PACIFIC JOURNAL OF MATHEMATICS
Vol 124, No. 1, 1986

ANALYTIC MULTIFUNCTIONS, THE $\bar{\partial}$-EQUATION, AND A PROOF OF THE CORONA THEOREM

B. BERNDTSSON AND T. J. RANSFORD

The purpose of this article is to give some applications of a recent theorem by Alexander-Wermer and Slodkowski on the structure of certain polynomial hulls. We want to show that this theorem gives a useful method of constructing analytic functions with prescribed properties in the disc. In particular it yields a rather easy proof of the Corona Theorem for two generators, and also implies Wolff's Theorem on the $\bar{\partial}$-equation.

1. Background on hulls and analytic multifunctions. We begin by formulating the theorem on polynomial hulls proved by Alexander and Wermer ([**1**]), and independently by Slodkowski [**16**]. In what follows, Δ denotes the open unit disc $\{\lambda \in \mathbf{C}: |\lambda| < 1\}$, and $\mathbf{T}$ denotes the unit circle $\{\zeta \in \mathbf{C}: |\zeta| = 1\}$.

THEOREM 1.1. *Let X be a compact subset of $\mathbf{T} \times \mathbf{C}$, and assume that for each $\zeta \in \mathbf{T}$ the fibre*

$$X_\zeta = \{z \in \mathbf{C}: (\zeta, z) \in X\}$$

is convex. Suppose moreover that $\hat{X}$, the polynomial hull of X, contains the point $(0, z_0)$ for some $z_0 \in \mathbf{C}$. Then there exists $h \in H^\infty(\Delta)$ such that for almost all $\zeta \in \mathbf{T}$

$$h^*(\zeta) \in X_\zeta,$$

where h^* denotes the radial boundary values of h.

Sketch proof (see [**1**] for further details). Let μ be a probability measure on X which represents $(0, z_0)$, in the sense that for any polynomial p,

$$p(0, z_0) = \int_X p(\zeta, z)\, d\mu(\zeta, z). \tag{1}$$

Then for $\lambda \in \Delta$ define

$$h(\lambda) = \int_X \frac{1 - |\lambda|^2}{|1 - \lambda\bar{\zeta}|^2} \cdot z \cdot d\mu(\zeta, z). \tag{2}$$

The proof consists of a verification that h has the required properties. First we observe that if $\lambda \in \Delta$ and $\zeta \in \mathbf{T}$ then

$$\frac{1-|\lambda|^2}{|1-\lambda\bar{\zeta}|^2} = \frac{1}{2}\left\{\frac{1+\lambda\bar{\zeta}}{1-\lambda\bar{\zeta}} + \frac{1+\bar{\lambda}\zeta}{1-\bar{\lambda}\zeta}\right\},$$

and the second term gives a constant contribution in (2), because of (1). It follows that h is *analytic* on Δ. Now for $\zeta \in \mathbf{T}$ let

$$\alpha(\zeta) = \sup\{\operatorname{Re} z\colon z \in X_\zeta\}.$$

From (2) we have

$$\operatorname{Re} h(\lambda) \leq \int_X \frac{1-|\lambda|^2}{|1-\lambda\bar{\zeta}|^2} \cdot \alpha(\zeta) \cdot d\mu(\zeta, z),$$

and it follows easily from (1), plus the fact that α is upper semicontinuous on $\mathbf{T}$, that for each $\zeta \in \mathbf{T}$

$$\limsup_{\substack{\lambda \to \zeta \\ \lambda \in \Delta}} \operatorname{Re} h(\lambda) \leq \alpha(\zeta).$$

Repeating the argument with h replaced by $e^{i\theta}h$, and using the fact that each X_ζ is convex, we deduce that for all $\zeta \in \mathbf{T}$

$$\operatorname{dist}(h(\lambda), X_\zeta) \to 0 \quad \text{as } \lambda \to \zeta,\ \lambda \in \Delta.$$

Consequently h is bounded on Δ, and $h^*(\zeta) \in X_\zeta$ a.e. on $\mathbf{T}$. □

In general it is not easy to decide when the hypothesis that $\hat{X}$ should contain some $(0, z_0)$ is satisfied (in fact is is equivalent to the assumption $\hat{X} \neq X$). However, one way of ensuring that it is fulfilled comes from the theory of analytic multifunctions, to which we now turn our attention.

We shall consider a mapping K which associates to each $\lambda \in \bar{\Delta}$ a non-empty compact subset $K(\lambda)$ of $\mathbf{C}$, and shall also assume that its *graph*, defined by

$$\Gamma(K) = \{(\lambda, z) \in \bar{\Delta} \times \mathbf{C}\colon z \in K(\lambda)\},$$

is a compact set. The multifunction K is said to be *analytic* on Δ if

$$U = (\Delta \times \mathbf{C}) \setminus \Gamma(K)$$

is a pseudoconvex set. We shall need the following important property of K which was first proved in **[13]** (see also **[4]**).

PROPOSITION 1.2. *If K is analytic on Δ, then $\Gamma(K)$ is contained in the polynomial hull of $\Gamma(K) \cap (\mathbf{T} \times \mathbf{C})$.*

Sketch proof. We shall assume that $(\Delta \times \mathbf{C}) \setminus \Gamma(K)$ is strictly pseudoconvex (which it always will be in our applications); the general case may be deduced via a standard approximation argument.

Take any point $q_0 = (\lambda_0, z_0)$ in $\Gamma(K) \cap (\Delta \times \mathbf{C})$ and let $p(\lambda, z)$ be any polynomial. We are done if we can show that for $\varepsilon > 0$ the function

$$u = |p(\lambda, z)| + \varepsilon\left(|\lambda|^2 + |z|^2\right)$$

cannot have a maximum on $\Gamma(K)$ at q_0. But the assumption of strict pseudoconvexity means that we can find a piece of an analytic manifold M passing through q_0 and lying entirely inside $\Gamma(K)$, so the desired conclusion follows from the maximum principle applied to the restriction of u to M, which is a strictly subharmonic function on M. □

We have now collected all the facts we need, and refer to **[2–5, 9–15]** for further information on analytic multifunctions, including the justification for their nomenclature.

Combining Theorem 1.1 and Proposition 1.2, we obtain the result which is the key to all that follows.

THEOREM 1.3. *Let K be a nowhere-empty multifunction on $\bar{\Delta}$ such that $\Gamma(K)$ is compact and K is analytic on Δ. Assume moreover that $K(\zeta)$ is convex for each $\zeta \in \mathbf{T}$. Then there exists $h \in H^\infty(\Delta)$ such that*

$$h^*(\zeta) \in K(\zeta) \quad a.e.\ on\ \mathbf{T}. \qquad \square$$

Let us call such a function h on *analytic selector*. It is a natural question to ask whether one can always find an analytic h such that $h(\lambda) \in K(\lambda)$ for *all λ in the disc* Δ: we shall see in §3 that this may not be possible. Nevertheless, Theorem 1.3 gives a powerful way of constructing analytic functions on the disc with control over their boundary values: all one has to do is set up an appropriate multifunction. We now proceed to illustrate this idea by using it to prove the Corona Theorem.

2. The Corona Theorem. Let $f_1, \ldots, f_N \in H^\infty(\Delta)$. The Corona Theorem says that if

$$1 > |f|^2 = \sum_1^N |f_j|^2 > \delta^2 > 0$$

everywhere on Δ, then there exist $g_1, \ldots, g_N \in H^\infty(\Delta)$ such that

$$\sum_1^N f_j g_j = 1. \tag{3}$$

For background material on the Corona Theorem, we refer to Garnett's book [7, §VIII].

For simplicity we treat only the case $N = 2$. In the proof we shall assume that the functions f_1, f_2 extend analytically to a neighbourhood of $\overline{\Delta}$, and shall end up by obtaining solutions g_1, g_2 whose sup-norms depend only on δ. A passage to the limit via normal families then gives the theorem.

To start with, take any two functions h_1, h_2 analytic on a neighbourhood of $\overline{\Delta}$, such that

$$f_1 h_1 + f_2 h_2 = 1,$$

without a priori control on the sup-norms. Then it is easy to see that the general solution to (3) is

$$\begin{aligned} g_1 &= h_1 - h f_2 \\ g_2 &= h_2 + h f_1, \end{aligned} \tag{4}$$

where h is analytic on Δ. Thus our problem is to choose h so that g_1 and g_2 satisfy an a priori estimate depending only on δ. This function h will be obtained from Theorem 1.3 as a selector to the following analytic multifunction whose definition almost suggests itself. For $\lambda \in \overline{\Delta}$ set

$$K(\lambda) = \left\{ z \in \mathbf{C} \colon |h_1(\lambda) - z f_2(\lambda)|^2 + |h_2(\lambda) + z f_1(\lambda)|^2 \leq r(\lambda) \right\}. \tag{5}$$

Here $r(\lambda)$ is a smooth positive function which has yet to be determined: indeed the crux of the proof lies in making a suitable choice. If we can choose $r(\lambda)$ to make K analytic on Δ and moreover so that $r(\lambda)$ satisfies an L^∞-estimate on $\mathbf{T}$ depending only on δ, then we are done, because any analytic selector h will give a solution (4) to (3).

The simplest choice would be to take $r(\lambda)$ equal to a large constant; however, this does not work. We shall explain where the difficulty lies. As mentioned in the proof of Proposition 1.2, to demand that K be analytic means essentially that for any point $q_0 = (\lambda_0, z_0)$ in $\Gamma(K) \cap (\Delta \times \mathbf{C})$ there should be a local analytic manifold lying inside $\Gamma(K)$ that passes through q_0. In our setting, any point $q = (\lambda, z)$ in $\Gamma(K)$ corresponds to a *pointwise* solution of the corona problem, i.e. to *numbers* β_1, β_2 such that

$$f_1(\lambda)\beta_1 + f_2(\lambda)\beta_2 = 1.$$

In the same way, a local analytic manifold in $\Gamma(K)$ corresponds to a local analytic solution to the corona problem. Thus we want to have a situation where *any* pointwise solution with norm

$$|\beta_1|^2 + |\beta_2|^2 \leq r(\lambda)$$

at $\lambda = \lambda_0$ can be extended to a local analytic solution whilst maintaining this inequality. Clearly this would violate the maximum principle if $r(\lambda)$ were constant. The problem is therefore to choose $r(\lambda)$ "subharmonic enough", while at the same time keeping it bounded.

To determine which functions r make K analytic, we use the following lemma whose proof is deferred until the end of the section.

LEMMA 2.1. *Assume that the multifunction K defined in* (5) *is nowhere empty on Δ. Then it is analytic if and only if $u = \log r$ satisfies the differential inequality*

$$\sqrt{(u_{\lambda\bar{\lambda}})} - \sqrt{(v_{\lambda\bar{\lambda}})} \geq \frac{|u_\lambda + v_\lambda|}{\sqrt{(e^{u+v} - 1)}}, \tag{6}$$

where $v = \log|f|^2 = \log(|f_1|^2 + |f_2|^2)$.

In particular one should observe that the condition (6) does not depend on the choice of the "arbitrary" functions h_1 and h_2, but only on f_1 and f_2. This condition also makes it plain that the "second simplest candidate" for $r(\lambda)$, namely $r = A|f|^2$ for some constant A, also fails, because it corresponds to taking $u = v +$ constant. So instead we try u of the form

$$u = v + F(v) \qquad (-\gamma < v < 0) \tag{7}$$

where $v = \log|f|^2$, $\gamma = \log(1/\delta^2)$, and F is a function defined on $[-\gamma, 0]$ satisfying

$$F'(v) > 0, \quad F''(v) > 0 \qquad (-\gamma < v < 0), \tag{8}$$

$$F(-\gamma) > 2\gamma. \tag{9}$$

Here (8) has been assumed for convenience in subsequent calculations, and (9) then guarantees that $r(\lambda) > 1/\delta^2$ on $\bar{\Delta}$, which implies in particular that the multifunction K takes non-empty values there. Adding $\sqrt{(v_{\lambda\bar{\lambda}})}$ to both sides of (6), squaring up, and substituting for u_λ and $u_{\lambda\bar{\lambda}}$ in terms of v_λ and $v_{\lambda\bar{\lambda}}$ (computed via (7)), we obtain

$$F'(v)v_{\lambda\bar{\lambda}} + F''(v)|v_\lambda|^2 \geq y + 2\sqrt{(v_{\lambda\bar{\lambda}})}\sqrt{(y)} \tag{10}$$

where

$$y = \frac{(2 + F'(v))^2|v_\lambda|^2}{e^{2v}e^{F(v)} - 1}.$$

Now

$$2\sqrt{(v_{\lambda\bar{\lambda}})}\sqrt{(y)} \leq F'(v)v_{\lambda\bar{\lambda}} + y/F'(v),$$

so for (10) to hold, it suffices that

$$F''(v) \geq \frac{(2 + F'(v))^2}{e^{2v}e^{F(v)} - 1}\left\{1 + \frac{1}{F'(v)}\right\},$$

or equivalently,

$$(11) \quad e^{2v}e^{F(v)} \geq \frac{(2 + F'(v))^2(1 + F'(v))}{F'(v)F''(v)} + 1 \qquad (-\gamma < v < 0).$$

Note that (8) and (11) together automatically imply (9). To sum up: in order to find u satisfying (6), it is enough to take $u = v + F(v)$, where F: $[-\gamma, 0] \to \mathbf{R}$ obeys (8) and (11).

There are many examples of such functions F; a convenient one is

$$F(v) = (1 + v/\gamma)^{3/2} + 2\gamma + 3\log\gamma + B,$$

where B is numerical constant independent of γ. Working backwards, we deduce the existence of a solution (g_1, g_2) to (3) such that

$$|g_1|^2 + |g_2|^2 \leq \sup\{r(\zeta) : \zeta \in \mathbf{T}\} \leq \frac{C(\log(1/\delta))^3}{\delta^4},$$

where C is a constant independent of δ. The proof of the Corona Theorem (for $N = 2$) is complete.

REMARK. The particular function F above was selected because it had the convenient property that $F'(v) \cdot F''(v) =$ constant. However, bearing (11) in mind, a more efficient choice of F can be made by solving the differential equation $F'(v) \cdot F''(v) = e^{-2v}$. This leads to the function

$$F(v) = (1/\gamma)\int_0^{\gamma+v} \sqrt{(1 - e^{-2s})}\, ds + 2\gamma + 2\log\gamma + B,$$

which indeed satisfies (8) and (11), and yields a solution (g_1, g_2) with the improved estimate

$$|g_1|^2 + |g_2|^2 \leq \frac{C(\log(1/\delta))^2}{\delta^4}.$$

Computation of a Levi form and the proof of Lemma 2.1. We are considering a domain of the form

$$U = \left\{(\lambda, z) \in \Delta \times \mathbf{C} : |h_1(\lambda) - zf_2(\lambda)|^2 + |h_2(\lambda) + zf_1(\lambda)|^2 > r(\lambda)\right\}$$

and want to know whether it is pseudoconvex. Now a domain in $\mathbf{C}^n$ of the kind

$$D = \{\zeta : \rho(\zeta) < 0\},$$

where $\rho \in C^2(\overline{D})$ and $d\rho \neq 0$ on ∂D, is pseudoconvex iff the hessian of the defining function ρ satisfies

$$\sum_{j,k} \frac{\partial^2 \rho}{\partial \zeta_j \partial \bar{\zeta}_k} a_j \bar{a}_k \geq 0$$

on ∂D, for all $\mathbf{a} \in \mathbf{C}^n$ satisfying

$$\sum_j a_j \frac{\partial \rho}{\partial \zeta_j} = 0.$$

It is easy to see, and of course well known, that this condition is invariant under biholomorphic transformations. To facilitate computations we shall first apply such a transformation. Define the manifold

$$M = \{(\lambda, w_1, w_2) \in \mathbf{C}^3 \colon |\lambda| < 1,\, w_1 f_1(\lambda) + w_2 f_2(\lambda) = 1\},$$

and consider the subdomain of M given by

$$V = \{(\lambda, w_1, w_2) \in M \colon r(\lambda) - |w_1|^2 - |w_2|^2 < 0\}.$$

The transformation

$$(\lambda, z) \mapsto (\lambda, h_1(\lambda) - z f_2(\lambda), h_2(\lambda) + z f_1(\lambda))$$

is biholomorphic from U onto V, so it is enough to determine when V is pseudoconvex in M. Now our function ρ is

$$\rho(\lambda, w_1, w_2) = r(\lambda) - |w_1|^2 - |w_2|^2,$$

so the hessian of ρ is

$$\begin{pmatrix} r_{\lambda\bar{\lambda}} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We want to apply the hessian to vectors (a_1, a_2, a_3) tangent both to M and to ∂V. This is a complex one-dimensional space generated by the vector product of the normals to M and ∂V respectively, so we can take

$$(a_1, a_2, a_3) = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ w \cdot f' & f_1 & f_2 \\ r_\lambda & -\overline{w}_1 & -\overline{w}_2 \end{vmatrix}$$

$$= \left(\overline{w}_1 f_2 - \overline{w}_2 f_1,\, \overline{w}_2(w \cdot f') + f_2 r_\lambda,\, -\overline{w}_1(w \cdot f') - f_1 r_\lambda\right)$$

(here $w \cdot f' = w_1 f_1' + w_2 f_2'$). Thus the condition for pseudoconvexity becomes

$$(12) \quad r_{\lambda\bar{\lambda}} |\overline{w}_1 f_2 - \overline{w}_2 f_1|^2 \geq |\overline{w}_2(w \cdot f') + f_2 r_\lambda|^2 + |\overline{w}_1(w \cdot f') + f_1 r_\lambda|^2$$

at all $(\lambda, w_1, w_2) \in \partial V$.

To simplify the left hand side, write

$$(w_1, w_2) = \xi(\bar{f}_1, \bar{f}_2) + \eta(f_2, -f_1). \tag{13}$$

Then on ∂V

$$r(\lambda) = |w|^2 = \left(|\xi|^2 + |\eta|^2\right)\left(|f_1|^2 + |f_2|^2\right) = \left(|\xi|^2 + |\eta|^2\right)|f|^2,$$

and $1 = w_1 f_1 + w_2 f_2 = \xi |f|^2$. Hence

$$\xi = 1/|f|^2, \qquad |\eta|^2 = r/|f|^2 - 1/|f|^4. \tag{14}$$

Moreover

$$\bar{w}_1 f_2 - \bar{w}_2 f_1 = \eta |f|^2,$$

so the left hand side of (12) becomes

$$r_{\lambda\bar{\lambda}}\left(r|f|^2 - 1\right).$$

For the right hand side, we expand and obtain

$$|w \cdot f'|^2 |w|^2 + |f|^2 |r_\lambda|^2 + 2\,\mathrm{Re}\left(\overline{w \cdot f'} r_\lambda\right),$$

where once again we have used $w_1 f_1 + w_2 f_2 = 1$. Now substitute $r = e^u$, and note that $|w|^2 = r = e^u$ on ∂V; the right hand side of (12) then becomes

$$\begin{aligned} e^u|w \cdot f' + u_\lambda|^2 &+ |f|^2 e^{2u}|u_\lambda|^2 - |u_\lambda|^2 e^u \\ &= e^u|w \cdot f' + u_\lambda|^2 + \left(|f|^2 e^u - 1\right)|u_\lambda|^2 e^u. \end{aligned}$$

Observing that

$$r_{\lambda\bar{\lambda}} = \left(u_{\lambda\bar{\lambda}} + |u_\lambda|^2\right) e^u,$$

we see that (12) reduces to

$$u_{\lambda\bar{\lambda}}\left(|f|^2 e^u - 1\right) \geq |w \cdot f' + u_\lambda|^2 \quad \text{on } \partial V. \tag{15}$$

For the final step, we again exploit the decomposition (13): setting $v = \log|f|^2$, we have

$$\begin{aligned} w \cdot f' &= \xi\left(f_1'\bar{f}_1 + f_2'\bar{f}_2\right) + \eta\left(f_1' f_2 - f_2' f_1\right) \\ &= v_\lambda + \eta\left(f_1' f_2 - f_2' f_1\right), \end{aligned}$$

the value of ξ having been determined using (14). Now the only restriction on η is also from (14); hence

$$\begin{aligned} \sup_{\partial V} |w \cdot f' + u_\lambda| &= |u_\lambda + v_\lambda| + \sqrt{\left(r|f|^2 - 1\right)}\, |f_1' f_2 - f_2' f_1|/|f|^2 \\ &= |u_\lambda + v_\lambda| + \sqrt{\left(r|f|^2 - 1\right)}\sqrt{(v_{\lambda\bar{\lambda}})}\,. \end{aligned}$$

So finally the inequality (15) is equivalent to

$$\sqrt{(u_{\lambda\bar{\lambda}})}\sqrt{(e^{u+v}-1)} \geq |u_\lambda + v_\lambda| + \sqrt{(e^{u+v}-1)}\sqrt{(v_{\lambda\bar{\lambda}})}$$

which is the same as (6). This completes the proof of Lemma 2.1. □

3. The $\bar{\partial}$-equation. The standard proofs of the Corona Theorem go via the $\bar{\partial}$-equation. In particular, not long ago, Wolff found a condition for the existence of bounded solutions of the $\bar{\partial}$-equation which greatly simplified the proof of the Corona Theorem. In this section we shall see how the selection theorem (1.3) applied to *disc-valued* multifunctions is equivalent to an existence theorem for the $\bar{\partial}$-equation, and that this theorem is essentially the same as Wollf's.

Consider a disc-valued multifunction on $\bar{\Delta}$ which is analytic on Δ, with variable centre $c(\lambda) \in \mathbf{C}$ and radius $r(\lambda) \geq 0$; thus

$$K(\lambda) = \{z \in \mathbf{C}: |z - c(\lambda)| \leq r(\lambda)\}. \tag{16}$$

Set

$$a = \frac{\partial c}{\partial \bar{\lambda}}$$

and let h be an analytic selector to K in the sense of Theorem 1.3. Then

$$b = c - h$$

solves

$$\frac{\partial b}{\partial \bar{\lambda}} = a(\lambda) \qquad (\lambda \in \Delta), \tag{17}$$

and satisfies the boundary estimate

$$|b^*(\zeta)| \leq r(\zeta) \quad \text{a.e. on } \mathbf{T}. \tag{18}$$

Conversely, say we want to solve (17) with (18) satisfied on $\mathbf{T}$: then we can define an analytic multifunction K by (16), where c is an arbitrary solution to the $\bar{\partial}$-equation, and our problem becomes equivalent to finding an analytic selector to K. To see what this means concretely, we need to know when a disc-valued multifunction is analytic. The answer is given by the following analogue of Lemma 2.1, whose proof is likewise deferred to the end of the section.

LEMMA 3.1.[1] *Let c: $\bar{\Delta} \to \mathbf{C}$ and r: $\bar{\Delta} \to (0, \infty)$ be C^2-functions. Then the disc-valued multifunction K defined by* (16) *is analytic on Δ if and only if*

$$u_{\lambda\bar{\lambda}} \geq e^{-u}|a_\lambda - 2au_\lambda| + e^{-2u}|a|^2, \tag{19}$$

where again $a = c_{\bar{\lambda}}$ and $u = \log r$.

[1] This lemma was obtained by Z. Slodkowski (unpublished). See [**9**, p. 58].

From our previous remarks and Theorem 1.3 we immediately deduce the following consequence.

THEOREM 3.2. *Suppose* $u \in C^2(\bar{\Delta})$ *and* $a \in C^1(\bar{\Delta})$ *satisfy* (19) *on* Δ. *Then there is a solution b to*

$$\frac{\partial b}{\partial \bar{\lambda}} = a(\lambda) \qquad (\lambda \in \Delta),$$

such that

$$|b^*(\zeta)|e^{-u(\zeta)} \leq 1 \quad a.e. \text{ on } \mathbf{T}. \qquad \square$$

Incidentally, this theorem has some formal similarities to Hörmander's L^2-theorem for the $\bar{\partial}$-equation ([**8**, §4]). In the one-variable case, this theorem ways that if

$$\int_\Delta \frac{|a|^2 e^{-2u}}{2u_{\lambda\bar{\lambda}}} \leq 1,$$

then there is a solution b to (17) such that

$$\int_\Delta |b|^2 e^{-2u} \leq 1.$$

Thus Theorem 3.2 is an L^∞-analogue of this result, and the price one has paid to pass from L^2 to L^∞ is an extra condition on the first derivative of a.

Let us now recall Wolff's Theorem.

THEOREM 3.3. ([7]). *Suppose that* $a \in C^1(\bar{\Delta})$ *satisfies the two conditions*:

(i) $(1 - |\lambda|^2)|a|^2$ *is a Carleson measure*;
(ii) $(1 - |\lambda|^2)|a_\lambda|$ *is a Carleson measure*.

Then there is a solution b to

$$\frac{\partial b}{\partial \bar{\lambda}} = a(\lambda) \qquad (\lambda \in \Delta)$$

such that $b^* \in L^\infty(\mathbf{T})$.

(For the definition of Carleson measure, see [7, p. 238]). The link with Theorem 3.2 is given by the following lemma.

LEMMA 3.4. (a) *Suppose that u is subharmonic and bounded on* Δ. *Then*

$$\mu = \left(1 - |\lambda|^2\right) u_{\lambda\bar{\lambda}}$$

is a Carleson measure.

(b) *Conversely, suppose that μ is a Carleson measure which is absolutely continuous with respect to Lebesgue measure m on Δ,*

$$d\mu = k\,dm,$$

where for some constant C

$$|k(\lambda)| \le C/(1 - |\lambda|^2) \qquad (\lambda \in \Delta).$$

Then there exists a bounded subharmonic function u on Δ such that

$$(1 - |\lambda|^2) u_{\lambda\bar{\lambda}} = \mu.$$

Proof. According to a lemma of Garnett ([7, p. 239]), a positive measure μ on Δ is a Carleson measure if and only if the potential

$$p(w) = \int_\Delta \frac{1 - |w|^2}{|1 - w\bar{\lambda}|^2} \cdot d\mu(\lambda)$$

is bounded on Δ. On the other hand, a positive measure ν is the Laplacian of a bounded function precisely when the Green potential

$$G(w) = \int_\Delta \log\left|\frac{1 - w\bar{\lambda}}{w - \lambda}\right|^2 d\nu(\lambda)$$

is bounded on Δ. Now use the identity

$$\left|\frac{w - \lambda}{1 - w\bar{\lambda}}\right|^2 = 1 - \frac{(1 - |w|^2)(1 - |\lambda|^2)}{|1 - w\bar{\lambda}|^2}$$

and the estimate

$$\log\left(\frac{1}{1 - x}\right) \ge x \quad \text{for } x < 1,$$

with approximate equality if x stays away from 1. These give

$$G(w) \ge \int_\Delta \frac{(1 - |w|^2)(1 - |\lambda|^2)}{|1 - w\bar{\lambda}|^2} d\nu(\lambda) \tag{20}$$

and

$$\begin{aligned} G(w) \approx &\int_F \log\left|\frac{1 - \bar{\lambda}w}{w - \lambda}\right|^2 d\nu(\lambda) \\ &+ \int_{\Delta\setminus F} \frac{(1 - |w|^2)(1 - |\lambda|^2)}{|1 - w\bar{\lambda}|^2} d\nu(\lambda), \end{aligned} \tag{21}$$

where

$$F = \left\{ \lambda \in \Delta : \left| \frac{w - \lambda}{1 - w\bar{\lambda}} \right|^2 \le \frac{1}{10} \right\}.$$

Now (20) shows that the Laplacian of a bounded function, multiplied by $(1 - |\lambda|^2)$, is a Carleson measure; and (21) shows that the converse also holds, provided we can estimate the first term in the decomposition of G. This requires an extra condition, but it is easy to see that the one we have assumed is more than enough. □

Thus Wolff's Theorem says that if

$$|a|^2 \le u_{\lambda\bar{\lambda}} \quad \text{and} \quad |a_\lambda| \le u_{\lambda\bar{\lambda}} \tag{22}$$

for some bounded function u on Δ, then the $\bar{\partial}$-equation with right hand side a has a bounded solution. It is not difficult to see that this follows from Theorem 3.2, but actually Theorem 3.2 gives a more precise, pointwise conclusion. The price we pay for this is to replace Wolff's clean hypothesis by the slightly messier one (19).

REMARK. The Carleson measures that are relevant to Wolff's proof of the Corona Theorem are

$$\mu = \left(1 - |\lambda|^2\right)\left(|f_1'|^2 + |f_2'|^2\right),$$

and are thus a priori of the form

$$\mu = \left(1 - |\lambda|^2\right) u_{\lambda\bar{\lambda}}$$

where $u = |f|^2$ is bounded and subharmonic on Δ. Lemma 3.4 is therefore not necessary for the proof, but rather shows that the notion of Carleson measures can be dispensed with in the context. This has already been noted by Gamelin [**6**].

We conclude our remarks on disc-valued multifunctions by fulfilling the promise made in §1, namely to show that Theorem 1.3 cannot be extended to guarantee the existence of analytic functions h which are selectors in the whole disc Δ. To this end we shall prove the following interpolation lemma.

LEMMA 3.5. *Let E be a compact polar subset of Δ and let f be any function analytic on a neighbourhood of E. Then there exists a disc-valued multifunction K on $\bar{\Delta}$, analytic on Δ, such that*

$$K(\lambda) = \{f(\lambda)\} \quad \textit{for } \lambda \in E.$$

Before proving the lemma, let us indicate how to deduce the promised counterexample. Recall that E is polar means that there exists a subharmonic function w on $\mathbf{C}$ such that

$$E \subset \{z \in \mathbf{C} \colon w(z) = -\infty\}. \tag{23}$$

Now take $E = E_0 \cup E_1$, where

$$E_0 = \{1/2\} \cup \{n/(2n+1) \colon n \geq 1\},$$
$$E_1 = \{-1/2\} \cup \{-n/(2n+1) \colon n \geq 1\}.$$

This E is certainly polar: for example, it satisfies (23) with w equal to $w_0 + w_1$, where

$$w_0(z) = \log\left|z - \frac{1}{2}\right| + \sum_{n \geq 1} 2^{-n} \log\left|z - \frac{n}{2n+1}\right|,$$
$$w_1(z) = \log\left|z + \frac{1}{2}\right| + \sum_{n \geq 1} 2^{-n} \log\left|z + \frac{n}{2n+1}\right|.$$

Now if we take disjoint open neighbourhoods N_0 of E_0 and N_1 of E_1, and define f to be 0 on N_0 and 1 on N_1, then the hypotheses of the Lemma are fulfilled, and we deduce there exists a disc-valued analytic multifunction K on $\overline{\Delta}$ such that

$$K(\lambda) = \begin{cases} \{0\}, & \lambda \in E_0 \\ \{1\}, & \lambda \in E_1. \end{cases}$$

But then any selector h of K on Δ has to satisfy

$$h(\lambda) = \begin{cases} 0, & \lambda \in E_0 \\ 1, & \lambda \in E_1 \end{cases}$$

which is clearly impossible if h is to be analytic, because E_0 and E_1 both possess limit points within Δ.

Proof of Lemma 3.5. Choose open sets V and W with

$$E \subset V \subset \overline{V} \subset W$$

such that f is analytic on a neighbourhood of $\overline{W}$. Let $c \in C^2(\Delta)$ be any function which agrees with f on W. As E is polar, there exists a subharmonic function w on $\mathbf{C}$ satisfying (23); moreover because E is compact we may take w to be harmonic on $\mathbf{C} \setminus E$ (this follows from standard potential theory, but is true in any case for the specific w we used above). Define K on $\overline{\Delta}$ by

$$K(\lambda) = \left\{z \in \mathbf{C} \colon |z - c(\lambda)| \leq \exp\left(w(\lambda) + |\lambda|^2 + A\right)\right\},$$

where A is a constant yet to be chosen. That K is analytic on W is straightforward, since if we express the subharmonic function $u = w + |\lambda|^2 + A$ as the limit of a decreasing sequence of smooth subharmonic functions u_m, then for each m the multifunction

$$K_m(\lambda) = \{z \in \mathbf{C}: |z - c(\lambda)| \le \exp(u_m(\lambda))\}$$

is plainly analytic on W by Lemma 3.1 (because $c_{\bar\lambda} = f_{\bar\lambda} \equiv 0$ there), and it follows easily that K itself must also be. On $\Delta \setminus \bar{V}$, we check analyticity directly using Lemma 3.1: the condition that needs to be satisfied is

$$w_{\lambda\bar\lambda} + 1 \ge e^{-A}\left\{e^{-w-|\lambda|^2} \cdot |c_{\lambda\bar\lambda} - 2c_{\bar\lambda}(w_\lambda + \bar\lambda)|\right\}$$
$$+ e^{-2A}\left\{e^{-2w-2|\lambda|^2} \cdot |c_{\bar\lambda}|^2\right\}.$$

Now the terms in curly brackets are both uniformly bounded on $\bar\Delta \setminus V$, say by B, and also $w_{\lambda\bar\lambda} \ge 0$ because w is subharmonic. Consequently K will be analytic on $\Delta \setminus \bar{V}$ provided A is chosen large enough so that

$$1 \ge e^{-A} \cdot B + e^{-2A} \cdot B.$$

Finally, note that if $\lambda \in E$ then $w(\lambda) + |\lambda|^2 + A = -\infty$, so that, as desired,

$$K(\lambda) = \{c(\lambda)\} = \{f(\lambda)\}. \qquad \square$$

Computation of a second Levi form and the proof of Lemma 3.1 This time we are considering a domain

$$u = \{(\lambda, z) \in \Delta \times \mathbf{C}: |z - c(\lambda)| > r(\lambda)\}$$

to determine whether it is pseudoconvex. Since U has the form

$$u = \{(\lambda, z) \in \Delta \times \mathbf{C}: \rho(\lambda, z) < 0\}$$

with

$$\rho(\lambda, z) = 2u(\lambda) - \log|z - c(\lambda)|^2$$

(where we have set $u = \log r$), the condition that needs to be checked is

$$(a_1\ a_2)\begin{pmatrix} \rho_{\lambda\bar\lambda} & \rho_{\lambda\bar z} \\ \rho_{z\bar\lambda} & \rho_{z\bar z} \end{pmatrix}\begin{pmatrix} \bar a_1 \\ \bar a_2 \end{pmatrix} \ge 0$$

on $\partial U \cap (\Delta \times \mathbf{C})$, for all $\mathbf{a} \in \mathbf{C}^2$ satisfying

$$a_1\rho_\lambda + a_2\rho_z = 0.$$

In other words, we need

$$(-\rho_z, \rho_\lambda)\begin{pmatrix} \rho_{\lambda\bar\lambda} & \rho_{\lambda\bar z} \\ \rho_{z\bar\lambda} & \rho_{z\bar z} \end{pmatrix}\begin{pmatrix} -\bar\rho_z \\ \bar\rho_\lambda \end{pmatrix} \ge 0 \quad \text{on } \partial U \cap (\Delta \times \mathbf{C}).$$

Now $\rho_z = -1/(z-c)$ and $\rho_{z\bar{z}} = 0$, so this condition can be restated as

(24) $$\rho_{\lambda\bar{\lambda}} + 2\,\mathrm{Re}\{(z-c)\rho_\lambda\rho_{z\bar{\lambda}}\} \geq 0 \quad \text{on } \partial U \cap (\Delta \times \mathbf{C}).$$

Computation of the derivatives yields

$$\rho_\lambda = 2u_\lambda + c_\lambda/(z-c) + \overline{c_{\bar{\lambda}}/(z-c)},$$

$$\rho_{\lambda\bar{\lambda}} = 2\,\mathrm{Re}\{u_{\lambda\bar{\lambda}} + c_{\lambda\bar{\lambda}}/(z-c) + c_\lambda c_{\bar{\lambda}}/(z-c)^2\},$$

$$\rho_{z\bar{\lambda}} = -c_{\bar{\lambda}}/(z-c)^2.$$

Hence (24) becomes

$$\mathrm{Re}\left\{u_{\lambda\bar{\lambda}} + \frac{c_{\lambda\bar{\lambda}}}{z-c} + \frac{c_\lambda c_{\bar{\lambda}}}{(z-c)^2} - \frac{c_{\bar{\lambda}}}{z-c}\left(2u_\lambda + \frac{c_\lambda}{z-c} + \overline{\frac{c_{\bar{\lambda}}}{z-c}}\right)\right\} \geq 0$$

$$\text{on } \partial U \cap (\Delta \times \mathbf{C}).$$

This simplifies to

$$u_{\lambda\bar{\lambda}} - |c_{\bar{\lambda}}/(z-c)|^2 + \mathrm{Re}\{(c_{\lambda\bar{\lambda}} - 2c_{\bar{\lambda}}u_\lambda)/(z-c)\} \geq 0$$

$$\text{on } \partial U \cap (\Delta \times \mathbf{C}).$$

Now the only restriction on $(z - c(\lambda))$ is that $(\lambda, z) \in \partial U \cap (\Delta \times \mathbf{C})$, or in other words, that $|z-c| = e^u$. Thus the last inequality is equivalent to

$$u_{\lambda\bar{\lambda}} - e^{-2u}|c_{\bar{\lambda}}|^2 - e^{-u}|c_{\lambda\bar{\lambda}} - 2c_{\bar{\lambda}}u_\lambda| \geq 0,$$

which is the same as (19). This completes the proof. □

We should like to thank UCLA for its hospitality and support while this paper was written.

Note: After we had derived Wolff's theorem from the selection theorem, but before we had completed our proof of the Corona theorem without the $\bar{\partial}$-equation, Z. Slodkowski informed one of us that the Corona theorem for two generators follows directly from merging his version of the selection theorem, with his Th. 4.3 of [13]. Unfortunately, we failed to fully realize the impact of his comment, and for this reason developed the alternate proof in §2. As it turns out the two proofs are quite similar. Slodkowski's proof is spelled out in [17], where it is also shown how his argument can be generalized to an arbitrary number of generators.

Finally, we would also like to express our thanks to T. Wolff, who was the first to suggest to us that we try to get rid of the $\bar{\partial}$-operator in the proof.

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Received April 11, 1985

CHALMERS UNIVERSITY OF TECHNOLOGY
AND THE UNIVERSITY OF GÖTEBORG
S-41296 GÖTEBORG, SWEDEN

UNIVERSITY OF LEEDS
LEEDS LS2 9JT, ENGLAND

Current address of both authors: Department of Mathematics
University of California
Los Angeles, CA 90024, U.S.A.