{D} \cap A$.

We may thus assume that $k \cap A = \varnothing$ and $\overline{M} \cap A$ consists of vertical arcs that miss $\bigcup V_i$.

Step 2. Assume that if $i < r$ then $\overline{M} \cap V_i$ and $\overline{M} \cap A$ consists of vertical arcs that miss the vertical edges of $\bigcup V_i$. Furthermore assume that there are no excellent arcs in $\overline{D} \cap (A \cup \bigcup_{i<r} V_i)$. Isotope $\overline{M}$ relative to $A \cup V_1 \cdots \cup V_{r-1}$ so that $\overline{M}$ is transverse to V_r and the number of components of $\overline{M} \cap V_r$ is minimal. It is possible to remove and straighten arcs of intersection in $\overline{D} \cap V_r$ as we did with the annulus. It is worth noting that the arcs in Case 2, situation b cannot occur. All the arguments in the other cases still go through. If there are no excellent arcs in $\overline{D} \cap V_r$ continue the process for V_{r+1}.

Suppose that after performing our normalization process for all V_i that there are no excellent arcs in $\overline{D} \cap (A \cup \bigcup_i V_i)$. If we cut $F \times I$ along $A \cup \bigcup_i V_i$ we get a manifold $\widetilde{F \times I}$ that is homeomorphic to $D^2 \times I$ in such a way that $\partial D^2 \times I$ is the image of the frontier of a regular neighborhood $N(A \cup \bigcup_i V_i)$, and the images of the arcs $\overline{M} \cap \partial N(A \cup \bigcup_i V_i)$ are vertical. Let $\tilde{\overline{M}}$ denote the image of $\overline{M}$ in $\widetilde{F \times I}$. Since k is disjoint from $A \cup \bigcup_i V_i$ we have that $\tilde{\overline{M}}$ consists of one component $\tilde{M}'$ that is a singular disk, and some components that are properly embedded disks. Notice that $\partial^-(\widetilde{F \times I} - N(k))$ is a torus and each component of $^-(\tilde{M}' - N(k))$ is a compressing disk. Hence $^-(\widetilde{F \times I} - N(k))$ is a solid torus. Notice that returning $N(k)$ to

$^{-}(\widetilde{F \times I} - N(k))$ is the same as glueing a plate to the solid torus to obtain a three ball. Hence if w is a nontrivial simple closed curve on $\operatorname{fr} N(k)$ then w has algebraic intersection $\pm I$ with any component of $^{-}(\tilde{M}' - N(k))$. This implies that each arc of $\partial\tilde{M}'$ that lies in the vertical part of $\partial F \times I$ is an excellent arc of intersection between $\overline{M}$ and A or $\overline{M}$ and some V_i. Since there is at least one such arc we have a contradiction. □

2. Least genus one-sided Heegaard splittings of the three-torus. The surface of nonorientable genus h, denoted U_h, is the connected sum of h projective planes. Bredon and Wood **[B-W]**, showed that if F is a closed orientable surface of positive genus, then U_4 is the least genus nonorientable surface that embeds in $S' \times F$. The three-torus possesses a one-sided Heegaard splitting of genus 4. To see this, let T be an incompressible torus in $S' \times S' \times S'$. Notice that the three-torus cut along T is isomorphic to $T \times I$. Let k be an unknotted arc in $T \times I$ whose ends are not identified when $T \times I$ is glued back together to form the three-torus. Construct U_4 as follows. Let $N(k)$ be a small regular neighborhood of k in $S' \times S' \times S'$. Remove $N(k) \cap T$ from T and replace it by the closure of the annulus component of $\partial N(k) - T$. From §1 we see that this copy of U_4 that we have constructed is a one-sided Heegaard splitting of the three-torus. Since U_4 is the least genus one-sided surface that embeds in the three-torus we have constructed a least genus one-sided Heegaard splitting of the three-torus. If we perform the same construction starting with an incompressible torus T' that is not $\mathbf{Z}_2$ homologous to T we obtain a one-sided Heegaard splitting of the three-torus by U_4 that cannot be isotopic to our original one-sided Heegaard splitting. We can however show the following.

THEOREM 2.1. *Least genus one-sided Heegaard splittings of the three-torus are unique up to homeomorphism.*

Proof. We will show that if K is a least genus Heegaard splitting of the three-torus then K is obtained by surgering an incompressible torus $T \subseteq S' \times S' \times S'$ along an arc that is unknotted in $T \times I$, the topological uniqueness of least genus one-sided Heegaard splittings of the three-torus will then be evident.

In **[Fr]** it is shown that an orientable S'-bundle over a closed orientable surface contains a one-sided incompressible surface if and only if it has even nonzero Euler class. Hence $K \subseteq S' \times S' \times S'$ is compressible. Let D be a compressing disk for K. Let h: $D \times [0,1] \to S' \times S' \times S'$

be an embedding such that $h|_{D\times\{1/2\}}(D \times \{\frac{1}{2}\}) = D$ and $h^{-1}(K) = \partial D \times [0,1]$. Notice that $T = K - h(\tilde{M}\partial D \times [0,1]) \cup h(D \times \{0,1\})$ must be orientable (Bredon and Wood) and nonseparating (surgering along a disk does not change the $\mathbf{Z}_2$ homology class). Hence T is an incompressible torus. Furthermore K is obtained by surgering T along $h(* \times I)$, where $* \in \operatorname{int} D$. Since $S' \times S' \times S' - K$ is a handlebody, we have that $S' \times S' \times S' - K - h(D^2 \times I)$ is a handlebody. By Lemma 1.1 we have that $h(* \times I)$ is unknotted in $T \times I$. □

It is worth noting that if K is a genus 4 one-sided Heegaard splitting of the three-torus M, then the double cover of the three-torus corresponding to the orientable double cover of K is again a three-torus $\tilde{M}$ and the double cover F of K in $\tilde{M}$ is a genus three Heegaard splitting of the three-torus $\tilde{M}$. Theorem 2.1 implies the topological uniqueness of one-sided genus 4 minimal surfaces in a flat three-torus.

3. Genus three minimal surfaces in a flat three-torus. A flat three-torus is $S^1 \times S^1 \times S^1$ equipped with a Riemannian metric having all its sectional curvatures equal to zero. Given a flat three-torus M the universal cover of M can be realized geometrically as $\mathbf{R}^3$ so that the deck transformations are translations by the vectors in some cocompact lattice $\Gamma \subseteq \mathbf{R}^3$. Notice that M inherits a group structure from a realization of its universal cover, just take the group structure on $\mathbf{R}^3/\Gamma$. Inversion ϕ in such a group structure can be lifted to $\tilde{\phi}: \mathbf{R}^3 \to \mathbf{R}^3$, $\tilde{\phi}(x_1, x_2, x_3) = -(x_1, x_2, x_3)$. Meeks [**M1**, **M2**], shows that a minimal surface F of negative Euler characteristic in a flat three-torus is a Heegaard splitting; furthermore if F has genus three, after a suitable choice of the identity for a group structure on the three-torus, $\phi(F) = F$ and $\phi|_F: F \to F$ is the hyperelliptic involution of the conformal structure on F inherited from the three-torus. A hyperelliptic involution on a closed orientable surface F_g is an involution that is the deck transformation of a branched cover $b: F_g \to S^2$. All hyperelliptic involutions on a given surface are topologically equivalent. If χ is a simple closed curve on S^2 that is disjoint from the branch set of b then χ lifts to F_g if and only if a component of $S^2 - \chi$ contains an even number of branch points.

The quotient space of $S^1 \times S^1 \times S^1$ under the action of ϕ is a three-manifold with 8 singularities which are cones on $\mathbf{P}^2$. Denote by Q the quotient space with the interiors of small regular neighborhoods of the singularities removed. The image of F in Q is a planar surface 0 with eight boundary components, each one a nontrivial curve in one of the eight boundary components of Q. In Lemma 3.4 we will show that 0 is a compressible surface, but for the sake of exposition we assume Lemma 3.4 for the time being and proceed.

THEOREM 3.1. *Up to topological equivalence there is only one genus three surface F in the three-torus that can be a minimal surface in a flat three-torus.*

Proof. From **[M1]** we know that if F is a minimal surface in some flat three-torus M then F is a Heegaard splitting of M. We will show that F is a "standard" Heegaard splitting, which is enough to prove our result.

Let 0 and Q be as above. By Lemma 3.4 the surface 0 is compressible in Q. Let D be a compressing disk for 0. Since ∂D lifts to F we have that ∂D on F/ϕ ($\cong S^2$) must separate F/ϕ into two disks each containing an even number of branch points. This can be seen from the monodromy representation for the hyperelliptic involution. Let D_1 and D_2 be the lifts of D to M. Notice that D_1 and D_2 lie on opposite sides of F. Suppose that one of the components C of $F/\phi - D$ contained only two branch points, then C lifts to an annulus A on F with $\partial A = \partial D_1 \cup \partial D_2$. Notice that $D_1 \cup A \cup D_2$ is a nonseparating sphere in M. This is absurd; hence we may assume that ∂D separates F/ϕ into two components each containing 4 branch points. Thus ∂D_1 and ∂D_2 separate F into two tori having two boundary components apiece. Let h_i: $D_i \times I \to M$ with $D_i = h(D_i \times \{\frac{1}{2}\})$ be embeddings such that $h_i^{-1}(M) = \partial D_i \times I$ & $h_1^{-1}(h_2(D_2 \times I)) = \varnothing$. Surger F by replacing $h_1(\partial D_1 \times I)$ and $h_2(\partial D_2 \times I)$ by $D_1 \times \{0,1\}$ and $D_2 \times \{0,1\}$. The result is two parallel nonseparating tori T_1 and T_2. Hence T_1 and T_2 separate M into two copies of the cartesian product of a torus and an interval. Let $T_i \times I$ denote the copy of $S' \times S' \times I$ containing D_i. Furthermore if $k_i = h_i(* \times I)$ with $* \in \operatorname{int} D_i$ then F is obtained by surgering $T_1 \cup T_2$ along k_1 and k_2. For F to be a Heegaard splitting $^{-}(T_i \times I - N(k_i))$ must be a handlebody for $i = 1, 2$. By Lemma 1.1 the arc k_i must be unknotted in $T_i \times I$. This is enough to show that F is topologically unique. □

REMARK. Using a good picture of F and the fact that the mapping class group of the three-torus is generated by Dehn twists along appropriately chosen tori, we may in fact show that F is unique up to isotopy.

It can be seen that if G is the double cover of a one-sided Heegaard splitting of the three-torus by U_4 then G is isotopic to the surfaces F described above.

It still remains to show that 0 is compressible. To do this we will assume that 0 is incompressible and isotope 0 into a "standard" position that will make the absurdity of our assumption obvious. Let $h\colon S^1 \times S^1 \to S^1 \times S^1$ be the hyperelliptic involution. If we view S^1 as $\mathbf{R}/\mathbf{Z}$ then $h(x_1, x_2) = -(x_1, x_2)$. Let h_0 and h_1 be the maps induced on $S^1 \times S^1 \times \{0\}$ and $S^1 \times S^1 \times \{1\}$ via the obvious identification. Then Q is homeomorphic to $S^1 \times S^1 \times I/h_0, h_1$ with the interior of small regular neighborhoods of the fixed points of h_0 and h_1 removed. Let P_0 and P_1 be the planar surfaces in Q corresponding to $S^1 \times S^1 \times \{0\}$ and $S^1 \times S^1 \times \{1\}$ respectively.

LEMMA 3.3. *Q is irreducible.*

Proof. Realize the universal cover $\tilde{Q}$ of Q as $\mathbf{R}^3$ with small open balls removed from around the points of the form $\frac{1}{2}(a, b, c)$ with $a, b, c \in \mathbf{Z}$, so that the group of deck transformations is generated by the integral translations and the map $\tilde{\phi}(x_1, x_2, x_3) = -(x_1, x_2, x_3)$. (Note that this realization is only topological. Incidentally the surfaces P_0 and P_1 can be chosen to be the images of $x_3 = 0$ and $x_3 = \frac{1}{2}$ in Q.) Let S be a sphere embedded in Q. Let $\tilde{S}$ be a lift of S to $\tilde{Q}$. Notice that since S is embedded if $d\tilde{S} \cap \tilde{S} = \varnothing$ for some deck transformation then d is the identity. Since $\tilde{Q} \subseteq \mathbf{R}^3$ we have that $\tilde{S}$ bounds a punctured ball $\tilde{B}$ in $\tilde{Q}$. If the ball is genuinely punctured then there is a nontrivial deck transformation preserving a puncture of $\tilde{B}$ and consequently either $d\tilde{B} \subseteq \tilde{B}$ or $\tilde{B} \subseteq d\tilde{B}$. Since d is nontrivial the valid inclusion is strict, but then d does not preserve volume on $\mathbf{R}^3$. This contradicts our choice of the universal cover of Q. Therefore $\tilde{B}$ is a ball and S bounds the ball in Q that is the image of $\tilde{B}$.

LEMMA 3.4. *The surface* 0 *is compressible.*

Proof. Assume that 0 is incompressible. Isotope 0 so that it is transverse to P_0 and P_1 and $0 \cap (P_0 \cup P_1)$ has the smallest number of components possible. Since both $\partial 0$ and $\partial(P_0 \cup P_1)$ consist of a single nontrivial curve on each boundary component of Q, we can isotope 0 without increasing the number of components of $0 \cap (P_0 \cup P_1)$ so that each boundary curve of 0 intersects $\partial(P_0 \cup P_1)$ exactly once in a point of transverse intersection. Hence $0 \cap P_i$ contains exactly two arcs k_{1i}, k_{2i} and they link distinct components of ∂P_i. Let $\overline{Q}$ denote Q cut along P_0 and P_1. Since $0 \cap (P_0 \cup P_1)$ is minimal the image $\overline{0}$ of 0 in $\overline{Q}$ is incompressible. Since Q is irreducible and $0 \cap (P_0 \cup P_1)$ is minimal we have that there are no curves in $0 \cap P_i$ that are trivial on P_i. Hence $\partial \bar{0}$ consists of nontrivial simple closed curves on $\partial \bar{Q}$. From the choice of P_0 and P_1 we have that $\bar{Q}$ is homeomorphic to the cartesian product of a torus and the unit interval. Since $\bar{0}$ is incompressible, and $\partial \bar{0}$ consists of nontrivial simple closed curves, a result of Waldhausen [**W3**] implies that $\bar{0}$ is a family of annuli. Hence the Euler characteristic of $\bar{0}$ is zero. Since 0 is obtained from $\bar{0}$ by identifying 4 pairs of arcs in $\partial \bar{0}$, and some pairs of circles, we have that the Euler characteristic of 0 is −4. Since 0 is a connected planar surface with eight boundary components, it has Euler characteristic −6. This is a contradiction. □

References

[B] E. M. Brown, *Unknotting in $M^2 \times I$*, Trans. Amer. Math. Soc., **123** (1966), 480–505.

[Fe] C. D. Feustal, *Isotopic unknotting in $F \times I$*, Trans. Amer. Math. Soc., **179** (1973), 227–238.

[Fr] C. D. Frohman, *One-sided incompressible surfaces in Seifert fibered spaces*, to appear in Topology Appls.

[L] H. B. Lawson, *The unknottedness of minimal embeddings*, Invent. Math., **11** (1970), 183–187.

[M1] W. H. Meeks, *Lectures on Plateu's problem*, IMPA, 1978.

[M2] _____, *The geometry and conformal structure of triply periodic minimal surfaces in* 3, Doctoral Thesis, U. C. Berkeley, 1975.

[W1] F. Waldhausen, *Heegaard Zerlegungen der* 3-*sphere*, Topology, **7** (1968), 195–203.

[W2] _____, *Eine Klasse von* 3-*deminsionale Mannigfaltigkeiten I*, Invent. Math., **3** (1967), 308–333.

[W3] _____, *On irreducible* 3-*manifolds that are sufficiently large*, Ann. of Math., **87** (1968), 56–58.

Received April 20, 1985.

University of California
Santa Barbara, CA 97106