PACIFIC JOURNAL OF MATHEMATICS
Vol. 124, No. 2, 1986

REGULARITY OF CAPILLARY SURFACES OVER DOMAINS WITH CORNERS: BORDERLINE CASE

LUEN-FAI TAM

Consider the solutions of capillary surface equation with contact angle boundary condition over domains with corners. It is known that if the corner angle 2α satisfies $0 < 2\alpha < \pi$ and $\alpha + \gamma > \pi/2$ where $0 < \gamma \leq \pi/2$ is the contact angle, then solutions are regular. It is also known that no regularity holds in case $\alpha + \gamma < \pi/2$. In this paper we show that solutions are still regular for the borderline case $\alpha + \gamma = \pi/2$ at the corner.

It was proved by Concus and Finn in [**1**] that the behavior of a capillary surface near a corner over a wedge can change discontinuously. They proved that if the contact angle is $\gamma > 0$ and the interior angle at the corner is 2α, then all solutions for which $\alpha + \gamma \geq \pi/2$ are bounded near the corner, while all solutions are unbounded if $\alpha + \gamma < \pi/2$. Later in [**9**], Simon went further and investigated the regularity near the corner.

Let Ω be a domain contained in $B_R = \{x \in \mathbf{R}^2 \,|\, |x| < R\}$ for some $R > 0$, such that $\partial\Omega$ consists of a circular arc of ∂B_R and two smooth Jordan arcs intersecting at the origin. Each arc makes an angle α with the positive x^1-axis, so that the interior angle at the origin is 2α. See Figure 1. Let u be a bounded function satisfying

$$(0.1) \qquad \begin{cases} \operatorname{div} Tu = H(x, u(x)) & \text{in } \Omega \\ Tu = \dfrac{Du}{\sqrt{1 + |Du|^2}} & \\ Tu \cdot \nu = \cos\gamma & \text{on } \Gamma = (\partial\Omega - \{0\}) \cap B_R \end{cases}$$

where $H(x, t)$ is a locally bounded function in $\overline{\Omega} \times \mathbf{R}$, $\pi/2 > \gamma > 0$ is a constant angle and ν is the unit outward normal of Γ. If u is smooth in $(\overline{\Omega} - \{0\})$ and if $\pi/2 > \alpha > \pi/2 - \gamma$, then Simon [**9**] proved that u actually extends to be a C^1 function in $\overline{\Omega}$. It is known that no regularity holds if $\alpha + \gamma < \pi/2$. Our aim is to examine the borderline case $\alpha + \gamma = \pi/2$. In this case, one cannot expect Du to be continuous or even bounded in $\overline{\Omega}$, as one can easily construct counterexamples. Note also that

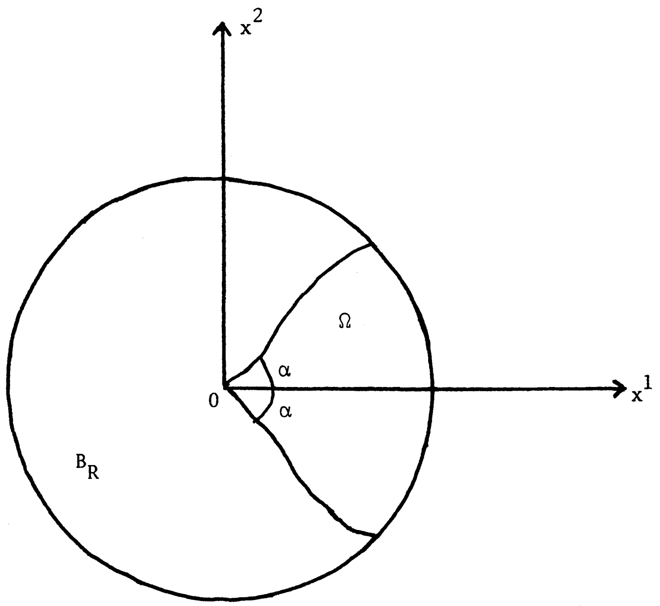

FIGURE 1

if $2\alpha > \pi$, then there are examples which show that u may be discontinuous at the corner, see [5]. In this paper we want to prove the following theorem:

THEOREM. *Let $u \in C^2(\overline{\Omega} - \{0\}) \cap L^\infty(\Omega)$ be a solution of* (0.1). *If $\alpha + \gamma = \pi/2$, then u and $(Tu, -1/\sqrt{1+|Du|^2})$ extend to be continuous functions in $\overline{\Omega}$ with values in $\mathbf{R}$ and $\mathbf{R}^3$ respectively.*

Since $H(x,t)$ is locally bounded in $\overline{\Omega} \times \mathbf{R}$ and $u \in L^\infty(\Omega)$, so we may assume that u satisfies:

$$\tag{0.2} \begin{cases} \operatorname{div} Tu = H & \text{in } \Omega \\ Tu \cdot \nu = \cos\gamma & \text{on } \Gamma \end{cases}$$

for some bounded continuous function $H = H(x)$ in Ω.

1. Continuity of u at the corner. Let $(0, a) \in \mathbf{R}^2 \times \mathbf{R} = \mathbf{R}^3$ be any point lying in the closure of the graph of u over Ω.

Define $v(x) = u(x) - a$.

THEOREM 1.1. *Under the above assumptions, we have*

$$\tag{1.1} \lim_{\substack{x\to 0\\ x\in\Omega}} \frac{v(x)}{x^1} = -\infty \quad \text{where } x = (x^1, x^2) \in \mathbf{R}^2.$$

Note that if x is close enough to the origin, we have $x^1 > 0$. Therefore without loss of generality, we may assume that $x^1 > 0$ for all $x \in \Omega$.

Proof. Suppose that (1.1) is not true, then there exists a real number M and a sequence of points $x_j \in \Omega$ such that $\lim_{j\to\infty} x_j = 0$ and

$$\frac{v(x_j)}{x_j^1} \geq M. \tag{1.2}$$

We want to get a contradiction from this. For this purpose we need several lemmas.

With minor modifications, the proofs of Lemma 1.2–1.6 in the following can be found in the literature. So we shall not prove them, but only give the references. We state them here for the convenience of the reader.

Let $\varepsilon_j = x_j^1$, then $\lim_{j\to\infty} \varepsilon_j = 0$. Define $v_j(x) = v(\varepsilon_j x)/\varepsilon_j$. Then $v_j(x)$ satisfies:

$$\begin{cases} \operatorname{div} Tv_j = \varepsilon_j H & \text{in } \Omega_j = \left\{ x \in \mathbf{R}^2 | \varepsilon_j x \in \Omega \right\}, \\ Tv_j \cdot \nu_j = \cos\gamma & \text{on } \Gamma_j = \left\{ x \in \mathbf{R}^2 | \varepsilon_j x \in \Gamma \right\}, \end{cases} \tag{1.3}$$

where ν_j is the unit outward normal of Γ_j. Notice that $v_j \in C^2(\overline{\Omega}_j - \{0\}) \cap L^\infty(\Omega_j)$ for all j.

Let $\Omega_\infty = \lim_{j\to\infty} \Omega_j = \{x \in \mathbf{R}^2 | |x^2| < (\tan\alpha)x^1\}$.

As shown in [9] (see also [3] and [10]), noting that $\varepsilon_j H$ tend to zero everywhere in Ω_∞, and $\varepsilon_j H$ are uniformly bounded, using the terminology in [3] we have:

LEMMA 1.2. *We can find a subsequence of v_j which converges locally to a generalized solution v_∞ in Ω_∞ of*

$$\mathscr{F}(w) \equiv \int_{\Omega_\infty} \sqrt{1 + |Dw|^2} - \cos\gamma \int_{\partial\Omega_\infty} w\, dH_1 \tag{1.4}$$

where H_k is the k-dimensional Hausdorff measure in $\mathbf{R}^n$, $k \leq n$. That is to say, if $V_\infty = \{(x, t) \in \Omega_\infty \times \mathbf{R} | t < v_\infty(x)\}$ is the subgraph of v_∞, then for any compact set $K \subset \mathbf{R}^3$, and for any Caccioppoli set (set of locally finite perimeter) E, such that $\operatorname{spt}(\varphi_{V_\infty} - \varphi_E) \subset K$, we have

$$F_K(V_\infty) \leq F_K(E) \tag{1.5}$$

where

$$F_K(W) \equiv \int_{(\Omega'_\infty \times \mathbf{R}) \cap K} |D\varphi_W| - \cos\gamma \int_{(\partial\Omega_\infty \times \mathbf{R}) \cap K} \varphi_W\, dH_2, \tag{1.6}$$

and where φ_W denotes the characteristic function of W.

A sequence of functions f_j is said to converge locally to a function f in a domain D, if the characteristic functions of the subgraphs of f_j converge almost everywhere to the characteristic function of the subgraph of f in $D \times \mathbf{R}$.

Note that v_∞ may take the value ∞ or $-\infty$.

Define

$$P = \{ x \in \Omega_\infty | v_\infty(x) = \infty \} \tag{1.7}$$

$$N = \{ x \in \Omega_\infty | v_\infty(x) = -\infty \}. \tag{1.8}$$

As in **[3]** (see also **[9, 10]**), we know that P minimizes

$$G(A) \equiv \int_{\Omega_\infty} |D\varphi_A| - \cos\gamma \int_{\partial\Omega_\infty \cap K} \varphi_A \, dH_1 \tag{1.9}$$

for Caccioppoli set $A \subset \Omega_\infty$. That is, for any compact set $K \subset \mathbf{R}^2$, and any Caccioppoli set with $\operatorname{spt}(\varphi_A - \varphi_P) \subset K$, we have

$$G_K(P) \equiv \int_{\Omega_\infty \cap K} |D\varphi_P| - \cos\gamma \int_{\partial\Omega_\infty \cap K} \varphi_P \, dH_1 \le G_K(A). \tag{1.10}$$

Similarly, N minimizes

$$G'(A) \equiv \int_{\Omega_\infty} |D\varphi_A| + \cos\gamma \int_{\partial\Omega_\infty} \varphi_A \, dH_1. \tag{1.11}$$

We want to know the structure of P and N, and we have:

LEMMA 1.3. *If $L \subset \Omega_\infty$ minimizes $G(A)$ defined in* (1.9), *then L equals to Ω_∞, $\varnothing$ or some $\triangle OAB$ bounded by $\partial\Omega_\infty$ and $x^1 = a$ for some $a > 0$. (See Figure* 2.)

The proof of the lemma is similar to the proof of Theorem 2.4 for the case $\alpha + \gamma > \pi/2$ in **[10]**. In that case, the conclusion is that $L = \Omega_\infty$ or $\varnothing$. In our case, it is possible to have $L = \triangle OAB$ described in the lemma because $2\alpha + 2\gamma = \pi$. We shall omit the proof. Similarly we have:

LEMMA 1.4. *If L minimizes $G'(A)$ defined by* (1.11), *then L equals to Ω_∞, $\varnothing$ or $\Omega_\infty - \triangle OAB$ for some $\triangle OAB$ described in Lemma* 1.3.

Since P minimizes $G(A)$ and N minimizes $G'(A)$, we conclude that

(1.12) $P = \Omega_\infty$, $\varnothing$ or $\triangle OAB$ which is bounded by $\partial\Omega_\infty$ and $x^1 = a$ for some $a > 0$.

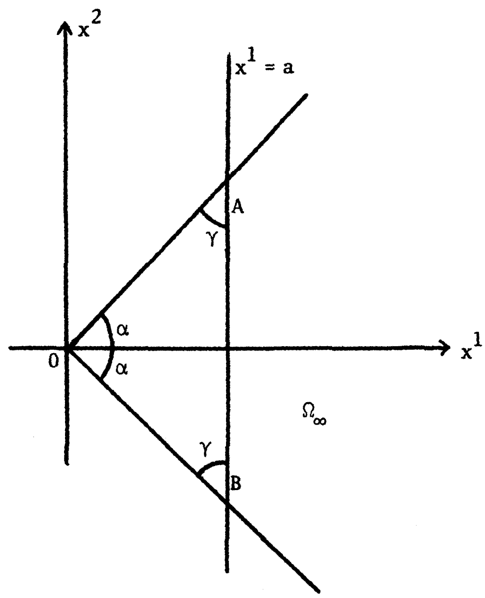

FIGURE 2

(1.13) $N = \Omega_\infty$, $\varnothing$ or $\Omega_\infty - \triangle OA'B'$ for some $\triangle OA'B'$ which is bounded by $\partial\Omega_\infty$ and $x^1 = a'$ for some $a' > 0$.

It is not hard to see from the proof of Lemma 3.1 in **[11]** that the following estimates are true. (See also **[3]**.) Let V_j be the subgraph of v_j.

LEMMA 1.5. *There exists $r_0 > 0$, $C > 0$ not depending on j such that for all $t \in \mathbf{R}$, the following is true*:

(1.14) *if $|V'_{j,r}(0,t)| > 0$ for all $r > 0$ then $|V'_{j,r}(0,t)| \geq Cr^3$ for all $0 < r \leq r_0$, where $C_r(x_0, t_0) = \{(x,t) \in \mathbf{R}^3 \,|\, |x - x_0| < r$ and $|t - t_0| < r\}$ and $V'_{j,r}(0,t) = C_r(0,t) - V_j$.*

LEMMA 1.6. *For any $0 < \tau_1 < \tau_2 < \infty$, there exist positive integer j_0 and positive numbers r_1 and C_1 such that for all $j \geq j_0$ and $(x,t) \in \Omega_j \cap \{x \in \mathbf{R}^2 \,|\, \tau_1 \leq x^1 \leq \tau_2\}$, the following are true*:

(1.15) *if $|V_{j,r}(x,t)| > 0$ for all $r > 0$, then $|V_{j,r}(x,t)| \geq C_1 r^3$, for all $0 < r \leq r_1$;*

(1.16) *if $V'_{j,r}(x,t) > 0$ for all $r > 0$, then $|V'_{j,r}(x,t)| \geq C_1 r^3$ for all $0 < r \leq r_1$,*

where $V_{j,r}(x,t) = C_r(x,t) \cap V_j$ and $V'_{j,r}(x,t) = C_r(x,t) - V_j$.

Notice that even though we do not have a similar result as (1.15) at the corner (because of the fact that $\alpha + \gamma = \pi/2$), we still have (1.14) since $\cos\gamma > 0$, as one can see from the proof of Lemma 3.1 in **[11]**.

Using the above lemmas, we can prove:

LEMMA 1.7. *$P = \{x \in \Omega_\infty | v_\infty(x) = \infty\}$ is empty.*

Proof. If $P \neq \varnothing$, then by Lemma 1.3, $P = \Omega_\infty$ or some $\triangle OAB$ which is bounded by $\partial\Omega_\infty$ and $x^1 = a$ for some $a > 0$. In any case, there is $\bar{r} > 0$ such that

$$|V'_{\infty,r}(0,0)| = |C_r(0,0) - V_\infty| = 0 \quad \text{for all } 0 < r \le \bar{r}. \tag{1.17}$$

By Lemma 1.5 and the fact that $(0,0) \in \mathbf{R}^3$ lies in the closure of the graph of v_j and that v_j is regular in $\overline{\Omega}_j - \{0\}$, we have:

$$|V'_{j,r}(0,0)| > 0 \quad \text{for all } r > 0, \quad \text{and so}$$
$$|V'_{j,r}(0,0)| \ge Cr^3 \quad \text{for all } 0 < r \le r_0.$$

In particular, if we take $r = \min(\bar{r}, r_0) > 0$, then

$$|V'_{j,r}(0,0)| \ge Cr^3.$$

Let $j \to \infty$, noting that φ_{V_j} converges to φ_{V_∞} almost everywhere in $\Omega_\infty \times \mathbf{R}$, we have

$$|V'_{\infty,r}(0,0)| \ge Cr^3 > 0.$$

This contradicts (1.17). Therefore P must be empty and the lemma is proved. $\square$

LEMMA 1.8. *If $N = \{x \in \Omega_\infty | v_\infty = -\infty\}$, then $N = \Omega_\infty$.*

Proof. By (1.13) and Lemma 1.7, if $N \neq \Omega_\infty$, then there exists $\tau > 0$ such that v_∞ is finite almost everywhere in $\{x \in \Omega_\infty | 0 < x^1 < \tau\}$. We claim that there is a positive integer j_0 such that

$$\sup_{j \ge j_0} \sup_{\substack{x \in \Omega_j \\ \tau/4 < x^1 < 3\tau/4}} |v_j(x)| < \infty. \tag{1.18}$$

Let j_0, r_1, and C_1 be the constants in Lemma 1.6 corresponding to $\tau_1 = \tau/4$, and $\tau_2 = 3\tau/4$.

Since each v_j is bounded in Ω_j, if (1.18) is not true, then we can find a subsequence of v_j, which we also call v_j, and $\bar{x}_j \in \Omega_j$, $\tau/4 < \bar{x}_j < 3\tau/4$, such that

$$\lim_{j \to \infty} |v_j(\bar{x}_j)| = \infty.$$

Passing to a subsequence if necessary, we may assume that $\lim_{j\to\infty} \bar{x}_j = z = (z^1, z^2)$ which is in $\bar{\Omega}_\infty$, with $\tau/4 \le z^1 \le 3\tau/4$, and such that

$$\lim_{j\to\infty} v_j(\bar{x}_j) = \infty, \quad \text{or} \qquad (1.19)$$
$$\lim_{j\to\infty} v_j(\bar{x}_j) = -\infty.$$

Suppose that $\lim_{j\to\infty} v_j(\bar{x}_j) = \infty$. Then for any $t > 0$, if j is large enough, we have

$$\left|V_{j,r}(\bar{x}_j, t)\right| > 0 \quad \text{for all } r > 0.$$

Hence by (1.15), if j is large enough, we have

$$\left|V_{j,r}(\bar{x}_j, t)\right| \ge C_1 r^3 \quad \text{for all } 0 < r \le r_1.$$

Let $j \to \infty$, we get

$$\left|V_{\infty,r}(z, t)\right| \ge C_1 r^3 \quad \text{for all } 0 < r \le r_1.$$

Since t can be arbitrarily large, this contradicts the fact that $P = \varnothing$.

Suppose that $\lim_{j\to\infty} v_j(\bar{x}_j) = -\infty$, then for any $t < 0$, if j is large enough, we have

$$\left|V'_{j,r}(\bar{x}_j, t)\right| > 0 \quad \text{for all } r > 0.$$

By (1.16), we have

$$\left|V'_{j,r}(\bar{x}_j, t)\right| \ge C_1 r^3 \quad \text{for all } 0 < r \le r_1.$$

Take $\bar{r} = \min(\frac{1}{4}\tau, r_1) > 0$ and let $j \to \infty$, we get

$$\left|V'_{\infty,\bar{r}}(z, t)\right| \ge C_1 \bar{r}^3 \quad \text{for all } t < 0.$$

Since t can be arbitrarily small, this contradicts the fact that v_∞ is finite almost everywhere in $\{x \in \Omega_\infty \,|\, 0 < x^1 < \tau\}$.

In any case, we have a contradiction. Therefore (1.18) is true.

By Theorem 3 in [7], v_∞ is regular in $D = \{x \in \Omega_\infty \,|\, \tau/4 < x^1 < 3r/4\}$ after modification by a set of measure zero. By the results of [6], we have

$$\begin{cases} \lim_{j\to\infty} v_j(x) = v(x) \\ \lim_{j\to\infty} Dv_j(x) = Dv(x) \end{cases} \qquad (1.20)$$

for $x \in D$. Integrating $\operatorname{div} Tv_j = \varepsilon_j H$ over $D_j = \{x \in \Omega_j \,|\, 0 < x^1 < \tau/2\}$, using (1.3) and let $\eta = (-1, 0, 0)$, we have, for j large enough:

$$\int_{\Gamma_j \cap \{0 < x^1 < \tau/2\}} Tv_j \cdot \nu_j \, dH_1 = \int_{D_j} \varepsilon_j H \, dx + \int_{D_j \cap \{x^1 = \tau/2\}} Tv_j \cdot \eta \, dH_1.$$

Since $Tv_j \cdot \nu_j = \cos\gamma$ on Γ_j, and $\lim_{j\to\infty} \varepsilon_j H = 0$, if we let $j \to \infty$, we get

$$\cos\gamma \cdot H_1\left(\partial\Omega_\infty \cap \left\{0 < x^1 < \frac{\tau}{2}\right\}\right) = \int_{D\cap\{x^1=\tau/2\}} Tv_\infty \cdot \eta \, dH_1.$$

But

$$\cos\gamma \cdot H_1\left(\partial\Omega_\infty \cap \left\{0 < x^1 < \frac{\tau}{2}\right\}\right) = H_1\left(D \cap \left\{x^1 = \frac{\tau}{2}\right\}\right).$$

Since $|Tv_\infty \cdot \eta| \le 1$, we conclude that $Tv_\infty \cdot \eta = 1$ H_1-almost everywhere on $D \cap \{x^1 = \tau/2\}$. This contradicts the fact that v_∞ is regular in D. Hence we must have $N = \Omega_\infty$. □

REMARK. We may simplify the proof by using the fact that V_∞ is a cone with vertex at the origin. But in the next section we shall use a similar argument, so we do it this way.

Conclusion of the proof of Theorem 1.1. Using the fact that $N = \Omega_\infty$ and using (1.15) and similar method of proof of (1.18), we can conclude that

$$\lim_{j\to\infty} \sup_{\substack{x\in\Omega_j \\ 1\le x^1\le 3/2}} v_j(x) = -\infty.$$

In particular, we have

$$\lim_{j\to\infty} \frac{v(x_j)}{x_j^1} = \lim_{j\to\infty} v_j\left(1, \frac{x_j^2}{x_j^1}\right) = -\infty.$$

This contradicts (1.2), and the proof of Theorem 1.1 is complete. □

Now we can prove the continuity of u.

THEOREM 1.9. *u extends to be a continuous function in $\overline{\Omega}$.*

Proof. If this is not true, then there exist real numbers $b > a$, such that $(0, a)$ and $(0, b)$ are both in the closure of the graph of u. Let $v = u - a$. By Theorem 1.1, we have

$$\lim_{\substack{x\to 0 \\ x\in\Omega}} \frac{v(x)}{x^1} = -\infty.$$

In particular, there exists $r > 0$, such that if $x \in \Omega$ and $|x| < r$, then $v(x)/x^1 < 0$. Therefore $u(x) < a$ for such x. Since $(0, b)$ also lies in the closure of the graph of u, we can always find $x \in \Omega$ with $0 < |x| < r$ and $u(x) > a$. This leads to contradiction and the theorem follows. □

2. Continuity of the normal. Let us proceed and examine the continuity of the normal of the graph of u over Ω. Since u is continuous at the origin, by adding a constant to u, we may assume that $u(0) = 0$. u still satisfies (0.2). We want to prove:

$$\lim_{\substack{x \to 0 \\ x \in \Omega}} \left(Tu, \frac{-1}{\sqrt{1 + |Du|^2}} \right) = (-1, 0, 0).$$

Since $u \in C^2(\bar{\Omega} - \{0\})$, it is sufficient to prove that for any sequence $x_j \in \Omega$, converging to 0, we have

$$\lim_{j \to \infty} \left(Tu(x_j), \frac{-1}{\sqrt{1 + |Du(x_j)|^2}} \right) = (-1, 0, 0). \tag{2.1}$$

First, we shall establish (2.1) for any sequence x_j tending to the origin non-tangentially to $\partial\Omega$. More precisely, we assume that there is ε with $0 < \varepsilon < \tan\alpha$, such that $x_j = (x_j^1, x_j^2)$ lies between the straight lines $x^2 = \pm(\tan\alpha - \varepsilon)x^1$.

THEOREM 2.1. *Let $x_j = (x_j^1, x_j^2) \in \Omega$ be a sequence of points approaching the origin such that $|x_j^2| < (\tan\alpha - \varepsilon)x_j^1$ for all j for some ε with $0 < \varepsilon < \tan\alpha$. Then* (2.1) *holds.*

Proof. If we can prove that for any subsequence of x_j, we can find a subsequence of the subsequence such that (2.1) is true for that subsequence, then we are done.

Since every subsequence of x_j also satisfies the assumptions of the theorem, so we may assume that the subsequence is $\{x_j\}$ itself.

Since $x_j^1 > 0$ for all j, if we set $\varepsilon_j = x_j^1$ and define

$$u_j(x) = \frac{1}{\varepsilon_j} u(\varepsilon_j x) - \frac{1}{\varepsilon_j} u(x_j),$$

then as in §1, u_j satisfies:

$$\begin{cases} \operatorname{div} Tu_j = \varepsilon_j H & \text{in } \Omega_j \\ Tu_j \cdot \nu_j = \cos\gamma & \text{on } \Gamma_j. \end{cases} \tag{2.2}$$

Also if

$$\bar{x}_j = \left(1, x_j^2/\varepsilon_j\right) = \left(1, x_j^2/x_j^1\right),$$

then

$$u_j(\bar{x}_j) = 0. \tag{2.3}$$

We may also assume that

$$\lim_{j\to\infty} \bar{x}_j = z = (1, z^2) \in \Omega_\infty \quad \text{with } |z^2| \le \tan\alpha - \varepsilon. \tag{2.4}$$

As in §1, we can find a subsequence of u_j, which we also call u_j, converging locally to a generalized solution u_∞ of $\mathscr{F}(w)$ defined by (1.4). Let

$$P = \left\{x \in \Omega_\infty \middle| u_\infty(x) = +\infty\right\}$$

and

$$N = \left\{x \in \Omega_\infty \middle| u_\infty(x) = -\infty\right\}.$$

As in §1, we know that $P = \Omega_\infty$, $\varnothing$ or some $\triangle OAB$ bounded by $\partial\Omega_\infty$ and $x^1 = a$ for some $a > 0$; and $N = \Omega_\infty$, $\varnothing$ or $\Omega_\infty - \triangle OA'B'$ for some $\triangle OA'B'$ bounded by $\partial\Omega_\infty$, and $x^1 = a'$ for some $a' > 0$.

Note that Lemma 1.6 is still true for the subgraph U_j of u_j. That is to say for any $0 < \tau_1 < \tau_2 < \infty$, there exist a positive integer j_0 and positive numbers r_1 and C_1 not depending on j such that for $j \ge j_0$ and for any $(x,t) \in \overline{\Omega}_j \cap \{x \in R^2 | \tau_1 < x^1 < \tau_2\}$, (1.15) and (1.16) are still true if we replace V_j by U_j.

Suppose that $\Omega_\infty - (P \cup N) \ne \varnothing$, because of the structures of P and N, there exist $0 < a < b < \infty$ such that u_∞ is finite almost everywhere in $\{x \in \Omega_\infty | a < x^1 < b\}$. Using (1.15) and (1.16) as in the proof of Lemma 1.8, we shall arrive at a contradiction.

Hence we must have $\Omega_\infty = P \cup N$.

Let U_∞ be the subgraph of u_∞. Since $u_j(\bar{x}_j) = 0$ so $(\bar{x}_j, 0)$ belongs to the boundary of U_j. Using (1.15), (1.16), the fact that $\lim_{j\to\infty} \bar{x}_j = z$, $u_j \in C^2(\overline{\Omega} - \{0\})$, and that φ_U converge to φ_{U_∞} almost everywhere in $\Omega_\infty \times \mathbf{R}$, we have:

$$|U_{\infty,r}(z,0)| \ge C_1 r^3, \quad \text{and} \quad |U'_{\infty,r}(z,0)| \ge C_1 r^3 \tag{2.5}$$

for all $0 < r \le r_1$. Hence $P \ne \Omega_\infty$ and $N \ne \Omega_\infty$. Combining this with the fact that $\Omega_\infty = P \cup N$, we conclude that there is an $a > 0$ such that if OAB is the triangle bounded by $\partial\Omega_\infty$ and $x^1 = a$, then $P = \triangle OAB$ and $N = \Omega_\infty - \triangle OAB$. So $U_\infty = \triangle OAB \times \mathbf{R}$.

In fact, we must have $a = 1$. Otherwise, as $z = (1, z^2)$, $a < 1$ will give a contradiction to the first inequality of (2.5), while $a > 1$ will give a contradiction to the second inequality of (2.5).

The inward normal of ∂U_∞ at $(z, 0) \in \mathbf{R}^3$ is $(-1, 0, 0)$, and the inward normal of ∂U_j at $(\bar{x}_j, u(\bar{x}_j))$ is $(Tu_j(\bar{x}_j), -1/\sqrt{1 + |Du_j(\bar{x}_j)|^2})$. Since $\lim_{j\to\infty}(\bar{x}_j, u_j(\bar{x}_j)) = (z, 0)$, so by Theorem 3 in [6], we have:

$$\lim_{j\to\infty}\left(Tu_j(\bar{x}_j), \frac{-1}{\sqrt{1 + |Du_j(\bar{x}_j)|^2}}\right) = (-1, 0, 0).$$

From the definitions of u_j and $\bar{x}_j$, we conclude that

$$\lim_{j\to\infty}\left(Tu(x_j), \frac{-1}{\sqrt{1 + |Du(x_j)|^2}}\right) = (-1, 0, 0). \qquad \square$$

Finally, we consider the case when x_j approaches the origin tangentially along $\partial\Omega_\infty$. We want to prove:

THEOREM 2.2. *Under the above assumptions*, (2.1) *is still true*, *namely*:

$$\lim_{j\to\infty}\left(Tu(x_j), \frac{-1}{\sqrt{1 + |Du(x_j)|^2}}\right) = (-1, 0, 0).$$

Proof. As in Theorem 2.1, it is sufficient to prove that (2.1) is true for a subsequence of x_j.

Define u_j and $\bar{x}_j$ as in Theorem 2.1. We also assume that $\lim_{j\to\infty}\bar{x}_j = z = (1, z^2)$ which lies in $\bar{\Omega}_\infty$, with $z^2 = \pm\tan\alpha$.

We can extract a subsequence of u_j, which we also denote by u_j, such that u_j converges locally to a generalized solution of $\mathscr{F}(w)$ in Ω_∞.

Using similar method as in Theorem 2.1, we can prove that the subgraph U_∞ of u_∞ is $\triangle OAB \times \mathbf{R}$ for some $\triangle OAB$ bounded by $\partial\Omega_\infty$ and $x^1 = 1$. Up to this point, the proof is exactly the same as the proof in Theorem 2.1. However, in this case $z \in \partial\Omega_\infty$ and we cannot apply the results of [6]. So we need some modifications. Before we proceed further, let us prove the following lemma.

LEMMA 2.3. (a) *For any* $0 < \tau_1 < \tau_2 < 1$, *we have*

$$\lim_{j\to\infty}\ \inf_{\substack{x\in\bar{\Omega}_j\\ \tau_1 < x^1 < \tau_2}} u_j(x) = \infty; \quad \textit{and} \tag{2.6}$$

(b) *For any* $1 < \tau_3 < \tau_4 < \infty$, *we have*

$$\lim_{j\to\infty} \sup_{\substack{x\in\overline{\Omega}_j \\ \tau_3 < x^1 < \tau_4}} u_j(x) = -\infty. \tag{2.7}$$

Proof. We shall prove (a) only, because the proof of (b) is similar.

Suppose that (2.6) is not true. Since $u_j \in C^2(\overline{\Omega}_j - \{0\})$, therefore we can find a real number M, a subsequence of u_j (which we also call u_j) and a sequence of points $y_j \in \Omega_j$, $\tau_1 < y_j^1 < \tau_2$ such that

$$u_j(y_j) \le M.$$

We may also assume that $\lim_{j\to\infty} y_j = y \in \overline{\Omega}_\infty$. Note that $\tau_1 \le y^1 \le \tau_2$. By (1.16) as before, we have

$$\left|U'_{j,r}(y_j, M)\right| \ge C_1 r^3$$

for all $0 < r \le r_1$ if j is large enough, where C_1, and r_1 are positive constants not depending on j. Now let $j \to \infty$, we have

$$\left|U'_{\infty,r}(y, M)\right| \ge C_1 r^3 \quad \text{for all } 0 < r \le r_1.$$

This contradicts the fact that $U_\infty = \triangle OAB \times \mathbf{R}$ and that $0 < \tau_1 < \tau_2 < 1$, bearing in mind the definition of $\triangle OAB$. The lemma is then proved. □

We now continue our proof of Theorem 2.2. By Lemma 2.3, since u_j is continuous in $\overline{\Omega}_j - \{0\}$, there exists j_0 such that for every $j \ge j_0$ we can find $y_j \in \partial\Omega_j$ with $u_j(y_j) = 0$ and $\lim_{j\to\infty} y_j = z$.

Let $Y_j = (y_j, u_j(y_j)) = (y_j, 0) \in \mathbf{R}^3$. By the results of **[12]**, there exist $r_2 > 0$, $C_2 > 0$ and $1 > \alpha > 0$ not depending on j such that if $\eta_j(X)$ is the unit inward normal of ∂U_j at the point $X \in \partial U_j \cap \Omega_j$ we have

$$\left|\eta_j(X) - \eta_j(\overline{X})\right| \le C_2 |X - \overline{X}|^\alpha \tag{2.8}$$

for any X, $\overline{X}$ belong to $\partial U_j \cap \Omega_j$ and $B_{r_2}(Y_j) = \{X \in \mathbf{R}^3 \,|\, |X - Y_j| < r_2\}$.

For any $r_2/2 > r > 0$, use Lemma 2.3 again, we can find $z_j \in \Omega_j$ and ε with $\tan\alpha > \varepsilon > 0$ not depending on j such that if j is large enough, we have

$$\begin{cases} |z_j^2| < (\tan\alpha - \varepsilon) z_j^1 \\ u_j(z_j) = 0 \\ |z_j - z| < r \\ \lim\limits_{j\to\infty} z_j^1 = 1. \end{cases} \tag{2.9}$$

Let $Z_j = (z_j, u_j(z_j)) = (z_j, 0)$, $Z = (z, 0)$ and $\bar{X}_j = (\bar{x}_j, u_j(\bar{x}_j)) = (\bar{x}_j, 0)$.

Then $\lim_{j\to\infty} Y_j = Z = \lim_{j\to\infty} \bar{X}_j$. If j is large enough, then we have

$$|\bar{X}_j - \bar{Y}_j| < r_2$$

and

$$|Z_j - Y_j| \le |Z_j - Z| + |Z - Y_j| < r + \frac{r_2}{2} < r_2.$$

By (2.8) we obtain

$$|\eta(Z_j) - \eta_j(\bar{X}_j)| \le C_2 |Z_j - \bar{X}_j|^\alpha. \tag{2.10}$$

Since $\lim_{j\to\infty} z_j^1 = 1$, and $|z_j^2| < (\tan\alpha - \varepsilon) z_j^1$, so by Theorem 3 of **[6]**, for any subsequence $\bar{Z}_j$ of Z_j, we can always find a subsequence $\bar{Z}_j'$ of $\bar{Z}_j$ such that $\lim_{j\to\infty} \eta_j(\bar{Z}_j') = (-1, 0, 0)$.

Therefore $\lim_{j\to\infty} \eta_j(Z_j) = (-1, 0, 0) = \eta$.

Also, it is easy to see from (2.9) that

$$\limsup_{j\to\infty} |Z_j - \bar{X}_j| \le r.$$

Let $j \to \infty$ in (2.10), we then have

$$\limsup_{j\to\infty} |\eta - \eta_j(\bar{X}_j)| \le C_2 r^\alpha.$$

Now let $r \to 0$, we conclude that $\lim_{j\to\infty} |\eta - \eta_j(\bar{X}_j)| = 0$. The proof of Theorem 2.2 is then completed. □

Combining Theorems 2.1 and 2.2, we get

THEOREM. *The unit normal vector* $(Tu, -1/\sqrt{1 + |Du|^2})$ *extends to be continuous on* $\bar{\Omega}$. *More precisely,*

$$\lim_{\substack{x\to 0 \\ x\in\bar{\Omega}-\{0\}}} \left(Tu(x), \frac{-1}{\sqrt{1 + |Du(x)|^2}} \right) = (-1, 0, 0).$$

Acknowledgment. I wish to thank Robert Finn and Brian White for useful discussions.

REFERENCES

[1] P. Concus and R. Finn, *Capillary free surfaces in a gravitational field*, Acta Math., **132** (1974), 207–223.

[2] R. Finn, *Existence criteria for capillary free surfaces without gravity*, Indiana Univ. Math. J., **32** (1983), 439–460.

[3] E. Giusti, *Generalized solutions of mean curvature equations*, Pacific J. Math., **88** (1980), 297–321.

[4] _____, *Minimal surfaces and functions of bounded variation*. Notes on pure mathematics. Australian National Univ., Canberra (1977).

[5] N. J. Korevaar, *On the behavior of a capillary surface at a re-entrant corner*, Pacific J. Math., **88** (1980), 379–385.

[6] U. Massari and L. Pepe, *Sulle successioni convergenti di superfici a curvatura media assegnata*, Rend. Sem. Mat. Padova, **53** (1975), 53–68.

[7] M. Miranda, *Un principio di massimo forte per le frontiere minimali ecc.*, Rend. Sem. Mat. Padova, **45** (1971), 355–366.

[8] _____, *Superfici minime illimitate*, Ann. Scuola Norm. Sup. Pisa, (4) **4** (1977), 313–322.

[9] L. Simon, *Regularity of capillary surfaces over domains with corners*, Pacific J. Math., **88** (1980), 363–377.

[10] L.-F. Tam. *The behavior of capillary surfaces as gravity tends to zero*, to appear in Comm. in Partial Differential Equations.

[11] _____, *Existence criteria for capillary free surfaces without gravity*, to appear in Pacific J. Math.

[12] J. Taylor, *Boundary regularity for solutions to various capillarity and free boundary problems*, Comm. in Partial Differential Equations, **2** (1977), 323–357.

Received February 7, 1984.

PURDUE UNIVERSITY
WEST LAFAYETTE, IN 47907