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QUADRATIC FORMS OVER DYADIC VALUED FIELDS I, THE GRADED WITT RING

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This paper gives a detailed account of the arithmetic of quadratic forms over a field F of characteristic 0, carrying a 2-Henselian discrete valuation with residue field of characteristic 2. We give an analogue of Springer's Theorem for the graded Witt ring of such a field, and describe new counterexamples to the amenabilitiy problem for multiquadratic extensions. The sequel to this paper will contain an axiomatic approach to the results contained herein, and will treat the Galois cohomology of such fields.

0. Introduction. This paper is devoted to the concrete calculation of the graded Witt ring of a 2-Henselian dyadic, discretely valued field. Section 1 is devoted to some computations necessary to test for the isotropicity of n-fold Pfister forms over such fields. These results, which are quite general and are proved for arbitrary dyadic valued fields, are of interest in their own right. Section 2 is the computational section, where we find bases for some specific ideal quotients needed in §3. We encourage the reader to read the statements of the results in §3 before reading §2, to clarity the goals of that section.

Section 3 contains the main result, which is the analogue of Springer's Theorem for the graded Witt ring of a 2-Henselian dyadic, discretely valued field (cf. [W] for the non-dyadic version). In particular, we compute $GW(F)$ for such a field F in terms of $GW(\mathscr{F})$, $\mathscr{F}$, and $v(2) \in \mathbf{Z}$, where $\mathscr{F}$ is the residue class field of F. The result is stated for the case where $\mathscr{F}$ has a finite 2-basis, the infinite 2-basis case can be obtained from this in an obvious way. The final §4 is devoted to some specific applications of §3, answering questions concerning the "amenability problem".

There have been several papers in the literature (cf. [T], in addition to others) on the behavior of quadratic forms over dyadic valued fields. None of these treatments are complete (in the sense that Springer's [S] non-dyadic treatment is complete), presumably because the problem is so very complicated and messy. In recent years, a need for a detailed account of the behavior of quadratic forms over dyadic valued fields has arisen, largely for two reasons (as far as this author is aware). First, dyadic valued fields provided a source of counterexamples for the phenomenon of "amenability". See [ELW1, 2], [ELTW], and [STW]. One needs to understand what is going on with these examples more exactly. Secondly, Kato has recently shown in [K2], that an important problem of Milnor's [M1] has an affirmative answer for complete, dyadic, discretely valued fields. His proofs, which involve K-theory and Galois cohomology, do not illuminate what's really happening to quadratic forms over such fields, so one desires to have more explicit information.

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1. Some representation computations. In this section we establish notation and prove some basic results needed for the rest of this paper. Throughout this paper $F^{\bullet} = F - \{0\}$ and $v\colon F^{\bullet} \to G$ will denote an additively written (Krull) valuation on the field F with value group G. F will always have characteristic 0, while the residue field $\mathscr{F}$ will always have characteristic 2. We recall that the set of squares $\mathscr{F}^2$ forms a subfield of $\mathscr{F}$. Also, if $t_1, \ldots, t_n \in \mathscr{F}$ one says that the set $\{t_1, \ldots, t_n\}$ is 2-independent in $\mathscr{F}$ if $\mathscr{F}^2 \neq \mathscr{F}^2(t_1) \neq \mathscr{F}^2(t_1, t_2) \neq \cdots \neq \mathscr{T}^2(t_1, \ldots, t_n)$. If, in addition, $\mathscr{F} = \mathscr{F}^2(t_1, \ldots, t_n)$ the $\{t_1, \ldots, t_n\}$ are called a 2-basis for $\mathscr{F}$.

We denote by O_v, M_v, U_v, the valuation ring, maximal ideal, and units of $v\colon F^{\bullet} \to G$ respectively. The subscripts will be deleted when no confusion may arise. We define $U^{\gamma} := \{x \in F \mid v(1-x) \geq \gamma\}$ for $\gamma \in G$, $\mathring{U}^{\gamma} := \{x \in F\colon v(1-x) > \gamma\}$, and $\hat{U}^{\gamma} := U^{\gamma} - \mathring{U}^{\gamma}$. For any $t_1, \ldots, t_n \in U$, with images $\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_n$ in $\mathscr{F}$, we define $\mathscr{F}_0^2(\bar{t}_1, \ldots, \bar{t}_n)$ to be the *additive subgroup* of $\mathscr{F}$ consisting of elements of the form $\sum_{\alpha \neq (0,\ldots,0)} \bar{t}^{\alpha} x_{\alpha}^2$, where the $x_{\alpha} \in \mathscr{F}$ and where $\bar{t}^{\alpha}$ means $\bar{t}_1^{i_1}\bar{t}_2^{i_2} \cdots \bar{t}_n^{i_n}$ whenever $\alpha = (i_1, i_2, \ldots, i_n)$ for $i_1, i_2, \ldots, i_n \in \{0, 1\}$. So if $\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_n$ are 2-independent $\mathscr{F}_0^2(\bar{t}_1, \ldots, \bar{t}_n)$ is a $2^n - 1$ dimensional $\mathscr{F}^2$-subspace of $\mathscr{F}$ that does not contain $\mathscr{F}^2$. Finally, $\wp(x) = x^2 + x$ will denote the characteristic 2 Artin-Schreier operator, which as the reader will recall defines an additive homomorphism $\wp\colon \mathscr{F} \to \mathscr{F}$.

For the rest of this section we adopt the following:

Standing Hypothesis 1.1. We fix $u \in U$ with $v(1-u) = \gamma$, where $\gamma \in \Delta := \{\delta \in G \mid 0 < \delta \leq v(4)\}$. We fix $t_1, \ldots, t_j$ to be units of F such that their residues $\{\bar{t}_1, \ldots, \bar{t}_j\}$ are 2-independent in $\mathscr{F}$. Elements $\pi_1, \ldots, \pi_m$ of F are also fixed where we assume that the residues of $v(\pi_1), \ldots, v(\pi_m)$ in $G/2G$ are independent in this $\mathbf{Z}/2\mathbf{Z}$-vector space. Depending upon γ

there are three possible cases which we always refer to as cases (i), (ii), (iii) throughout the rest of this section

Case (i). Here $\gamma \notin 2G$. In this case we *additionally* assume that

$$\gamma \notin 2G(\pi_1, \dots, \pi_k)$$

where $2G(\pi_1, \dots, \pi_m)$ denotes the subgroup of G generated by $2G$ and $\pi_1, \dots, \pi_m$.

Case (ii). Here $\gamma = v(\pi^2) \neq v(4)$ for some $\pi \in F$. In this case we *additionally* assume that the residue

$$\overline{\pi^{-2}(1-u)} \notin \mathscr{F}^2(\bar{t}_1, \dots, \bar{t}_j).$$

Case (iii). Here $\gamma = v(4)$. In this case we *additionally* assume that

$$\overline{(1-u)/4} \notin \wp(\mathscr{F}) + \mathscr{F}_0^2(\bar{t}_1, \dots, \bar{t}_j).$$

LEMMA 1.2. *Suppose u and γ satisfy* 1.1 *above. Let $x \in F$. Then $v(1 - ux^2) \leq \gamma$. If $v(1 - ux^2) < \gamma$ then $v(1 - ux^2) = v(\pi^2)$ for some $\pi \in F$ with $\overline{\pi^{-2}(1 - ux^2)} \in \mathscr{F}^2$. Moreover*:

Case (i). *If $v(1 - ux^2) = \gamma \notin 2G$, then $\overline{(1-u)^{-1}(1 - ux^2)} \in \mathscr{F}^2$.*

Case (ii). *If $v(1 - ux^2) = v(\pi^2) < v(4)$ then $\overline{\pi^{-2}(1 - ux^2)}$ lies in the $\mathscr{F}^2$-coset $\mathscr{F}^2 + \overline{\pi^{-2}(1-u)}$.*

Case (iii). *If $v(1 - ux^2) = \gamma = v(4)$, then $\overline{\pi^{-2}(1 - ux^2)}$ lies in the $\wp(\mathscr{F})$ coset $\wp(\mathscr{F}) + \overline{(1-u)/4}$.*

Proof. First suppose that $v(1 - ux^2) > \gamma$. Then as $1 - ux^2 = (1-u) + (u - ux^2)$ we conclude that $\gamma = v(u - ux^2) = v(1 - x^2)$. Expressing $x = 1 + \pi'$, we find $\gamma = v(1 - x^2) = v(\pi'^2 + 2\pi')$. If $\gamma < v(4)$ then necessarily $v(\pi') < v(2)$ so that $\gamma = v(\pi'^2)$. But now $v(1 - ux^2) > \gamma$ implies $\overline{\pi'^{-2}(1-u)} = \overline{\pi'^{-2}u(1-x^2)} = \overline{\pi'^{-2}u(\pi'^2 + 2\pi')} = 1$, contrary to the choice of u. In case $\gamma = v(4)$, then $v(\pi') = v(2)$ and we find that $\overline{(1-u)/4} = \overline{u(1-x^2)/4} = \overline{(\pi'^2 + 2\pi')/4} \in \wp(\mathscr{F})$, again contradicting the choice of u.

We now assume that $v(1 - ux^2) = \gamma' < \gamma$. Then as $(1-u) = (1 - ux^2) + u(x^2 - 1)$ we find that $v(1 - ux^2) = v(x^2 - 1) = \gamma' < \gamma \leq v(4)$. Again setting $x = 1 + \pi'$, we find as above that $v(\pi') < v(2)$ and that $\gamma' = v(\pi'^2)$. As

$$\overline{\pi'^{-2}(1-ux^2)} = \overline{\pi'^{-2}u(x^2-1)} = \overline{\pi'^{-2}(\pi'^2+2\pi')} = 1 \in \mathscr{F}^2,$$

the desired conclusion follows.

In case $v(1-ux^2)=\gamma$ and $\gamma \notin 2G$, as $1-ux^2=(1-u)+u(1-x^2)$, we see that $v(1-x^2)\geq\gamma$. As $\gamma<v(4)$, in case $v(1-x^2)=\gamma$ we conclude that $\gamma=v(\pi'^2)$ for $x=1+\pi'$ as above, a contradiction to the fact that $\gamma\notin 2G$. Thus $v(1-x^2)>\gamma$ and hence $\overline{(1-u)^{-1}(1-ux^2)}=1\in\mathscr{F}^2$ which settles case (i).

Finally in case $v(1-ux^2)=\gamma=v(\pi^2)$ we find that $v(1-x^2)=v((1-ux^2)-(x^2(1-u)))\geq\gamma$. Setting $x=1+\pi'$ this means that $v(2\pi'+\pi'^2)\geq\gamma=v(\pi^2)$ and hence $v(\pi')\geq v(\pi)$ as $v(\pi)\leq v(2)$. Thus

$$\begin{aligned}&\overline{\pi^{-2}(1-ux^2)}\\ &\quad=\overline{\pi^{-2}x^2(1-u)}+\overline{\pi^{-2}(1-x^2)}=\overline{\pi^{-2}(1-u)}+\overline{\pi^{-2}(\pi'^2+2\pi')},\end{aligned}$$

as $\bar{x}=1$. If $\gamma<v(4)$ then $\overline{\pi^{-2}(\pi'^2+2\pi')}$ is 0 if $v(\pi')>v(\pi)$ and equals $\overline{\pi^{-2}\pi'^2}\in\mathscr{F}^2$ if $v(\pi')=v(\pi)$. Case (ii) follows. If $\gamma=v(4)$ we may set $\pi=2$, and then $\overline{\pi^{-2}(\pi'^2+2\pi')}\in\wp(\mathscr{F})$ so that Case (iii) follows. This proves Lemma 1.2. $\square$

Recall that $\langle\langle a_1,a_2,\ldots,a_n\rangle\rangle$ denotes the n-fold Pfister form $\otimes_{i=1}^n\langle 1,a_i\rangle$, which is a 2^n-dimensional quadratic form. For any quadratic form α, $D(\alpha)$ denotes the subset of $F^{\bullet}$ represented by α. See [L] for details.

LEMMA 1.3. *Assume that u, γ, $t_1,t_2,\ldots,t_j$ satisfy* 1.1 *above. Then* $v(D\langle\langle -u,-t_1,\ldots,-t_j\rangle\rangle)\subseteq 2G(\gamma)$.

Proof. By the 2-independence properties of $t_1,t_2,\ldots,t_j$, it is clear that $v(D\langle\langle -t_1,\ldots,-t_j\rangle\rangle)\subseteq 2G$. If $w\in D\langle\langle -u,-t_1,\ldots,-t_j\rangle\rangle$ we can express $w=w_1-uw_2$ where $w_i\in D\langle\langle -t_1,\ldots,-t_j\rangle\rangle$. Evidently we may assume that $v(w_1)=v(w_2)=0$, and further as $w_1^{-1}\in D\langle\langle -t_1,\ldots,-t_j\rangle\rangle$ we may assume that $w_1=1$. Thus we are reduced to computing $v(1-uw_2)$.

Express $w_2=x_1^2-t_1x_2^2+\cdots+(-1)^jt_1\cdots t_jx_{2^j}^2$, and note that $1-uw_2=(1-ux_1^2)-u(-t_1x_2^2+\cdots+(-1)^jt_1\cdots t_jx_{2^j}^2)$. We see that as $v(1-ux_1^2)\in 2G(\gamma)$ (by Lemma 1.2) in order for $v(1-uw_2)\notin 2G(\gamma)$ we must have $v(1-ux_1^2)=v(-t_1x_2^2+\cdots+(-1)^jt_1\cdots t_jx_{2^j}^2)$. Now choose x_k, where $2\leq k\leq 2^j$ so that

$$v\left(x_k^2\right)=v\left(-t_1x_2^2+\cdots(-1)^jt_1\cdots t_jx_{2^j}^2\right).$$

In case $v(1 - ux_1^2) < v(4)$ we find that

$$\overline{x_k^{-2}(1 - ux_1^2)} \notin \mathscr{F}_0^2(\bar{t}_1, \bar{t}_2, \dots, \bar{t}_j)$$

by Lemma 1.2, and by the Standing Hypothesis 1.1. Thus when $\gamma < v(4)$ we find that

$$v(1 - uw_2) = v(1 - ux_1^2) \quad \text{or} \quad v\left(-t_1x_2^2 + \cdots + (-1)^j t_1 \cdots t_j x_{2^j}^2\right),$$

both of which lie in $2G(\gamma)$. In case $v(1 - ux_1^2) = v(4)$, then as $\overline{(1 - u)/4} \notin \wp(\mathscr{F}) + \mathscr{F}_0^2(\bar{t}_1, \bar{t}_2, \dots, \bar{t}_j)$ by 1.1, and as $\overline{(1 - ux_1^2)/4} \in \wp(\mathscr{F}) + \overline{(1 - u)/4}$, it follows that $\overline{x_k^{-2}(1 - ux_1^2)} \notin \mathscr{F}_0^2(\bar{t}_1, \bar{t}_2, \dots, \bar{t}_j)$. (Note that $\mathscr{F}_0^2(\bar{t}_1, \bar{t}_2, \dots, \bar{t}_j)$ is closed under multiplication by elements of $\mathscr{F}^2$.) Thus we conclude that $v(1 - uw_2) = v(4) \in 2G$. This proves Lemma 1.3. □

LEMMA 1.4. *Suppose that* $u, \gamma, t_1, t_2, \dots, t_j, \pi_1, \pi_2, \dots, \pi_k$ *satisfy* 1.1. *Then*

$$v\left(D\left\langle\left\langle -u, -t_1, \dots, -t_j, -\pi_1, \dots, -\pi_k\right\rangle\right\rangle\right) \subseteq 2G(\gamma, v(\pi_1), \dots, v(\pi_k)).$$

Proof. We proceed by induction on k. If $k = 0$, this is Lemma 1.3. So assume that

$$v\left(D\left\langle\left\langle -u_1, -t_1, \dots, -t_j, -\pi_1, \dots, -\pi_{k-1}\right\rangle\right\rangle\right) \subseteq 2G(\gamma, v(\pi_1), \dots, v(\pi_{k-1})).$$

Then note that if $w \in D\langle\langle -u, -t_1, \dots, -t_j, -\pi_1, \dots, -\pi_k\rangle\rangle$ then $w = w_1 - \pi_k w_2$ for $w_1, w_2 \in D\langle\langle -u, -t_1, \dots, -t_j, -\pi_1, \dots, -\pi_{k-1}\rangle\rangle$. By induction, as $v(\pi_k) \notin 2G(\gamma, v(\pi_1), \dots, v(\pi_{k-1}))$ the desired result now follows. □

Recall that if $\alpha = \langle\langle a_1, \dots, a_r\rangle\rangle$ is an r-fold Pfister form, that $\alpha' = \langle\langle a_1, \dots, a_r\rangle\rangle'$ denotes the so-called pure subform of α. This is the $2^r - 1$ dimensional subform $\langle a_1, a_2, \dots, a_1 \cdots a_r\rangle$ of α. See [L, p. 278] for more discussion of α'. Recall that $\wp(\mathscr{F}) + \overline{(1 - u)/4}$ denotes the $\wp(\mathscr{F})$-coset in the additive group $\mathscr{F}^+$.

LEMMA 1.5. *Suppose* $u, \gamma, t_1, t_2, \dots, t_j$ *satisfy Hypothesis* 1.1. *Let* $w \in D(\langle\langle -u\rangle\rangle \otimes \langle\langle -t_1, \dots, -t_j\rangle\rangle)$. *Then we have the following*:

Case (i). *If* $\gamma \notin 2G$ *and* $v(w) = v(\pi^2)$ *for some* $\pi \in F$, *then* $\overline{\pi^{-2}w} \in \mathscr{F}_0^2(\bar{t}_1, \bar{t}_2, \dots, \bar{t}_j)$. *In particular,* $\overline{\pi^{-2}w} \notin \mathscr{F}^2$. *If* $\gamma \notin 2G$ *and* $v(w) = \gamma$, *then* $\overline{(1 - u)^{-1}w} \notin \mathscr{F}^2$.

Case (ii). *If* $\gamma = v(\pi^2) \neq v(4)$ *and* $v(w) = v(\pi'^2)$ *for* $\pi, \pi' \in F$, *then* $\overline{\pi'^{-2}w} \in \mathscr{F}^2(\overline{\pi^{-2}(1-u)})_0(\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_j)$. *In particular* $\overline{\pi'^{-2}w} \notin \mathscr{F}^2(\overline{\pi^{-2}(1-u)})$.

Case (iii). *If* $\gamma = v(4)$ *and* $v(w) = v(\pi^2)$, *then* $\overline{\pi^{-2}w} \notin \mathscr{F}^2 \cup (\wp(\mathscr{F}) + \overline{(1-u)/4})$.

In all cases if $v(w) = v(\pi^2)$, *then* $\overline{\pi^{-2}w} \notin \mathscr{F}^2$.

Proof. We express $w = w_1 - uw_2$, where $w_i \in D\langle\langle -t_1, \ldots, -t_j\rangle\rangle'$. If $v(w) = v(w_1)$ or $v(w_2)$, then as $v(w_i) \in 2G$ by the 2-independence of $\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_j$ we find that $v(w) \in 2G$. Next note that $v(w_i) = v(\pi^2)$ implies $\overline{\pi^{-2}w_i} \in \mathscr{F}_0^2(\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_j)$. Since $\bar{u} = 1$ this shows that $\overline{\pi^{-2}w} \in \mathscr{F}_0^2(\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_j)$. Case (i) is now clear in this situation. Likewise, for Case (ii), as $\overline{\pi^{-2}(1-u)} \notin \mathscr{F}^2(\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_j)$ by 1.1, the result is also clear. For case (iii), clearly $\overline{\pi^{-2}w} \notin \mathscr{F}^2$, so assume $\overline{\pi^{-2}w} \in \wp(\mathscr{F}) + \overline{(1-u)/4}$. But then $\overline{(1-u)/4} \in \wp(\mathscr{F}) + \mathscr{F}_0^2(\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_j)$ a contradiction to 1.1. Thus we may assume in all cases that $v(w) > v(w_1) = v(w_2)$.

In what follows we shall express w as a sum of many terms with the property that the value of the sum must be the smallest value among the terms. The desired conclusions of the Lemma will then follow by inspecting the residues of these summands. We express

$$w_i = \sum_{\alpha \neq (0, \ldots, 0)} (-t)^\alpha x_{i\alpha}^2$$

where the $x_{i\alpha} \in F$. Then we find that

$$w = \sum_{\alpha \neq (0, \ldots, 0)} (-t)^\alpha \left(x_{1\alpha}^2 - ux_{2\alpha}^2\right).$$

Using the 2-independence of the $\bar{t}_i$'s over $\mathscr{F}^2$, and the facts that $v(w) > v(w_1) = v(w_2)$ we conclude that whenever $v(x_{i\alpha}^2) = v(w_i)$ we must have that $v(x_{1\alpha}) = v(x_{2\alpha})$ and that $x_{2\alpha} = x_{1\alpha}(1 + \pi_\alpha)$ for some π_α with $v(\pi_\alpha) > 0$. For those $x_{i\alpha}$ with $v(x_{i\alpha}^2) > v(w_i)$, we express $x_{2\alpha} = x_{1\alpha}(1 + \pi_\alpha)$ with $v(\pi_\alpha) > 0$ if this is possible, otherwise we do not. Setting $u = 1 + \pi^\gamma$, $v(\pi^\gamma) = \gamma$, we are able to express

$$w = \sum_{\alpha \in X_1} (-t)^\alpha \left(x_{1\alpha}^2 - (1 + \pi^\gamma)(1 + \pi_\alpha)^2 x_{1\alpha}^2\right) + \sum_{\alpha \in X_2} (-t)^\alpha \left(x_{1\alpha}^2 - ux_{2\alpha}^2\right)$$

where $X_1 \dot{\cup} X_2 = 2^j - \{(0, 0, \ldots, 0)\}$, and

$$v\left(x_{1\alpha}^2 - ux_{2\alpha}^2\right) = \inf\left\{v\left(x_{1\alpha}^2\right), v\left(x_{2\alpha}^2\right)\right\} \quad \text{for all } \alpha \in X_2.$$

We investigate the terms

$$\begin{aligned} A_\alpha &:= -\left(x_{1\alpha}^2 - (1+\pi^\gamma)(1+\pi_\alpha)^2 x_{1\alpha}^2\right)x_{1\alpha}^{-2} \\ &= \left(\pi^\gamma + \pi_\alpha^2 + 2\pi_\alpha + \pi_\alpha\left(\pi_\alpha^2 + 2\pi_\alpha\right)\right) \end{aligned}$$

where $\alpha \in X_1$ and the terms $B_\alpha := (x_{1\alpha}^2 - ux_{2\alpha}^2)$ where $\alpha \in X_2$. As $\gamma \leq v(4)$, in case $v(\pi_\alpha) > v(2)$, then we find that $v(A_\alpha) = v(\pi^\gamma) = \gamma$ and thus one of the following three things must occur:

(a) $\overline{(1-u)^{-1}A_\alpha} = 1 \in \mathcal{F}^2$ if $\gamma \notin 2G$, or

(b) $\overline{\pi^{-2}A_\alpha} \in \mathcal{F}^2(\overline{\pi^{-2}(1-u)})$ if $\gamma = v(\pi^2)$, or

(c) $\overline{A_\alpha/4} = \overline{\pi^\gamma/4} = \overline{(1-u)/4}$ when $\gamma = v(4)$.

In case $v(\pi_\alpha) \leq v(2)$ and $\gamma \neq v(\pi_\alpha^2)$ we find that $v(A_\alpha) = \inf\{\gamma, v(\pi_\alpha^2)\}$ and that in case $v(\pi_\alpha^2) < \gamma$ the following occurs:

(a′) $\overline{\pi_\alpha^{-2}A_\alpha} \in \mathcal{F}^2$ where $v(\pi_\alpha^2) < \gamma$.

In case $\gamma < v(\pi_\alpha^2)$ we find that conclusions (a) or (b) above must hold.

Now we assume that $v(\pi_\alpha) < v(2)$, while $v(\pi_\alpha^2) = \gamma$. But as $\overline{\pi_\alpha^{-2}\pi^\gamma} \notin \mathcal{F}^2$ by 1.1, we find that $\overline{\pi_\alpha^{-2}\pi^\gamma} \neq 1 \in \mathcal{F}^2$. Thus $v(A_\alpha) = v(\pi_\alpha^2)$ and we conclude:

(b′) $\overline{\pi_\alpha^{-2}A_\alpha} \in \mathcal{F}^2(\overline{\pi_\alpha^{-2}(1-u)})$ where $\gamma = v(\pi_\alpha^2)$.

Finally, in case $\gamma = v(\pi_\alpha^2) = v(4)$, as $\overline{\pi^\gamma/4} \notin \wp(\mathcal{F})$ by 1.1, we see that $\overline{\left(\pi^\gamma + \pi_\alpha^2 + 2\pi_\alpha\right)/4} \neq 0 \in \mathcal{F}$, so that $v(A_\alpha) = v(4)$. Hence we may conclude that

(c′) $\overline{A_\alpha/4} \in \overline{(1-u)/4} + \wp(\mathcal{F})$ where $\gamma = v(4)$.

Finally we note that whenever $\alpha \in X_2$, $v(B_\alpha) = \inf\{v(x_{1\alpha}^2), v(x_{2\alpha}^2)\}$. Thus if this inf is realized by $v(x_{i\alpha})$ we conclude

(d) $\overline{x_{i\alpha}^{-2}B_\alpha} \in \mathcal{F}^2$ where $v(x_{i\alpha}^2) = v(B_\alpha)$.

The proof of the Lemma is concluded as follows: If $\gamma < v(4)$ and $\gamma \notin 2G$, then as the $\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_j$ are 2-independent over $\mathcal{F}^2$, we find using (a), (a′), and (d) above that necessarily $v(w) = \inf\{v(x_\alpha^2 A_\alpha), v(B_\alpha)\}$, and that the desired conclusions of case (i) immediately follow. Similarly, in case $\gamma \in 2G$ where $\gamma < v(4)$ one uses (b), (a′), (b′), and (d) together with 1.1 to see that case (ii) follows. Finally in case $\gamma = v(4)$, using (c), (a′), (c′), and (d) together with 1.1 and Lemma 1.6 below, the desired conclusions of case (iii) follow. This proves Lemma 1.5. □

LEMMA 1.6. *Let $\mathcal{F}$ be a field of characteristic 2, and let $f, t_1, t_2, \ldots, t_j \in \mathcal{F}$ be such that $t_1, t_2, \ldots, t_j$ are 2-independent and $f \notin \wp(\mathcal{F}) + \mathcal{F}_0^2(t_1, t_2, \ldots, t_j)$. Then the equation $g_1 + t_1 g_2 + \cdots + t_1 \cdots t_j g_{2^j} = 0$ has no solutions with $g_i \in \mathcal{F}^2 \cup \mathcal{F}^2 \cdot (f + \wp(\mathcal{F}))$ and some $g_i \neq 0$.*

Proof. As the $t_1, t_2, \ldots, t_j$ are 2-independent over $\mathscr{F}^2$, we see that some $g_i \notin \mathscr{F}^2$ in some such solution. Multiplying the expression by a square and some product among the $t_1, t_2, \ldots, t_j$ if necessary, we can assume that $g_1 = f + \wp(z_1)$. We then obtain an expression of the form:

$$f + \wp(z_1) + \sum_{\alpha \in X_1} t^\alpha w_\alpha^2 (f + \wp(z_\alpha)) + \sum_{\alpha \in X_2} t^\alpha z_\alpha^2 = 0$$

for $X_1 \dot{\cup} X_2 = 2^j - \{(0,0,\ldots,0)\}$. Solving for f we find that:

$$f = \left(\wp(z_1) + \sum_{\alpha \in X_1} t^\alpha w_\alpha^2 \wp(z_\alpha) + \sum_{\alpha \in X_2} t^\alpha z_\alpha^2\right) \Big/ \left(1 + \sum_{\alpha \in X_1} t^\alpha w_\alpha^2\right).$$

Note that as $X_1 \cap X_2 = \varnothing$, we have that

$$\sum_{\alpha \in X_1} t^\alpha z_\alpha^2 \Big/ \left(1 + \sum_{\alpha \in X_2} t^\alpha w_\alpha^2\right) \in \mathscr{F}_0^2(t_1, t_2, \ldots, t_j).$$

Next consider a term of the form:

$$t^{\alpha'} w_{\alpha'}^2 \wp(z_{\alpha'}) \Big/ \left(1 + \sum_{\alpha \in X_1} t^\alpha w_\alpha^2\right) \quad \text{where } \alpha' \in X_1.$$

Multiplying both the numerator and the denominator by $(t^{\alpha'} w_{\alpha'}^2)^{-1}$, this term actually equals an expression of the form

$$\wp(z_{\alpha'}) \Big/ \left(1 + \sum_{\alpha \in X'} t^\alpha w_\alpha'^2\right),$$

where again $\alpha' \in X'$ and X' is a subset of $2^j - \{(0,0,\ldots,0)\}$. We now apply the formula $\wp(r)/(1+\tau) = \wp(r/(1+\tau)) + (r^2/(1+\tau^2))\tau$ to conclude that the preceding expression lies in $\wp(\mathscr{F}) + \mathscr{F}_0^2(t_1, t_2, \ldots, t_j)$. From this it follows that $f \in \wp(\mathscr{F}) + \mathscr{F}_0^2(t_1, t_2, \ldots, t_j)$, a contradiction, so the Lemma is proved. □

THEOREM 1.7. *Suppose that u, $t_1, t_2, \ldots, t_j$, $\pi_1, \pi_2, \ldots, \pi_k$ satisfy the hypotheses* 1.1. *Then for any unit*

$$-w \in D\left(\left\langle\left\langle -u, -t_1, \ldots, -t_j, -\pi_1, \ldots, -\pi_k \right\rangle\right\rangle'\right)$$

we have $v(1-x) \leq \gamma$.

Proof. We proceed by induction on k. First assume that $k = 0$. We express $-w = -ux^2 + w'$ where $w' \in D(\langle\langle -u \rangle\rangle \otimes \langle\langle -t_1, -t_2, \ldots, -t_j \rangle\rangle')$. Suppose that $v(ux^2) < 0$. But according to Lemma 1.5 we have that $\overline{x^{-2}w'} \notin \mathscr{F}^2$ while $\bar{u} = 1 \in \mathscr{F}^2$. This contradicts the assumption that w is a unit and shows that $v(ux^2) \geq 0$. If $v(ux^2) > 0$, then $v(w') = 0$ and $\overline{-w} = \overline{-w'} \notin \mathscr{F}^2$ by Lemma 1.5, so we conclude $v(1-w) = 0$. Thus we may assume $v(ux^2) = 0$ and $v(w') \geq 0$.

Now suppose that $v(1-w) > \gamma$. As $1 - w = 1 - ux^2 + w'$, and as $v(1-ux^2) = \gamma' \leq \gamma$ by Lemma 1.2, we conclude that $v(w') = \gamma' \leq \gamma$. But further, in case $\gamma' < \gamma$, we know by Lemma 1.2 that $v(1-ux^2) = v(\pi^2)$ for some $\pi \in F$ and that $\overline{\pi^{-2}(1-ux^2)} \in \mathscr{F}^2$. Thus, as $\overline{\pi^{-2}w'} \notin \mathscr{F}^2$ by Lemma 1.5, we conclude that $v(1-w) = \gamma'$ in this case. Hence we are reduced to considering the case where $v(1-ux^2) = \gamma = v(w')$. However, comparing Lemma 1.2 Cases (i), (ii), (iii) with Lemma 1.5 Cases (i), (ii), (iii) respectively shows that $v(1-w) = \gamma$. This takes care of the case where $k = 0$.

Now assume the result is true for $k - 1$. If

$$-w \in D\big(\langle\langle -u, -t_1, \ldots, -t_j, -\pi_1, \ldots, -\pi_k\rangle\rangle'\big)$$

is a unit, we express $-w = -w_1 - \pi_k w_2$ where

$$-w_1 \in D\big(\langle\langle -u, -t_1, \ldots, -t_j, -\pi_1, \ldots, -\pi_{k-1}\rangle\rangle'\big)$$

and

$$w_2 \in D\big(\langle\langle -u, -t_1, \ldots, -t_j, -\pi_1, \ldots, -\pi_{k-1}\rangle\rangle\big).$$

By Lemma 1.4 we find that $v(\pi_k w_2) \in v(\pi_k) + 2G(\gamma, v(\pi_1), \ldots, v(\pi_{k-1}))$, and hence $v(w_1) \neq v(\pi_k w_2) \neq 0$. Thus $-w_1$ is a unit as $-w_1 - \pi_k w_2$ is a unit. Hence by induction we find that $v(1-w_1) \leq \gamma$. But also by Lemma 1.4, as $1 - w_1 \in D(\langle\langle -u, -t_1, \ldots, -t_j, -\pi_1, \ldots, -\pi_{k-1}\rangle\rangle)$, $v(1-w_1) \in 2G(\gamma, v(\pi_1), \ldots, v(\pi_{k-1}))$. It now necessarily follows that

$$v(1-w) = v(1 - w_1 - \pi_k w_2) = \inf\{v(1-w_1), v(\pi_k w_2)\} \leq \gamma.$$

This proves the Theorem. □

As an important consequence of Theorem1.7 we now have:

COROLLARY 1.8. (i) *Suppose that* $u, \gamma, t_1, t_2, \ldots, t_j, \pi_1, \pi_2, \ldots, \pi_k$ *satisfy Hypothesis* 1.1, *or that* $v(1-u) = v(\pi_1)$, *where* $\bar{t}_1, \ldots, \bar{t}_j$ *are* 2-*independent over* $\mathscr{F}^2$, $\overline{(1-u)/\pi_1} \notin \mathscr{F}^2(\bar{t}_1, \ldots, \bar{t}_j)$, *and* $\pi_1, \ldots, \pi_k$ *are independent in* $G/2G$. *Then the* $j + k + 1$-*fold Pfister form* $\langle\langle -u, -t_1, \ldots, -t_j, -\pi_1, \ldots, -\pi_k\rangle\rangle$ *is anisotropic over* F.

(ii) *In particular, if* $u = 1 + 4g$, *if* $\bar{t}_1, \ldots, \bar{t}_j$ *are* 2-*independent in* $\mathscr{F}$, *if* $\pi_1, \ldots, \pi_k$ *are independent in* $G/2G$, *and if* $\langle\langle -(1+4g), -t_1, \ldots, -t_j, -\pi_1, \ldots, -\pi_k\rangle\rangle$ *is isotropic, then* $\bar{g} \in \wp(\mathscr{F}) + \mathscr{F}_0^2(\bar{t}_1, \ldots, \bar{t}_j)$.

Proof. (i) In case $u, \gamma, t_1, \ldots, t_j, \pi_1, \ldots, \pi_k$ satisfy Hypothesis 1.1, then the result is an immediate consequence of Theorem 1.7. In the second case we observe that $\langle\langle -u, -\pi_1\rangle\rangle \cong \langle\langle -u, -\pi_1^{-1}(1-u)\rangle\rangle$. Since $\overline{\pi_1^{-1}(1-u)} \notin \mathscr{F}^2(\bar{t}_1, \ldots, \bar{t}_j)$, we find that $u, \gamma, t_1, \ldots, t_j, \pi_1^{-1}(1-u)$,

$\pi_2, \ldots, \pi_k$ satisfy Hypothesis 1.1. Thus the result follows from Theorem 1.7 in this case as well.

(ii) This follows from (i) above, negating standing Hypothesis 1.1 case (iii). □

We now give one last technical result, which is essential for the proofs of the main results of §2.

THEOREM 1.9. *Suppose that* u, γ, $t_1, t_2, \ldots, t_j$, $\pi_1, \pi_2, \ldots, \pi_k$ *satisfy Hypothesis* 1.1. *Let* $w \in D(\langle\langle -u, -t_1, \ldots, -t_j, -\pi_1, \ldots, -\pi_k \rangle\rangle')$. *Then* $v(w) \in 2G(\gamma, v(\pi_1), \ldots, v(\pi_k))$ *and moreover*:

Case (i). *If* $\gamma \notin 2G$ *and* $v(w) = v(w'^2(1-u))$ *for* $w' \in F$, *then* $(w'^2(1-u))^{-1} w \in \mathscr{F}_0^2(\bar{t}_1, \ldots, \bar{t}_j)$. *If* $v(w) = 0$ *then* $w \in \mathscr{F}^2(t_1, \ldots, t_j)$.

Case (ii). *If* $\gamma = v(\pi^2) < v(4)$ *for some* $\pi \in F$, *and if* $v(w) = 0$, *then* $\pi^{-2}(1-u)w \in \mathscr{F}_0^2(\bar{t}_1, \ldots, \bar{t}_j, \pi^{-2}(1-u))$.

Proof. The first statement is clear by Lemma 1.4. The remaining conclusions will be proved by induction on k. We first assume that $k = 0$. We express $w = -ux^2 + w_1$ where $w_1 \in D\langle\langle -u \rangle\rangle \otimes \langle\langle -t_1, \ldots, -t_j \rangle\rangle'$. For the first part of case (i), if $\gamma \notin 2G$ and $v(w) = v(w'^2(1-u))$, then as $v(ux^2) \in 2G$ we conclude by the first part of case (i) of Lemma 1.5 that if $v(ux^2) = v(w_1)$, then $v(w) = v(ux^2)$. Thus as $v(w) \notin 2G$, we conclude that $v(w) = v(w_1) < v(ux^2)$. In particular $(w'^2(1-u))^{-2} w = (w'^2(1-u)^{-1}) w_1 \in \mathscr{F}_0^2(\bar{t}_1, \ldots, \bar{t}_j)$ by the second part of Case (i) of Lemma 1.5.

For the second part of Case (i), if $v(w) = 0$, then by Lemma 1.5 Case (i) we conclude that $v(ux^2)$, $v(w_1) \geq 0$, and that the desired result $\bar{w} = \overline{-ux^2 + w_1} \in \mathscr{F}^2(\bar{t}_1, \ldots, \bar{t}_j)$ follows. Finally for case (ii) we find as above, using Lemma 1.5 Case (ii), that $v(ux^2)$, $v(w_1) \geq 0$ and that $\bar{w} = \overline{-ux^2 + w_1} \in \mathscr{F}^2 + \mathscr{F}^2(\pi^{-2}(1-u))_0(\bar{t}_1, \ldots, \bar{t}_j)$. The desired conclusion follows from this. This takes care of the Theorem where $k = 0$.

We now assume that the result is true for $k-1$ and prove it for k. In each case the argument is the same. Express $w = w_1 - \pi_k w_2$ where $w_1 \in D(\langle\langle -u, -t_1, \ldots, -t_j, -\pi_1, \ldots, -\pi_{k-1} \rangle\rangle')$ and where $w_2 \in D(\langle\langle -u, -t_1, \ldots, -t_j, -\pi_1, \ldots, -\pi_{k-1} \rangle\rangle)$. By Lemma 1.4 $v(\pi_k w_2) \in v(\pi_k) + 2G(\gamma, v(\pi_1), \ldots, v(\pi_{k-1}))$. Hence $v(w_1 - \pi_k w_2) = \inf\{v(w_1), v(\pi_k w_2)\}$ and this must equal $v(w_1)$ in order to satisfy the hypotheses of the Theorem. The desired conclusions now follow by induction, proving Theorem 1.9. □

There is no suitable analogue of Theorem 1.9 in the case of $\gamma = v(4)$. This is primarily due to the technicalities of Lemma 1.6, and how they affect the proof of Theorem 1.5. In order to obtain the results necessary for the next section, we must prove some special results for the $\gamma = v(4)$ case. Our main goal is Theorem 1.13 below, but first we need some technical lemmas about fields of characteristic 2. The first such lemma is an analogue of Springer's Theorem on odd degree extensions (c.f. [**L**, p. 197]) in the characteristic 2 case. In fact, the proof is simply the obvious generalization of the well-known proof of that theorem.

In the sequel we shall adopt the following notation. For any collection $u_1, \dots, u_r$ we denote by $A(u_1, \dots, u_r) = \{1, u_1, u_2, u_1u_2, \dots, u_1 \cdots u_r\}$ (the set of all 2^r products), and we denote by $A_0(u_1, \dots, u_r) = A(u_1, \dots, u_r) - \{1\}$. Whenever $S_1, S_2 \subseteq A(u_1, \dots, u_n)$, $S_2S_2 := \{s_1s_2 | s_1 \in S_1 \text{ and } s_2 \in S_2\}$.

LEMMA 1.10. *Suppose that $\mathscr{F}$ is a field of characteristic 2, that $t_1, \dots, t_n$ are 2-independent in $\mathscr{F}$, $0 \le s,\ r < 2^n$, and that $\tau_1, \dots, \tau_s$, $\eta_1, \dots, \eta_r \in A_0(t_1, \dots, t_n)$ are all distinct. Suppose that $k = \mathscr{F}(\alpha)$ is an odd degree extension of $\mathscr{F}$. Then*:

(i) $\mathscr{F} \cap (k^2 + \sum \tau_i k^2) = \mathscr{F}^2 + \sum \tau_i \mathscr{F}^2$

In particular, $t_1, \dots, t_n$ are 2-independent in k.

(ii) *If a, $a_j \in \mathscr{F}$, f, g_i, $h_j \in k$ and*

$$a = \wp(f) + \sum \tau_i g_i^2 + \left(\sum a_j h_j\right)^2 + \sum \eta_j h_j^2$$

then there exists f', g_i', $h_j' \in \mathscr{F}$ so that

$$a = \wp(f') + \sum \tau_i g_i'^2 + \left(\sum a_j h_j'\right)^2 + \sum \eta_j h_j'^2.$$

In particular, $\mathscr{F} \cap (\wp(k) + \sum \tau_i k^2) = \wp(\mathscr{F}) + \sum \tau_i \mathscr{F}^2$.

Proof. Suppose that for some $a \in \mathscr{F}$ there exists f, g_i, $h_j \in k$ so that:

$$a = q(f) + \sum \tau_i g_i^2 + \left(\sum a_j h_j\right)^2 + \sum \eta_j h_j^2$$

where $q(f) = f^2$, and there are no a_j's or η_j's in case (i), and $q(f) = f^2 + f$ in case (ii). Our job is to show that in either case a similar expression exists with f', g_i', $h_j' \in \mathscr{F}$. What we shall show is that in fact such an expression exists with f', g_i', $h_j' \in k'$, where k' is an odd degree extension of $\mathscr{F}$ with $[k':\mathscr{F}] < [k:\mathscr{F}]$. The result is then immediate by an obvious induction.

Let $\pi(X)$ the monic irreducible polynomial of α over $\mathscr{F}$. We choose $f(X)$, $g_i(X)$, $h_j(X) \in \mathscr{F}[X]$ each of degree less than $n = [k:\mathscr{F}] =$ degree$(\pi(X))$, so that $f(\alpha) = f$, $g_i(\alpha) = g_i$ and $h_j(\alpha) = h_j$ in k. Since $k \cong \mathscr{F}[X]/(\pi(X))$, this means that for some $d(X) \in \mathscr{F}[X]$:

For $R(X) := q(f(X)) + \sum \tau_i g_i(X)^2 + (\sum a_j h_j(X))^2 + \sum \eta_j h_j(X)^2$ we have $a = R(X) + \pi(X)d(X)$.

Each degree among $q(f(X))$, $\tau_i g_i(X)^2$, $(\sum a_j h_j(X))^2$, and $\eta_j h_j(X)^2$ is even and less than $2n - 1$. We claim that the degree of $R(X)$ is the maximum of the degrees of these summands. In case (i), $R(X) = f(X)^2 + \sum \tau_i g_i(X)^2$. Thus by the 2-independence of $t_1, \dots, t_n$, and by a leading coefficient argument, we conclude that the degree of $R(X)$ is the maximum degree of its summands. In (ii), in case the degree of one of $\tau_i g_i(X)^2$ or $\eta_j h_j(X)^2$ is maximal among the degrees of the summands of $R(X)$, then again by the 2-independence of $t_1, \dots, t_n$ and a leading coefficient argument, the desired conclusion follows. (Note that if $q(X)^2 + (\sum a_j h_j(X))^2$ attains maximal degree, its leading coefficient is a square.) If none of $\tau_i g_i(X)^2$ and $\eta_j h_j(X)^2$ attains the maximal degree, then $(\sum a_j h_j(X))^2$ cannot either, as its degree is necessarily $\leq$ the maximum of the degrees of the $\eta_j h_j(X)^2$. Thus, $q(X)$ is the only summand attaining the maximal degree, and the claim follows.

In case the degree of $R(X)$ is less than the degree of $\pi(X)$, we see that the degree of each of the polynomials $f(X)$, $g_i(X)$, $h_j(X)$ is zero, $d(X) = 0$, and the result is immediate. Hence we can assume that the degree of $R(X)$ is greater than n. Since the degree of $R(X)$ is even and less than $2n$, and since n is odd, we conclude that the degree of $d(X)$ is odd and less than n. Let $p(X)$ be an irreducible odd degree factor of $d(X)$. Let β be a root of $p(X)$, and set $k' = \mathscr{F}(\beta)$. We have that $[k':\mathscr{F}] < [k:\mathscr{F}]$. Set $f' = f(\beta)$, $g_i' = g_i(\beta)$, and $h_j' = h_j(\beta)$ in k'. In view of the equation $a = R(\beta) + \pi(\beta)d(\beta) = R(\beta) \in k'$, the proof of the Lemma is complete. □

The next Lemma, provides in characteristic 2, the information necessary to study the function fields that arise in Theorem 1.13.

LEMMA 1.11. *Suppose that $\mathscr{F}$ is a field of characteristic 2, and $K = \mathscr{F}(X_1, X_2, \dots, X_r)$ is a field of rational functions in r variables over $\mathscr{F}$. Assume that $t_1, \dots, t_n$ are 2-independent in $\mathscr{F}$ and that $\tau_1, \dots, \tau_s, \eta_1, \dots, \eta_r \in A_0(t_1, \dots, t_n)$ are all distinct. Suppose that a, $a_j \neq 0 \in \mathscr{F}$, and there* *exist* f, g_i, $h_j \in K$ *such that*:

$$(*) \qquad a = \wp(f) + \sum_{j=1}^{s} \tau_i g_i^2 + \left(\sum_{j=1}^{r} a_j X_j h_j \right)^2 + \sum_{j=1}^{r} \eta_j h_j^2.$$

Then $a \in \wp(\mathscr{F}) + \Sigma \tau_i \mathscr{F}^2$.

Proof. We proceed by induction on r. If $r = 0$, then the result is trivial. Assuming the result for $r - 1$, we prove it for r. Suppose that a, a_j, f, g_i, h_j are as in the hypotheses of the Lemma. Set $K_0 = \mathscr{F}(X_1, \ldots, X_{r-1})$. Let v: $K \to \mathbf{Z}$ be any valuation on K with $K^{\cdot 0} \subseteq U_v$, and for which $t_1, \ldots, t_n$ are 2-independent in the residue class field K_v. Let σ be a summand on the right hand side of $(*)$ and suppose that the value $v(\sigma) < 0$ for some such summand. Then necessarily (since $v(f^2) < v(f)$ whenever $v(f) < 0$) the minimum of all such $v(\sigma)$ must be even, say $-2m$. Let π be a uniformizing parameter for v. Multiplying $(*)$ by π^{2m}, using the fact that $v(f^2) < v(f)$ whenever $v(f) < 0$, we obtain by passing to residues that:

$$0 = \overline{f^2 \pi^{2m}} + \overline{\left(\sum_{j=1}^{r} a_j X_j h_j \right)^2 \pi^{2m}} + \sum_{i=1}^{s} \overline{\tau_i g_i^2 \pi^{2m}} + \sum_{j=1}^{r} \overline{\eta_j h_j^2 \pi^{2m}}.$$

By the 2-independence of the $t_1, \ldots, t_n$ in K_v we obtain that each of the residues $\overline{\tau_i g_i \pi^{2m}}$, $\overline{\eta_j h_j \pi^{2m}}$, and $\overline{\left[f^2 + \left(\Sigma_{j=1}^{r} a_j X_j h_j \right)^2 \right] \pi^{2m}}$ must be 0. In particular, $v(g_i) > -m$ and $v(h_j) > -m$ for each i, j. From this, as $v(\sigma) = -2m$, $v(f^2) = v((\Sigma a_j X_j h_j)^2) = -2m$. Since $v(a_j X_j h_j) > -m$ for $j \neq r$, we conclude $v(a_r X_r h_r) = -m$, i.e. $v(X_r) \leq -1$. Thus:

$(**)$ Whenever $v(X_r) \geq 0$ or $v(a_r X_r h_r) \geq 0$, $v(\sigma) \geq 0$ for every summand σ on the right-hand side of $(*)$.

Consider the discrete valuation v_1: $K \to \mathbf{Z}$, with residue class field K_0, with $v_1(1/X_r) = 1$. (The value of v_1 on a polynomial in $K_0[X_r]$ is the negative of its degree in X_r.) Suppose first that $v_1(a_r X_r h_r) > 0$. By $(**)$ each summand on the right-hand side of $(*)$ is v_1-integral. Also, as $a_r \neq 0$, $v_1(h_r) > 0$ as well, so passing to residues we find that in K_0:

$$a = \wp(\bar{f}) + \sum_{i=1}^{s} \overline{\tau_i g_i^2} + \left(\sum_{j=1}^{r-1} \overline{a_j X_j h_j} \right)^2 + \sum_{j=1}^{r-1} \overline{\eta_j h_j^2}.$$

The desired result now follows from our inductive hypothesis.

From this point on we may assume that $v_1(a_r X_r h_r) \leq 0$. We first show that $v_1(h_r)$ is odd. If $v_1(a_r X_r h_r) = 0$, then $v_1(h_r) = 1$. So now suppose that $v_1(a_r X_r h_r) < 0$. Assuming that $v_1(\sigma) = -2m < 0$ is the minimum value of any summand occurring on the right-hand side of $(*)$, we have already established that $v_1(g_i) > -m$, $v_1(h_j) > -m$, and $v_1(f) = v_1(a_rX_rh_r) = -m$, with the residue $\overline{[f^2 + (a_rX_rh_r)^2](1/X_r)^{2m}} = 0$ in K_0. It follows that one can express in K:

$$a_rX_rh_r = f + f', \quad \text{with } v_1(f') > -m.$$

Substituting this into $(*)$ we obtain:

$$a = f + (f')^2 + \left(\sum_{j=1}^{r-1} a_jX_jh_j\right)^2 + \sum_{i=1}^{s} \tau_i g_i^2 + \sum_{j=1}^{r} \eta_j h_j^2.$$

We know that $v_1(\sigma') \geq -2(m-1)$ for each summand σ' in the above, and that $v_1(h_r^2) = -2(m-1)$. Suppose $v_1(f) > -2(m-1)$. Then, as $v_1(f) = -m$, $m > 2$. Multiplying by $(1/X_r)^{2(m-1)}$ and passing to residues, we find that in K_0:

$$0 = \overline{\left(\sum_{i=1}^{s} \tau_i g_i^2 + \left(f' + \sum_{j=1}^{r-1} a_jX_jh_j\right)^2 + \sum_{j=1}^{r} \eta_j h_j^2\right)\left(\frac{1}{X_r}\right)^{2(m-1)}}.$$

By the 2-independence of $t_1, \ldots, t_n$ inside K_0, we find that $\overline{(\eta_r h_r^2)(1/X_r)^{2(m-1)}} = 0$, a contradiction since $v_1(h_r^2) = -2(m-1)$. Hence $v_1(f) = -2(m-1)$. Since $v_1(f^2) = -2m$, it follows that $m = 2$. Since $-2 = v_1(f) = v_1(a_rX_rh_r)$, we conclude that $v_1(h_r) = -1$, in particular it is odd.

We express $h_r = c(X_r)/d(X_r)$ where $c(X_r)$, $d(X_r) \in K_0[X_r]$ are relatively prime. It follows, since $v_1(h_r)$ is odd, that as polynomials in X_r, one of $c(X_r)$ or $d(X_r)$ must have odd degree. Thus, one of $c(X_r)$ or $d(X_r)$ has an odd degree irreducible (in $K_0[X_r]$) factor $\pi(X_r)$. We denote by v_π the discrete valuation on K, with $v_\pi(\pi(X_r)) = 1$, and residue class field $K_0(\alpha)$ where $\pi(\alpha) = 0$. By Lemma 1.10, we know that $t_1, \ldots, t_n$ are 2-independent in $K_0(\alpha)$. Since $v_\pi(X_r) \geq 0$, $(**)$ shows that for every term σ occurring on the right-hand side of $(*)$, $v_\pi(\sigma) \geq 0$. In particular, $v_\pi(h_r) \geq 0$, but the choice of π assures $v_\pi(h_r) \neq 0$. We may conclude that both $v_\pi(a_rX_rh_r) > 0$ and $v_\pi(h_r) > 0$. With this information, passing to residues we find that in $K_0(\alpha)$:

$$a = \wp(\bar{f}) + \sum_{i=1}^{s} \overline{\tau_i g_i^2} + \left(\sum_{j=1}^{r-1} \overline{a_jX_jh_j}\right)^2 + \sum_{j=1}^{r-1} \overline{\eta_j h_j^2}.$$

Since $[K_0(\alpha) : K_0]$ is odd, we can apply Lemma 1.10 to find that the above equation actually holds (for some possibly modified f, g_i, h_j) inside K_0. The result now follows by induction, proving the Lemma. $\square$

LEMMA 1.12. *Suppose that $\mathscr{F}$ is a field of characteristic* 2, $t_1, \ldots, t_n \in \mathscr{F}$, *and suppose that* $S \subseteq A(t_1, \ldots, t_{n-1})$. *Set* $\mathscr{F}' = \mathscr{F}(t_n^{1/2})$. *Then*:

$$\left(\sum_{\tau \in S} \tau \mathscr{F}'^2\right) = \sum_{\tau \in S} \tau \mathscr{F}^2 + \sum_{\tau \in t_n S} \tau \mathscr{F}^2,$$

and

$$\mathscr{F} \cap \left(\wp(\mathscr{F}') + \sum_{\tau \in S} \tau \mathscr{F}'^2\right) = \wp(\mathscr{F}) + \sum_{\tau \in S} \tau \mathscr{F}^2 + \sum_{\tau \in t_n S} \tau \mathscr{F}^2.$$

Proof. Suppose $g = \sum \tau \alpha_\tau^2$ where $g \in \mathscr{F}$ and $\alpha_\tau \in \mathscr{F}'$. Express $\alpha_\tau = a_\tau + b_\tau t_n^{1/2}$ where $a_\tau, b_\tau \in \mathscr{F}$. Then, $\alpha_\tau^2 = a_\tau^2 + b_\tau^2 t_n$, and the result follows substituting these expressions. □

We now return to our valued field F (of characteristic 0) with residue class field $\mathscr{F}$. Let $\pi \in F$, with $v(\pi) \notin 2G$. Recall that $t_1, \ldots, t_n \in F$ are units with 2-independent residues in $\mathscr{F}$. Whenever q is a n-fold Pfister form, say $q = \langle\langle -a_1, -a_2, \ldots, -a_n \rangle\rangle$, we consider the subform $q^* := \langle -a_1 \rangle \perp \langle\langle -a_2, \ldots, -a_n \rangle\rangle$ over F. A particular function field, $F(q^*)$ (denoted $F(q^*)_0$ in **[Kn]**) is defined by:

$$F(q^*) := F(X_1, \ldots, X_r)\left(\left(a_1 + \sum_{j=1}^{r} - \mu_j X_j^2\right)^{1/2}\right)$$

where $r = 2^{n-1} - 1$ and $\{\mu_1, \ldots, \mu_r\} = A_0(-a_2, \ldots, -a_n)$. Since q^* is a $(2^{n-1} + 1)$-dimensional subform of the n-fold Pfister form q, and since q^* becomes isotropic in $F(q^*)$, q vanishes inside $W(F(q^*))$. Further, since q^* is a Pfister neighbor of q, it is known that $\ker(W(F) \to W(F(q^*))) = qW(F)$ (**[Kn-Sc**, p. 29]).

THEOREM 1.13. *Let* $\phi = \langle\langle -(1 + 4g), -t_1, \ldots, -\hat{t}_k, \ldots, -t_n, -\pi \rangle\rangle$ *or* $\langle\langle -(1 + 4g), -t_1, \ldots, -\hat{t}_k, \ldots, -t_n \rangle\rangle$ *where* $1 \leq k \leq n$ *and* $v(g) = 0$. *Suppose that* $\phi \in \langle\langle -t_k, -t_{k+1}, \ldots, -t_n \rangle\rangle W(F)$. *Then*:

$$\begin{aligned} \bar{g} \in \wp(\mathscr{F}) &+ \mathscr{F}_0^2(\bar{t}_1, \ldots, \hat{\bar{t}}_k, \ldots, \bar{t}_n) \\ &+ \bar{t}_k \mathscr{F}_0^2(\bar{t}_1, \ldots, \bar{t}_{k-1}) \mathscr{F}^2(\bar{t}_{k+1}, \ldots, \bar{t}_n). \end{aligned}$$

Proof. We set $F' = F(q^*)$ where q is the n-fold Pfister form $\langle\langle -t_k, -t_{k+1}, \ldots, -t_n \rangle\rangle$ over F. The explicit description of F' we need in this case is:

$$F' = F(X_1, \ldots, X_r)\left(\left(t_k + \sum_{j=1}^{r} - \mu_j X_j^2\right)^{1/2}\right)$$

where $r = 2^{n-k} - 1$, $X_1, \dots, X_r$ are algebraically independent over F, and $\{\mu_1, \dots, \mu_r\} = A_0(-t_{k+1}, \dots, -t_n)$. The valuation $v\colon F \to G$ has an unramified extension $v\colon F' \to G$ for which the $X_1, X_2, \dots, X_r$ are units and the residue field is

$$\mathscr{F}' = \mathscr{F}(X_1, \dots, X_r)\left(\left(\bar{t}_k + \sum_{j=1}^{r} \bar{\mu}_j X_j^2\right)^{1/2}\right)$$

where $X_1, X_2, \dots, X_r$ are algebraically independent over $\mathscr{F}$. (We abuse notation by writing X_j for the residue of X_j inside $\mathscr{F}'$.)

Since $\phi = 0 \in W(F')$, we conclude from Corollary 1.8 (ii) that $\bar{g} \in \wp(\mathscr{F}') + \mathscr{F}_0'^2(\bar{t}_1, \dots, \hat{\bar{t}}_k, \dots, \bar{t}_n)$. We denote $K = \mathscr{F}(X_1, \dots, X_r) \subseteq \mathscr{F}'$, and $u = \bar{t}_k + \sum \bar{\mu}_j X_j^2 \in K$. Applying Lemma 1.12 we find that there exist f, g_i, $h_i \in K$ such that:

$$\bar{g} = \wp(f) + \sum_{i=1}^{s'} \bar{\tau}_i g_i^2 + \sum_{i=1}^{s'} u\bar{\tau}_i h_i^2$$

where $\{\tau_1, \dots, \tau_{s'}\} = A_0(t_1, \dots, \hat{t}_k, \dots, t_n)$. Assume that the τ_j are numbered so that $\tau_j = \pm\mu_j$ if $\tau_j \in A_0(t_{k+1}, \dots, t_n)$. We denote by

$$\{\tau_{s'+1}, \dots, \tau_s\} = A_0(t_1, \dots, t_{k-1}) t_k A(t_{k+1}, \dots, t_n).$$

Note that each $\pm\mu_j \in \{\tau_1, \dots, \tau_s\}$, but if we define $\eta_j = t_k \mu_j$ then each $\eta_j \notin \{\tau_1, \dots, \tau_s\}$. Expanding u according to its definition shows that (for $1 \le j \le s'$)

$$u\bar{\tau}_j h_j^2 \in \sum_{i=1}^{s} \bar{\tau}_i K^2, \quad \text{if } \tau_j \in A_0(t_1, \dots, t_{k-1}) A(t_{k+1}, \dots, t_n);$$

$$u\bar{\tau}_j h_j^2 = \bar{t}_k \bar{\tau}_j h_j^2 + \left(\bar{\tau}_j X_j h_j\right)^2 + H_j$$

where $H_j \in \sum_{i=1}^{s'} \bar{\tau}_i K^2$, if $\tau_j \in A_0(t_{k+1}, \dots, t_n)$. We obtain, modifying each g_i to g_i' for $i = 1, \dots, s'$, absorbing appropriate summands to create g_i's for $i = s' + 1, \dots, s$:

$$\bar{g} = \wp(f) + \sum_{i=1}^{s} \bar{\tau}_i g_i'^2 + \left(\sum \bar{\eta}_j X_j h_j\right)^2 + \sum \bar{\eta}_j h_j^2,$$

where the η_j range over $t_k A_0(t_{k+1}, \dots, t_n)$. The desired conclusion now follows from Lemma 1.11, proving the Theorem. □

2. Some ideal quotients. The object of this section is to define some subideals V_r^γ, $\mathring{V}_r^\gamma$ of $I^r F$, where $\gamma \in \Delta := \{\gamma \in G\colon 0 < \gamma \le v(4)\}$, and to determine the structure of the quotients $V_r^\gamma / \mathring{V}_r^\gamma$ in terms of $\mathscr{F}$ and

G. Throughout the rest of this paper we shall assume that $G = \mathbf{Z}$, and we shall fix $\pi \in F$ with $v(\pi) = 1 \in G = \mathbf{Z}$. All of the results of this section can be extended to the non-discrete value group case in an obvious way. However the author has not been able to extend the results of §3 to the non-discrete case, and for this reason the many extra calculations needed to generalize this section have been omitted. In view of the technical nature of this section, the author recommends to the readers that they familiarize themselves with the statements of the results of §3 before wading through the results proved here.

All notation established in §1 will remain in force throughout this section. We fix $t_1, t_2, \dots, t_n \in U$ chosen so that their residues $\bar{t}_1, \bar{t}_2, \dots, \bar{t}_n$ form a 2-basis of $\mathscr{F}$. Whenever $X = \{t_{i_1}, \dots, t_{i_s}\} \subseteq \{t_1, t_2, \dots, t_n\}$ we shall denote by $A(X)$ the 2^s products of distinct elements of X, i.e.

$$A(X) = \{1, t_{i_1}, \dots, t_{i_1} \cdots t_{i_s}\}.$$

We shall also denote $A_0(X) = A(X) - \{1\}$. Next, whenever $t_{i_1}, \dots, t_{i_j} \in X$ are distinct with $i_1 < i_2 < \cdots < i_j$, we define:

$$B(X, i_1, \dots, i_j) := \{b \in A_0(X)\colon \text{expressing } b = t_{s_1} \cdots t_{s_k}$$

$$\text{with } s_1 < s_2 \cdots < s_k \text{ then } s_1 \notin \{i_1, \dots, i_j\}\}$$

Note that $B(X, i_1, \dots, i_j) = \varnothing$ in case $X = \{i_1, \dots, i_j\}$. We similarly define

$$C(X, i_1, \dots, i_j) := \{c \in A_0(X) \mid \text{expressing } c = t_{s_1} \cdots t_{s_k} \text{ with}$$

$$s_1 < s_2 < \cdots < s_k \text{ then } s_1 \in \{i_1, \dots, i_j\}\}.$$

Evidently $C(X, i_1, \dots, i_j) = A_0(X) - B(X, i_1, \dots, i_j)$. For any $Y \subseteq A(X)$ we set $D(Y) := \sum_{\tau \in Y} \bar{\tau}\mathscr{F}^2$, an additive subgroup of $\mathscr{F}$. Thus for instance in this notation we have when $X = \{t_1, \dots, t_n\}$ that $D(A(X)) = \mathscr{F}$ and $D(A_0(X)) = \mathscr{F}_0^2(\bar{t}_1, \dots, \bar{t}_n)$. For convenience, when $X = \{t_1, \dots, t_n\}$ we shall denote $A(X)$ and $A_0(X)$ by A and A_0 respectively.

In what follows we fix r, $2 \le r$ and we fix $T = \{t_{s_1}, \dots, t_{s_{n'}}\} \subseteq \{t_1, \dots, t_n\}$ for $1 \le n' \le n$. For any $\alpha \in A_0(T)$, $\alpha = t_{s_1}^{i_1} \cdots t_{s_{n'}}^{i_{n'}}$, we set $N(\alpha) = \{s_j \mid i_j = 1\} \subseteq \{s_1, \dots, s_{n'}\}$, and we let $n(\alpha) =$ cardinality of $N(\alpha)$. We consider a "lifting" $(\ )_T^*\colon \mathscr{F} \to F$ which is chosen with the property that $(x_i)_T^* = x_i$ *and* such that $(\bar{\alpha} x_i^2)^* \in \alpha F^2$ for all $\alpha \in A(T)$, where the x_i give a fixed $\mathbf{Z}/2\mathbf{Z}$-basis of the additive vector space $\mathscr{F}^+$. Next we describe an ideal $V_r(T)$ whose definition depends upon $(\ )_T^*$, the T_i, some f_j's, g_j's, h_j's described below, as well as upon T. By definition $V_r(T)$ will be the subideal of I^rF generated by $I^{r+1}F$ and the set of r-folds described in the following list:

List 2.1. For all $\gamma \in \Delta$ we consider all $r-1$ or $r-2$ tuples $(i_1, \ldots, i_{r-1})$ and $(i_1, \ldots, i_{r-2})$ respectively with $t_{i_j} \in T$ for each i_j, $1 \leq i_1 < \cdots < i_{r-1} \leq n$ and $1 \leq i_1 < \cdots < i_{r-2} \leq n$ respectively. With the x_i's as above we list the following r-folds:

Type A. For $\gamma \in \Delta$ with $\gamma \notin 2G$ we have

(i) $\langle\langle -(1 + (-1)^{s(\alpha)}\pi^\gamma(\bar{\alpha}x_i^2)_T^*), -t_{i_1}, \ldots, -t_{i_{r-1}}\rangle\rangle$ for all $\alpha \in A_0(T)$ where $s(\alpha)$ is the cardinality of $N(\alpha) \cap \{i_1, \ldots, i_{r-1}\}$.

(ii) $\langle\langle -(1 - \pi^\gamma h_j), -t_{i_1}, \ldots, -t_{i_{r-1}}\rangle\rangle$ for $\bar{h}_j$ ranging over a basis of $\mathscr{F}^+$ mod $\mathscr{F}_0^2(\bar{t}_i\colon t_i \in T)$.

(iii) $\langle\langle -(1 - \pi^\gamma h_j), -t_{i_1}, \ldots, -t_{i_{r-2}}, -\pi\rangle\rangle$ for $\bar{h}_j$ ranging over a basis of $\mathscr{F}^+$ mod $\mathscr{F}^2(\bar{t}_i\colon t_i \in T)$.

Type B. For $\gamma \in \Delta$ with $\gamma \in 2G$ but $\gamma \neq v(4)$ we have

(i) $\langle\langle -(1 - \pi^\gamma(\bar{\tau}x_i^2)_T^*), -t_{i_1}, \ldots, -t_{i_{r-1}}\rangle\rangle$ for $\tau \in B(T\colon i_1, \ldots, i_{r-1})$.

(ii) $\langle\langle -(1 - \pi^\gamma(\bar{\tau}x_i^2)_T^*), -t_{i_1}, \ldots, -t_{i_{r-2}}, -\pi\rangle\rangle$ for $\tau \in B(T\colon i_1, \ldots, i_{r-2})$.

(iii) $\langle\langle -(1 - \pi^\gamma h_j), -t_{i_1}, \ldots, -t_{i_{r-1}}\rangle\rangle$ and $\langle\langle -(1 - \pi^\gamma h_j), -t_{i_1}, \ldots, -t_{i_{r-2}}, -\pi\rangle\rangle$ for $\bar{h}_j$ ranging over a basis of $\mathscr{F}$ mod $\mathscr{F}^2(\bar{t}_i\colon t_i \in T)$.

Type C. Here we consider $\gamma = v(4) \in \Delta$.

(i) $\langle\langle -(1 - 4g_j), -t_{i_1}, \ldots, -t_{i_{r-1}}\rangle\rangle$ where the g_j give a basis of $\mathscr{F}$ mod $(\wp(\mathscr{F}) + D(C(T, i_1, \ldots, i_{r-1})))$.

(ii) $\langle\langle -(1 - 4g_j), -t_{i_1}, \ldots, -t_{i_{r-2}}, -\pi\rangle\rangle$ where the g_j give a basis of $\mathscr{F}$ mod $(\wp(\mathscr{F}) + D(C(T, i_1, \ldots, i_{r-2})))$.

Throughout the rest of this section the subscript T on $(\ )_T^*$ will be dropped as the set T will always be clear from context. The condition that $(\bar{\alpha}x_i^2)^* \in \alpha F^2$ for $\alpha \in A(T)$ gives:

$$(*)\quad \left\langle\left\langle -\left(1 - \pi^\gamma\left(\alpha x_i^2\right)^*\right), -\beta\right\rangle\right\rangle = \left\langle\left\langle -\left(1 - \pi^\gamma\left(\alpha x_i^2\right)^*\right), -\beta\pi^\gamma\alpha\right\rangle\right\rangle.$$

This fact will be used often in what follows. We also remark that in case $r > n' + 2$, then none of the $r-1$ or $r-2$-tuples can exist so $V_r(T)$ becomes the ideal $I^{r+1}(F)$. When $r = n' + 2$, then the generators of $V_r(T)$ over $I^{r+1}(F)$ all have the form $\langle\langle -(1 - \pi^\gamma h), -t_{i_1}, \ldots, -t_{i_{n'}}, -\pi\rangle\rangle$ for some $h \in F$. Mod $I^{r+1}(F)$ any sum of such r-folds is another such r-fold. Thus by a straight-forward application of Corollary 1.8 we see that the generators listed above in Type A (iii), Type B (iii) and Type C (i), (ii) freely generate $V_r(T)/I^{r+1}(F)$ as a $\mathbf{Z}/2\mathbf{Z}$-vector space. We now show that the same holds more generally.

PROPOSITION 2.2. *Suppose $r = n' + 1$. Then $V_r(T)/I^{r+1}(F)$ is freely generated by the r-folds listed in* 2.1 *as a $\mathbf{Z}/2\mathbf{Z}$-vector space.*

Proof. Without loss of generality we may assume that $T = \{t_1, \dots, t_{n'}\}$. All of the generators of $V_r(T)$ over $I^{r+1}(F)$ can now be redescribed as below. In this new listing note that $(*)$ has been applied to the Type A (i) generators (here $s(\alpha) = n(\alpha)$), and the generators of Type B (i) and Type C (ii) become vacuous when $r = n' + 1$.

Type A. (i) $\langle\langle -(1 + (-1)^{s(\beta)}\pi^{\gamma}(\bar{\beta}x_i^2)^*), -t_1, \dots, -\hat{t}_k, \dots, -t_{n'}, -\pi\rangle\rangle$ where "hat" means delete, and $\beta \in t_k A(\{t_{k+1}, \dots, t_{n'}\})$ with $1 \leq k \leq n'$.

(ii) $\langle\langle -(1 - \pi^{\gamma}h_j), -t_1, \dots, -t_{n'}\rangle\rangle$ where the $\bar{h}_j$ give a basis of $\mathscr{F} \bmod \mathscr{F}_0^2(\bar{t}_1, \bar{t}_2, \dots, \bar{t}_{n'})$.

(iii) $\langle\langle -(1 - \pi^{\gamma}h_j), -t_1, \dots, -\hat{t}_k, \dots, -t_{n'}, -\pi\rangle\rangle$ where the $\bar{h}_j$ give a basis of $\mathscr{F} \bmod \mathscr{F}^2(\bar{t}_1, \bar{t}_2, \dots, \bar{t}_{n'})$.

Type B. (ii) $\langle\langle -(1 - \pi^{\gamma}(\bar{\tau}x_i^2)^*), -t_1, \dots, -\hat{t}_k, \dots, -t_{n'}, -\pi\rangle\rangle$ where $\tau \in t_k A(\{t_{k+1}, \dots, t_{n'}\})$ and $1 \leq k \leq n'$.

(iii) $\langle\langle -(1 - \pi^{\gamma}h_j), -t_1, \dots, -t_{n'}\rangle\rangle$ and $\langle\langle -(1 - \pi^{\gamma}h_j), -t_1, \dots, -\hat{t}_k, \dots, -t_{n'}, -\pi\rangle\rangle$ where the $\bar{h}_j$ give a basis of $\mathscr{F} \bmod \mathscr{F}^2(\bar{t}_1, \bar{t}_2, \dots, \bar{t}_{n'})$.

Type C. (i) $\langle\langle -(1 - 4f_j), -t_1, \dots, -t_{n'}\rangle\rangle$ where the $\bar{f}_j$ give a basis of $\mathscr{F} \bmod (\wp(\mathscr{F}) + D(A_0(T)))$.

(ii) $\langle\langle -(1 - 4g_j), -t_1, \dots, -\hat{t}_k, \dots, -t_{n'}, -\pi\rangle\rangle$ where the $\bar{g}_j$ give a basis of $\mathscr{F} \bmod (\wp(\mathscr{F}) + D(A_0 - t_k A(\{t_{k+1}, \dots, t_{n'}\})))$, and $1 \leq k \leq n'$.

In what follows we shall suppose that some sum σ of the above r-folds lies in $I^{r+1}(F)$. We shall examine σ in many quadratic extensions of F as outlined in the following two steps.

Step 1. Set $F_1 = F((\pi)^{1/2})$. It follows that over F_1 we have a sum σ_{F_1} of distinct r-folds of the following type that lies in $I^{r+1}(F_1)$:

Type A. $\langle\langle -(1 - \pi^{\gamma}h_j), -t_1, \dots, -t_{n'}\rangle\rangle$ where the $\bar{h}_j$ give a basis of $\mathscr{F} \bmod \mathscr{F}_0^2(\bar{t}_1, \dots, \bar{t}_n)$.

Type B. $\langle\langle -(1-\pi^\gamma h_j), -t_1, \ldots, -t_{n'}\rangle\rangle$ where the $\bar{h}_j$ give a basis of $\mathscr{F} \bmod \mathscr{F}^2(\bar{t}_1, \ldots, \bar{t}_{n'})$.

Type C. $\langle\langle -(1-4f_j), -t_1, \ldots, -t_{n'}\rangle\rangle$ where the $\bar{f}_j$ give a basis of $\mathscr{F} \bmod(\wp(\mathscr{F}) + D(A_0(\{t_1, \ldots, t_{n'}\})))$.

Using the multilinearity of r-folds $\bmod I^{r+1}(F)$, we find that σ is congruent to an r-fold of the form $\langle\langle -u, -t_1, \ldots, -t_{n'}\rangle\rangle$ where u is the product of the corresponding first terms. Note by construction that this implies $u \in \hat{U}^\gamma$ for some $\gamma \in \Delta$. Observe also that the conditions listed in Types A, B, and C above imply that the Hypotheses 1.1 apply to this r-fold. Thus as $\langle\langle -u, -t_1, \ldots, -t_{n'}\rangle\rangle = 0 \in W(F_1)$ by the Arason-Pfister Hauptsatz we have $\pi \in D_F(\langle\langle -u, -t_1, \ldots, -t_{n'}\rangle\rangle')$. According to the first part of Theorem 1.9 this can occur only when $\gamma = v(1-u) \notin 2G$. But then by the conditions on the $\bar{h}_j$ in (A) immediately above we see that $\pi^{-\gamma}(1-u) \notin D(A_0(T))$. This contradicts Theorem 1.9 Case (i). Hence $u = 1 \in F_1^2$ and we conclude that in the original σ each r-fold must contain a 'π'-term. This concludes step 1.

Step 2. From step 1 it follows that σ is a sum of r-folds of the form:

Type A. (i) $\langle\langle -(1+(-1)^{s(\beta)}\pi^\gamma(\beta x_i^2)^*), -t_1, \ldots, -\hat{t}_k, \ldots, -t_{n'}, -\pi\rangle\rangle$ where $\beta \in t_k A(\{t_{k+1}, \ldots, t_{n'}\})$.

(iii) $\langle\langle -(1-\pi^\gamma h_j), -t_1, \ldots, -\hat{t}_k, \ldots, -t_{n'}, -\pi\rangle\rangle$ where the $\bar{h}_j$ give a basis of $\mathscr{F} \bmod \mathscr{F}^2(\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_{n'})$.

Type B. (ii) $\langle\langle -(1-\pi^\gamma(\bar{\tau} x_i^2)^*), -t_1, \ldots, -\hat{t}_k, \ldots, -t_{n'}, -\pi\rangle\rangle$ where $\tau \in t_k A(\{t_{k+1}, \ldots, t_{n'}\})$.

(iii) $\langle\langle -(1-\pi^\gamma h_j), -t_1, \ldots, -\hat{t}_k, \ldots, -t_{n'}, -\pi\rangle\rangle$ where the $\bar{h}_j$ give a basis of $\mathscr{F} \bmod \mathscr{F}^2(\bar{t}_1, \ldots, \bar{t}_{n'})$.

Type C. (ii) $\langle\langle -(1-4g_j), -t_1, \ldots, -\hat{t}_k, \ldots, -t_{n'}, -\pi\rangle\rangle$ where the $\bar{g}_j$ give a basis of $\mathscr{F} \bmod (\wp(\mathscr{F}) + D(A_0(T) - t_k A(\{t_{k+1}, \ldots, t_{n'}\})))$.

We may now express σ as:

$$\begin{aligned}\sigma = {} & \langle\langle -u_1, -t_2, \ldots, -t_{n'}, -\pi\rangle\rangle \\ & + \cdots + \langle\langle -u_k, -t_1, \ldots, -\hat{t}_k, \ldots, -t_{n'}, -\pi\rangle\rangle \\ & + \cdots + \langle\langle -u_{n'}, -t_1, \ldots, -t_{n'-1}, -\pi\rangle\rangle,\end{aligned}$$

the u_k appropriate products. Note that if $u_k \in F^2$ then the 'kth' summands in σ are each 0. Let k be the largest such that $u_k \notin F^2$ and we proceed by induction on k to derive a contradiction. If $k = 1$ then

$\langle\langle -u_1, -t_2, \dots, -t_{n'}, -\pi\rangle\rangle \in I^{r+1}(F)$, i.e. it is hyperbolic. If σ is not the trivial sum, then $u_1 \in U^\gamma$ for some $\gamma \in \Delta$. If $\gamma \notin 2G$, then according to the type A generators listed at the end of step 1 we have that $\overline{\pi^{-\gamma}(1-u_1)} \notin \mathscr{F}^2(\bar{t}_2, \dots, \bar{t}_{n'})$. If $\gamma < v(4)$ $\gamma \in 2G$, then according to the Type B generators listed at the end of Step 1 we conclude that $\overline{\pi^{-\gamma}(1-u_1)} \notin \mathscr{F}^2(\bar{t}_2, \dots, \bar{t}_{n'})$. Lastly if $\gamma = v(4)$ we conclude that $\overline{(1-u_1)/4} \notin \wp(\mathscr{F}) + \mathscr{F}_0^2(\bar{t}_2, \dots, \bar{t}_{n'})$. Each of these contradicts Corollary 1.8, so we are done if $k = 1$.

We now assume the result for $k-1$ and examine k. There are three cases:

Case 1. In this case we assume $u_k \in \hat{U}^\gamma$ where $\gamma \notin 2G$. It follows from the Type A generators that either $\overline{\pi^{-\gamma}(1-u_k)} \notin \mathscr{F}^2(\bar{t}_1, \bar{t}_2, \dots, \bar{t}_{n'})$ or else $\overline{\pi^{-\gamma}(1-u_k)} \in \bar{t}_k\mathscr{F}^2(\bar{t}_{k+1}, \dots, \bar{t}_{n'})$. In case $\overline{\pi^{-\gamma}(1-u_k)} \notin \mathscr{F}^2(\bar{t}_1, \bar{t}_2, \dots, \bar{t}_{n'})$ we consider $F_k := F((t_k)^{1/2})$. As all but the last summand in σ vanishes in F_k we note that $\langle\langle -u_k, -t_1, \dots, -\hat{t}_k, \dots, -t_{n'}, -\pi\rangle\rangle \in I^{n'+2}(F_k)$, i.e. that $\langle\langle -u_k, -t_1, \dots, -\hat{t}_k, \dots, -t_{n'}, -\pi\rangle\rangle = 0 \in W(F_k)$. This contradicts Corollary 1.8.

We now may assume that $\overline{\pi^{-\gamma}(1-u_k)} \in \bar{t}_k\mathscr{F}^2(\bar{t}_{k+1}, \dots, \bar{t}_{n'})$. In view of this there exists some $w \in F$ with $\bar{w} = \overline{\pi^{-\gamma}(1-u_k)}$ and such that $w \in D(\langle\langle -t_k, \dots, -t_{n'}\rangle\rangle')$. According to the maximality of k we find that $\langle\langle -u_j, -t_1, \dots, -\hat{t}_j, \dots, -t_{n'}, -\pi\rangle\rangle = 0 \in W(F((w\pi)^{1/2}))$ whenever $j \neq k$. Thus we find as $\sigma \in I^{r+1}(F)$ that $\langle\langle -u_k, -t_1, \dots, -\hat{t}_k, \dots, -t_{n'}, -\pi\rangle\rangle = 0 \in W(F((w\pi)^{1/2}))$. However,

$$\begin{aligned}&\langle\langle -u_k, -t_1, \dots, -\hat{t}_k, \dots, -t_{n'}, -\pi\rangle\rangle \\ &\qquad \cong_F \langle\langle -u_k, \dots, -t_k, \dots, -t_{n'}, -\pi^{-\gamma}(1-u_k)\rangle\rangle\end{aligned}$$

where now $u_k, \underline{t_1, \dots, \hat{t}_k, \dots}, t_{n'}, \pi^{-\gamma}(1-u_k)$ satisfy Hypothesis 1.1 Case (i). But now as $\overline{w\pi^{-\gamma}(1-u_k)} \in \mathscr{F}^2$ and

$$-w\pi^{-\gamma} \in D_F\big(\langle\langle -u_k, -t_1, \dots, -\hat{t}_k, \dots, -t_{n'}, -\pi^{-\gamma}(1-u_k)\rangle\rangle'\big)$$

we have a contradiction to Theorem 1.9 Case (i). This concludes Case 1.

Case 2. Here we suppose that $u_k \in \hat{U}^\gamma$ where $\gamma \in 2G$ and $\gamma \neq v(4)$. In case $\overline{\pi^{-\gamma}(1-u_k)} \notin \mathscr{F}^2(\bar{t}_1, \bar{t}_2, \dots, \bar{t}_{n'})$, we note arguing as in Case 1 that $\langle\langle -u_k, -t_1, \dots, -\hat{t}_k, \dots, -t_{n'}, -\pi\rangle\rangle = 0 \in W(F(t_k)^{1/2})$, contradicting Corollary 1.8 applied to $F(t^{1/2})$. In view of the Type B generators listed at the end of Step 1 we can assume that $\overline{\pi^{-\gamma}(1-u_k)} \in \bar{t}_k\mathscr{F}^2(\bar{t}_{k+1}, \dots, \bar{t}_{n'})$. From this it now follows that there is some $w \in F$ with $\bar{w} = \overline{\pi^{-\gamma}(1-u_k)}$ and $-w \in D_F(\langle\langle -t_k, \dots, -t_{n'}\rangle\rangle')$. As in case 1 we find that $\langle\langle -u_k, -t_1, \dots, -\hat{t}_k, \dots, -t_{n'}, -\pi\rangle\rangle = 0 \in W(F(w^{1/2}))$. However this means that $-w \in D_F(\langle\langle -u_k, -t_1, \ldots, -\hat{t}_k, \ldots, -t_{n'}, -\pi\rangle\rangle')$ and as $w\pi^{-\gamma}(1-u_k) \in \mathscr{F}^2$ we have another contradiction to Theorem 1.9 Case (ii). This concludes Case 2.

Case 3. Finally we consider the situation where $u_k \in \hat{U}^{v(4)}$. By the Type C generators listed at the end of Step 1 we see that $u_k = (1+4g)$ where $\bar{g} \notin \wp(\mathscr{F}) + D(A_0(T) - t_k A(\{t_{k+1}, \ldots, t_{n'}\}))$. The maximality of k shows that

$$\langle\langle -u_k, -t_1, \ldots, -\hat{t}_k, \ldots, -t_{n'}, -\pi\rangle\rangle \in \langle\langle -t_k, \ldots, -t_{n'}\rangle\rangle W(F).$$

Thus, Theorem 1.13 shows that

$$\bar{g} \in \wp(\mathscr{F}) + D(A_0(T) - t_k A(t_{k+1}, \ldots, t_{n'})),$$

a contradiction. This concludes step 2 and the proof of the Proposition. □

In the following we denote the subideal of $V_r(F)$ generated by all the type C r-folds and $I^{r+1}(F)$ by $V_r^{v(4)}(F)$.

PROPOSITION 2.3. *Suppose* $r \leq n' + 1$. *Then* $V_r(T)/V_r^{v(4)}(F)$ *is freely generated by the Type* A *and Type* B *r-folds listed in* 2.1 *as a* $\mathbf{Z}/2\mathbf{Z}$*-vector space.*

Proof. We proceed by induction on $s = n' + 1 - r$. The case of $s = 0$ follows from Proposition 2.2, so we will assume the result for $s-1$ and prove it for s. We may assume that $T = \{t_1, \ldots, t_{n'}\}$. As in the proof of 2.2 we suppose that σ is a sum of generators that lies in $V_r^{v(4)}(F)$, and we show that σ is a trivial sum. By taking a suitable multiquadratic unramified extension F' of F we can assume that σ in fact lies in $I^{r+1}(F)$. To see this, observe that if one expresses σ as sum of Type C r-folds $\langle\langle -u_j, -t_{i_1}, \ldots\rangle\rangle \bmod I^{r+1}(F)$ then one can take F' to be the multiquadratic extension obtained by adjoining $(u_j)^{1/2}$ to F for all such u_j's. Such F' is unramified over F, the residue field of F' is a separable multiquadratic extension of $\mathscr{F}$. Over such a residue field, the elements $\bar{t}_1, \ldots, \bar{t}_n$ remain 2-independent, so the hypotheses of the Proposition apply to the field F'. With these reductions we now give a sequence of steps that shows σ was in fact the trivial sum.

Step 1. First we consider the valued field $F_{n'} := F((t_{n'})^{1/2})$. Then, over $F_{n'}$, σ is a sum of generators of the form $\langle\langle -u, -t_{i_1}, \ldots, -t_{i_{r-2}}, -\pi\rangle\rangle$ and $\langle\langle -u, -t_{i_1}, \ldots, -t_{i_{r-1}}\rangle\rangle$ where each $i_j < n'$. We claim that each of these generators can be viewed as generators of Types (A) or (B) for an ideal $V_r(T - \{t_{n'}\})$ over $F_{n'}$. There are two cases to consider:

Case 1. Let $(\ )^*_{T'}$: $\mathscr{F}(\bar{t}_{n'}^{1/2}) \to F_{n'}$ be any lifting which extends the lifting $(\ )^*_T$: $\mathscr{F} \to F$. We consider as a basis for the $\mathbf{Z}/2\mathbf{Z}$-vector space $\mathscr{F}(\bar{t}_{n'}^{1/2})$ the elements $\{x_i, (t_{n'}^{1/2})x_i\}$. Evidently for all $\alpha \in A(T - \{t_n\})$ we have that $(\bar{\alpha}x_i^2)^*_{T'} \in \alpha F_{n'}^2$ and $(\bar{\alpha}((\bar{t}_{n'})^{1/2}x_i)^2)^*_{T'} \in \alpha F_{n'}^2$ so that this lifting (which we shall from now on simply denote by *) satisfies all the desired properties. We also note that $\mathscr{F}(\bar{t}_{n'}^{1/2})$ has a 2-basis consisting of the n-elements $\{\bar{t}_1, \ldots, \bar{t}_{n'-1}, \bar{t}_{n'}^{1/2}, \bar{t}_{n'+1}, \ldots, \bar{t}_n\}$. From these remarks it is clear how to regard the original Type A generators as Type A generators for the new ideal $V_r(T - \{t_{n'}\})$ in $F_{n'}$. (Note that those type A (i) generators with $\alpha = t_{n'}$ become type A (ii) generators.)

Case 2. For the Type B generators of the form $\langle\langle -(1 - \pi^\gamma h_j), -t_{i_1}, \ldots, -t_{i_{r-2}}, -\pi \rangle\rangle$ where $i_j < n'$ and the h_j are independent mod $\mathscr{F}^2(t_1, \ldots, t_{n'})$ there is no problem arguing as in case 1 to see that such elements are Type B (iii) generators over $F_{n'}$. Also, in case $\tau \in B(T, i_1, \ldots, i_{r-2}) - t_{n'}A(\{t_1, \ldots, t_{n'-1}\})$, then clearly $\tau \in B(T - \{t_{n'}\}, i_1, \ldots, i_{r-2})$ so likewise $\langle\langle -(1 - \pi^\gamma(\bar{\tau}x_i^2)^*), -t_{i_1}, \ldots, -t_{i_{r-1}}, -\pi \rangle\rangle$ has the desired form as a Type B (ii) generator. Next we assume $\tau \in B(T, i_1, \ldots, i_{r-2}) \cap t_{n'}A(\{t_1, \ldots, t_{n'-1}\})$ and we set $\tau' := \tau/t_{n'} \in A(\{t_1, \ldots, t_{n'-1}\})$. Again in case $\tau' \in B(T - \{t_{n'}\}, i_1, \ldots, i_{r-2})$ over $F_{n'}$, there is no problem, for in this situation we may view $(\bar{\tau}x_i^2)^* = (\bar{\tau}'(\bar{t}_{n'}^{1/2}x_i)^2)^*$. It follows however from the definitions that necessarily $\tau' \in B(T - \{t_{n'}\}, i_1, \ldots, i_{r-2})$ occurs except in the case where $\tau = t_{n'}$. In examining this case we note that over $F_{n'}$ we may apply the identity $(1 - a^2)/(1 + a)^2 = 1 - 2(a + a^2)/(1 + a)^2$ with $a^2 = \pi^\gamma(\bar{t}_{n'}x_i^2)^*$ to obtain mod $F_{n'}^{\cdot 2}$ that:

$$\begin{aligned}
&\left(1 - \pi^\gamma\left(\bar{t}_{n'}x_i^2\right)^*\right) \\
&\quad \equiv 1 - \left[2\pi^{\gamma/2}\left(\left(\bar{t}_{n'}x_i^2\right)^{*1/2} + \pi^{\gamma/2}\left(\bar{t}_{n'}x_i^2\right)^*\right)\left(1 + \pi^{\gamma/2}\left(\bar{t}_{n'}x_i^2\right)^{*1/2}\right)^{-2}\right] \\
&\quad \equiv 1 - \pi^{v(2)+\gamma/2}h_i
\end{aligned}$$

where

$$h_i := c\left(\left(\bar{t}_{n'}x_i^2\right)^{*1/2} + \pi^{\gamma/2}\left(\bar{t}_{n'}x_i^2\right)^*\right)\left(1 + \pi^{\gamma/2}\left(\bar{t}_{n'}x_i^2\right)^{*1/2}\right)^{-2},$$

with $c = \pi^{v(2)}/2$. Since $\bar{h}_i = \overline{ct_{n'}^{1/2}}x_i$, these residues are all independent in $\mathscr{F}(\bar{t}_{n'}^{1/2})$ mod $\mathscr{F}$. Thus these r-folds can be viewed as either Type A (iii) or Type B (iii) generators for $V_r(T - \{t_{n'}\})$ over $F_{n'}$, depending upon whether

$v(2) + \gamma/2 \in 2G$ or not. One treats the case of the r-folds of the form $\langle\langle -u, -t_{i_1}, \dots, -t_{i_{r-1}}\rangle\rangle$ in exactly the same manner. This concludes Case 2.

It now follows from Cases 1 and 2 that all the r-folds in σ must have a "$t_{n'}$"-term as otherwise we would have a non-trivial relation amongst the generators of $V_r(T - \{t_{n'}\})$ in $F_{n'}$ contradicting our inductive hypotheses. This concludes Step 1.

Step 2. We now consider the field $F_{n'-1} := F(t_{n'-1}^{1/2})$, and we repeat the arguments of Cases 1 and 2 above. Case 1 goes through exactly as in Step 1 above. Case 2 goes through exactly as in Step 1 except that when $\tau \in B(T, i_1, \dots, i_{r-3}, n') \cap t_{n'-1}A(\{t_1, \dots, t_{n'-2}, t_{n'}\})$ with $\tau/t_{n'-1} = \tau' \notin B(T - \{t_{n'-1}\}, i_1, \dots, i_{r-3}, n')$ one has that either $\tau = t_{n'-1}$ or $\tau = t_{n'-1}t_{n'}$. The case where $\tau = t_{n'-1}$ can be handled exactly as in Step 1 Case 2 (where we had $\tau = t_{n'}$), to see that such a r-fold is either a Type A (iii) or a Type B (iii) generator for $V_r(T - \{t_{n'}\})$. This leaves the case where $\tau = t_{n'-1}t_{n'}$. One notes that

$$\left\langle\left\langle -\left(1 - \pi^{\gamma}\left(\bar{t}_{n'-1}\bar{t}_{n'}x_i^2\right)^*\right), -t_{n'}\right\rangle\right\rangle \cong \left\langle\left\langle -\left(1 - \pi^{\gamma}\left(\bar{t}_{n'-1}\bar{t}_{n'}x_i^2\right)^*\right), -t_{n'-1}\right\rangle\right\rangle$$

as $\gamma \in 2G$ so we find that such r-folds arising in this situation vanish in $F_{n'-1}$.

From this we can conclude that all the r-folds occurring in σ must have a $t_{n'}$-term and a $t_{n'-1}$-term *except* those of the form:

Type B. (i) $\langle\langle -(1 - \pi^{\gamma}(\bar{t}_{n'-1}\bar{t}_{n'}x_i^2)^*), -t_{i_1}, \dots, -t_{i_{r-3}}, -t_{n'}, -\pi\rangle\rangle$ and
(ii) $\langle\langle -(1 - \pi^{\gamma}(\bar{t}_{n'-1}\bar{t}_{n'}x_i^2)^*), -t_{i_1}, \dots, -t_{i_{r-2}}, -t_{n'}\rangle\rangle$.

We now consider the extension $F_{n'-1,n'} := F((-t_{n'-1}t_{n'})^{1/2})$. Clearly, all the r-folds in σ which contain both a $t_{n'}$-term and a $t_{n'-1}$-term vanish in $F_{n'-1,n'}$. By the argument of Step 1 Case 2 (where $\tau = t_{n'}$) we see that the remaining Type B r-folds in σ listed immediately above can be regarded as Type A (iii) or Type B (iii) for an ideal $V_r(T - \{t_{n'-1}\})$ over $F_{n'-1,n'}$. It now follows from the inductive hypothesis that such r-folds cannot occur in σ. We thus conclude that every r-fold in σ contains both a $t_{n'}$-term and a $t_{n'-1}$-term. This concludes Step 2.

Steps $3 - n'$. We now apply the arguments of Steps 1 and 2 to $F_{n'-2} := F(t_{n'-2}^{1/2})$ and so forth to find that each r-fold occurring in σ must be of the form $\langle\langle -u, -t_1, \dots, -t_{n'}\rangle\rangle$ where u is some unit in F. We have thus reduced to the case where $r = n' + 1$, i.e. where $s = 0$. This concludes the proof of Proposition 2.3. □

In the following we consider $(r-1)$-tuples $I = (i_1, \ldots, i_{r-1})$ of elements in $\{1, 2, \ldots, n'\}$ for which $i_1 < i_2 < \cdots < i_{r-1}$. If $J = (j_1, \ldots, j_{r-1})$ is another such $(r-1)$-tuple, we say that J dominates I if the tail end of J can be expressed as $J = (\ldots, j_s, i_{s+1}, \ldots, i_{r-1})$ (where $1 \le s \le r-1$) with $j_s > i_s$. (This is the same as saying that $J > I$ in the "right to left" lexicographic order.) We denote by $\langle\langle -t_I \rangle\rangle$ the $(r-1)$-fold Pfister form $\langle\langle -t_{i_1}, \ldots, -t_{i_{r-1}} \rangle\rangle$.

LEMMA 2.4. *Suppose $I = (i_1, \ldots, i_{r-1})$, $J_1, \ldots, J_u$ are $(r-1)$-tuples, and each J_k dominates I. Let $g \in U_v$. If*

$$\langle\langle -(1-4g) \rangle\rangle \langle\langle -t_I \rangle\rangle \in \sum_{k=1}^{u} \langle\langle -t_{J_k} \rangle\rangle I(F) + I^{r+1}(F), \quad or$$

$$\langle\langle -(1-4g) \rangle\rangle \langle\langle -t_I \rangle\rangle \langle\langle -\pi \rangle\rangle \in \sum_{k=1}^{u} \langle\langle -t_{J_k} \rangle\rangle I^2(F) + I^{r+2}(F),$$

then: $\bar{g} \in \wp(\mathscr{F}) + D(C(T, i_1, \ldots, i_{r-1}))$.

Proof. We treat only the first case of

$$\langle\langle -(1-4g) \rangle\rangle \langle\langle -t_I \rangle\rangle \in \sum_{k=1}^{u} \langle\langle -t_{J_k} \rangle\rangle I(F) + I^{r+1}(F),$$

the proof in the second case is entirely analogous. For each $(r-1)$-tuple J_k we denote by $J_k' := (j_v, j_{v+1}, \ldots, j_{r-1})$ the $(r-v)$-tuple of the last $(r-v)$-terms of J where $j_v \neq i_v$ but $(j_{v+1}, \ldots, j_{r-1}) = (i_{v+1}, \ldots, i_{r-1})$. (So, $i_v < j_v < j_{v+1} = i_{v+1}$.) For example, in case $j_{r-1} \neq i_{r-1}$ then $J_k' = (j_{r-1})$. Note that possibly some $J_i' = J_k'$ even though $J_i \neq J_k$. Additionally, note that no two different J_i' can have the same leftmost coefficient. Eliminating all duplications, we list the distinct J' sequences that occur as $J_1', \ldots, J_w'$, ordered from greatest to least leftmost coefficient. In other words, if $J_i' = (j_{i1}, j_{i2}, \ldots)$, $J_k' = (j_{k1}, j_{k2}, \ldots)$ and $i < k$, then $j_{i1} > j_{k1}$. For each k, $1 \le k \le w$, we denote by j_{k1} the leading term of J_k'. (This is denoted j_v in the definition of J_k' above, but from now on we need to keep track of which J_k' this leading term corresponds to.) We denote by $s_k \in \{1, 2, \ldots, r-1\}$ the unique integer such that $i_{s_k} < j_{k1} < i_{s_k+1}$ ($s_k = r-1$ if $i_{r-1} < j_k$). According to the ordering of the $J_1', \ldots, J_w'$ described above, we have that if $i < k$, then $s_i \ge s_k$ (s_k is completely determined by the length of J_k').

In the above notation $J_k' = (j_{k1}, i_{s_k+1}, \ldots, i_{r-1})$. Thus $\langle\langle -t_{J_k'} \rangle\rangle$ is the Pfister form $\langle\langle -t_{j_{k1}}, -t_{i_{s_k+1}}, \ldots, -t_{i_{r-1}} \rangle\rangle$. For each k with $1 \le k \le w$ we define F_k to be the iterated function field:

$$F_k = F\big(\langle\langle -t_{J_1'} \rangle\rangle^*\big)\big(\langle\langle -t_{J_2'} \rangle\rangle^*\big) \cdots \big(\langle\langle -t_{J_k'} \rangle\rangle^*\big)$$

where $\langle\langle -t_{J_l'}\rangle\rangle^*$, $F(\langle\langle -t_{J_l'}\rangle\rangle^*)$ are as described in Theorem 1.13. Our hypotheses, together with the Arason-Pfister Hauptsatz show that $\langle\langle -(1-4g), -t_{i_1}, \dots, -t_{i_{r-1}}\rangle\rangle = 0 \in W(F_w)$. We must show that this implies $\bar{g} \in \wp(\mathscr{F}) + D(C(T, i_1, \dots, i_{r-1}))$. In case $w = 0$, this is a consequence of Corollary 1.8 (ii). So we assume $w \geq 1$. We denote by v: $F_k \to \mathbf{Z}$ the unramified extension of v: $F \to \mathbf{Z}$, as in the proof of Theorem 1.13, except iterated. Additionally, we denote the residue class field F_k by $\mathscr{F}_k$, where for notational convenience $\mathscr{F}_0 = \mathscr{F}$. Over the field F_{w-1}, we have that $\langle\langle -(1-4g)\rangle\rangle\langle\langle -t_I\rangle\rangle \in \langle\langle -t_{J_w'}\rangle\rangle W(F_{w-1})$. Since $J_w' = (j_{w1}, i_{s_w+1}, \dots, i_{r-1})$ where $j_{w1} \notin \{i_1, \dots, i_{r-1}\}$, applying Theorem 1.13 (or Lemma 1.12 and Corollary 1.8 (ii) in case $J_w' = (j_{r-1})$) we obtain:

$$\bar{g} \in \wp(\mathscr{F}_{w-1}) + (\mathscr{F}_{w-1})_0^2(\bar{t}_{i_1}, \dots, \bar{t}_{i_{r-1}})$$

$$+ t_{j_w}(\mathscr{F}_{w-1})_0^2(\bar{t}_{i_1}, \dots, \bar{t}_{i_{s_w}})(\mathscr{F}_{w-1})^2(\bar{t}_{i_{s_w+1}}, \dots, \bar{t}_{i_{r-1}}).$$

For k with $1 \leq k \leq r-1$ we denote $A(k) := A(t_{i_{s_k+1}}, \dots, t_{i_{r-1}})$, and $A_0(k) := A(k) - \{1\}$. For all p with $1 \leq p \leq w$ we define C_p, $C_{p+1}^+ \subseteq C(T, i_1, \dots, i_{r-1})$ as follows:

$$C_p = \{\tau = t_{c_1} \cdots t_{c_s} \in C(T, i_1, \dots, i_{r-1}) | c_1 < c_2 < \cdots < c_s \text{ and}$$

$$\text{each } c_l \in \{i_1, \dots, i_{r-1}\} \cup \{j_{w1}, \dots, j_{p1}\} \text{ while } c_1 \in \{i_1, \dots, i_{r-1}\}\}$$

$$C_{p+1}^+ = \{\tau | \tau = t_{c_1} \cdots t_{c_q} \in C_{p+1} \text{ with } c_1 < j_{p1}\}.$$

It is readily checked that, in case $c_1 > j_{(p-1)1}$, as $j_{k1} < j_{(p-1)1}$ for $k > p-1$, necessarily $c_l \in \{i_1, \dots, i_{r-1}\}$ for all l. Hence $C_p = C_p^+ \dot{\cup} A_0(p-1)$ for all p.

Suppose $\eta \in A_0(p)$ and $\tau \in C_p$ where $\tau \neq \eta$. If τ lies in $A_0(t_{i_1}, \dots, t_{i_{r-1}})$, then so does $\tau\eta$ (modulo squares). If $\tau \in A_0(t_{i_1}, \dots, t_{i_{r-1}})$ then for some $k \geq p$, $\tau \in t_{j_{k1}} C_{k+1}^+ A(k)$. But now, as $s_k \leq s_p$ we see that $\tau\eta \in t_{j_{k1}} C_{k+1}^+ A(k)$ (modulo squares) as well. Altogether, whenever $\tau \in C_p$, $\eta \in A_0(p)$, and $\tau \neq \eta$, then $\tau\eta \in C_p$ (modulo squares).

We finally prove by (backwards) induction on p that:

$$\bar{g} \in \wp(\mathscr{F}_{p-1}) + D_{\mathscr{F}_{p-1}}(C_p).$$

The case of $p = w$, has been observed above. We have that

$$\mathscr{F}_p = \mathscr{F}_{p-1}(X_1, \dots, X_r)\left(\left(\bar{t}_{J_{p1}} + \sum_{l=1}^{r} \bar{\eta}_l X_l^2\right)^{1/2}\right)$$

where $J_p = (j_{p1}, i_{s_p+1}, \ldots, i_{r-1})$ and $\{\eta_1, \ldots, \eta_r\} = A_0(p)$. We assume the induction hypothesis that $\bar{g} \in \wp(\mathscr{F}_p) + D_{\mathscr{F}_p}(C_{p+1})$. Using this, it follows from Lemma 1.12 that there exist elements f, g_τ, $h_\tau \in K_p := \mathscr{F}_{p-1}(X_1, \ldots, X_r)$ such that:

$$(1)\quad \bar{g} = \wp(f) + \sum_{\tau \in C_{p+1}} \bar{\tau} g_\tau^2 + \sum_{\tau \in C_{p+1}} \alpha \bar{\tau} h_\tau^2, \quad \text{where } \alpha = \bar{t}_{j_p} + \sum_{l=1}^{r} \bar{\eta}_l X_l^2.$$

Each $\bar{\eta}_l \in A_0(p+1) \subseteq C_{p+1}$. Thus by the preceding paragraph for all l, τ: $\bar{\eta}_l \bar{\tau} h_\tau^2 X_l^2 \in D_{K_p}(C_{p+1})$ whenever $\eta_l \neq \tau$. From this:

$$(2)\qquad \sum_{\tau \in C_{p+1}} \alpha \bar{\tau} h_\tau^2 = \sum_{\tau \in C_{p+1}} \bar{t}_{j_p} \bar{\tau} h_\tau^2 + \sum_{\eta_l = \tau \in A_0(p)} (\bar{\eta}_l X_l h_\tau)^2 + H$$

where $H \in D_{K_p}(C_{p+1})$. As remarked earlier, $C_{p+1} = C_{p+1}^+ \dot{\cup} A_0(p)$. If $\tau \in C_{p+1}^+$, $t_{j_p}\tau \in C_p$, and if $\tau \in A_0(p)$, $t_{j_p}\tau \notin C_p$. The former correspond to summands of (2), that like H, all lie in $D_{K_p}(C_p)$. The latter $\bar{t}_{j_p} \bar{\tau} h_\tau^2$ summands of (2), cannot be cancelled by summands from H. With this in mind, substituting (2) into (1) and relabeling yields:

$$\bar{g} = \wp(f) + \sum_{\tau \in C_p} \bar{\tau} g_\tau'^2 + \left(\sum_{l=1}^{r} \bar{\eta}_l X_l h_l \right)^2 + \sum_{l=1}^{r} \bar{t}_{j_p} \bar{\eta}_l h_l^2.$$

According to Lemma 1.11, $g \in \wp(\mathscr{F}_{p-1}) + D_{\mathscr{F}_{p-1}}(C_p)$. This completes the induction. From the $p = 1$ case, we find that:

$$\bar{g} \in \wp(\mathscr{F}) + D_{\mathscr{F}}(C_1) \subseteq \wp(\mathscr{F}) + D(C(T, i_1, \ldots, i_{r-1}))$$

which completes the proof of the Lemma. □

Proposition 2.5. *Whenever $r \leq n' + 1$, the ideal quotient $V_r^{v(4)}(F)/I^{r+1}(F)$ is freely generated by the Type* C *generators listed in* 2.1.

Proof. Given any sum σ of generators for $V_r^{v(4)}(F)$, using the multilinearity of r-folds mod $I^{r+1}(F)$ we may combine all summands of the form $\langle\langle -(1 - g_j), -t_{i_1}, \ldots, -t_{i_{r-1}} \rangle\rangle$ for fixed $i_1, i_2, \ldots, i_{r-1}$ into a single r-fold of the form $\langle\langle -f_j, -t_{i_1}, \ldots, -t_{i_{r-1}} \rangle\rangle$ where we still have that $(1 - f_j)/4 \notin \wp(\mathscr{F}) + D(C(T, i_1, \ldots, i_{r-1}))$. In what follows we assume that only summands of forms of this type $\langle\langle -f_j, -t_{i_1}, \ldots, -t_{i_{r-1}} \rangle\rangle$ occur in σ. The general case where r-folds of the form $\langle\langle -(1 - g_j), -t_{i_1}, \ldots, -t_{i_{r-2}}, -\pi \rangle\rangle$ also occur is handled analogously; by first passing to $F(\pi^{1/2})$ to eliminate the r-folds not containing a "π"-term and then applying the analogue of the argument below to eliminate the summands with "π"-terms.

We shall now prove that if a sum of r-folds of this form lies in $I^{r+1}(F)$ then it is a trivial sum. From this the result will follow. We work with the $(r-1)$-tuples $(i_1, \dots, i_{r-1})$ in the linear order of domination described above. We fix $I = (i_1, \dots, i_{r-1})$ to be the least $(r-1)$-tuple occurring in σ. Let $\langle\langle -(1-4g)\rangle\rangle\langle\langle -t_I\rangle\rangle$ be the corresponding summand in σ. For the remaining $(r-1)$-tuples $J_1, \dots, J_v$ occurring in σ, with each J_k dominating I, we have

$$\langle\langle -(1-4g)\rangle\rangle\langle\langle -t_I\rangle\rangle \in \sum_{k=1}^{v} \langle\langle -t_{J_k}\rangle\rangle I(F) + I^{r+1}(F).$$

It follows from Lemma 2.4 that $\bar{g} \in \wp(\mathscr{F}) + D(C(T, i_1, \dots, i_{r-1}))$. This contradiction proves the Proposition. □

DEFINITION 2.6. We define $V_r(F) := V_r(\{t_1, \dots, t_n\})$, where as we recall $\{\bar{t}_1, \dots, \bar{t}_n\}$ forms a 2-basis for $\mathscr{F}$. We also define $V_r^\gamma(F)$, for $\gamma \in \Delta$, to be the subideal of $V_r(F)$ generated by $I^{r+1}(F)$, and all the Type A, B, or C r-folds, where the leading units lie in U^γ. Finally we define:

$$\mathring{V}_r^\gamma := \sum_{\delta > \gamma} V_r^\delta(F).$$

THEOREM 2.7. *The ideal* $V_r(F)$ *is freely generated* mod $I^{r+1}(F)$ *by the generators of Type* A, B, C *listed in* 2.1.

Proof. This is why we proved Propositions 2.3 and 2.5. □

In what follows we record the structure of the ideal quotients $V_r^\gamma(F)/\mathring{V}_r^\gamma(F)$. These quotients provide the key to all the results of the rest of this paper. These results can be viewed as the ideal theoretic analogues of the cohomological results of Kato (cf. [K]) obtained in the complete discrete case. As remarked in the beginning of this section, all of these results can be generalized to the non-discrete case using the general results of §1, however this has been omitted to free the reader from the many extra pages of cumbersome notation that would be involved.

The generators listed in List 2.1 were listed in a manner to accomodate the inductive nature of the proofs of Propositions 2.2 and 2.3. For future reference we record here the simplifications that occur in this list when $T = \{t_1, t_2, \dots, t_n\}$, which is only case we care about from this point on anyway. In this new list, since $\mathscr{F}^2(\bar{t}_i \colon t_i \in T) = \mathscr{F}$, the Type A (iii) and Type B (iii) generators become vacuous. Also, we have been able to drop the sign change factor $(-1)^{s(\alpha)}$ in the Type A (i) generators, this being a simple basis change modulo Type B and Typc C generators. Thus the Type A (i) and (ii) may be combined in one list.

List 2.8. Let $r \geq 2$. We consider all $r-1$ or $r-2$-tuples $(i_1, \ldots, i_{r-1})$ and $(i_1, \ldots, i_{r-2})$ with $1 \leq i_1 < \cdots < i_{r-1} \leq n$ (resp. $i_{r-2} \leq n$). Let x_i be a $\mathbf{Z}/2\mathbf{Z}$-basis of the additive vector space of $\mathscr{F}$. Then the following are the generators of the ideal $V_r(F)$ over $I^{r+1}(F)$.

Type A. For $\gamma \in \Delta$ with $\gamma \notin 2G$ we have $\langle\langle -(1-\pi^\gamma(\bar{\alpha}x_i^2)^*), -t_{i_1}, \ldots, -t_{i_{r-1}}\rangle\rangle$ for all $\alpha \in A(t_1, \ldots, t_n)$.

Type B. For $\gamma \in \Delta$ with $\gamma \in 2G$ and $\gamma < v(4)$:

(i) $\langle\langle -(1-\pi^\gamma(\bar{\tau}x_i^2)^*), -t_{i_1}, \ldots, -t_{i_{r-1}}\rangle\rangle$ for all $\tau \in B(i_1, \ldots, i_{r-1})$.

(ii) $\langle\langle -(1-\pi^\gamma(\bar{\tau}x_i^2)^*), -t_{i_1}, \ldots, -t_{i_{r-2}}, -\pi\rangle\rangle$ for all $\tau \in B(i_1, \ldots, i_{r-2})$.

Type C. Here $\gamma = v(4) \in \Delta$.

(i) $\langle\langle -(1-4f_j), -t_{i_1}, \ldots, -t_{i_{r-1}}\rangle\rangle$ where these $\bar{f}_j$ give a basis for $\mathscr{F} \bmod \wp(\mathscr{F}) + D(C(i_1, \ldots, i_{r-1}))$.

(ii) $\langle\langle -(1-4f_j), -t_{i_1}, \ldots, -t_{i_{r-2}}, -\pi\rangle\rangle$ where these $\bar{f}_j$ give a basis for $\mathscr{F} \bmod \wp(\mathscr{F}) + D(C(i_1, \ldots, i_{r-2}))$.

COROLLARY 2.9. *Suppose $\gamma \in \Delta$ and $\gamma \notin 2G$. Then there is an isomorphism*

$$\rho_\gamma \colon \mathscr{F}^{\binom{n}{r-1}} \to V_r^\gamma(F)/\mathring{V}_r^\gamma(F)$$

which is defined by mapping the multi-indexed $f_{i_1, \ldots, i_{r-1}} \in \mathscr{F}$ to the class of the r-fold $\langle\langle -(1-\pi^\gamma(f_{i_1, \ldots, i_{r-1}})^*), -t_{i_1}, \ldots, -t_{i_{r-1}}\rangle\rangle$.

Proof. This follows immediately from Theorem 2.8 and the list of generators for $V_r(F)$ as described in 2.8 once we see that the described map really is a group homomorphism. This follows from the observation that for fixed $i_1, \ldots, i_{r-1}$ the map from $\mathscr{F}$ to $V_r^\gamma(F)/\mathring{V}_r^\gamma(F)$ given by $f \mapsto \langle\langle -(1-\pi^\gamma(f)^*), -t_{i_1}, \ldots, -t_{i_{r-1}}\rangle\rangle$ (mod $\mathring{V}_r^\gamma(F)$) is a group homomorphism by Fact 3.3 (iv) proved in the next section. □

COROLLARY 2.10. *Suppose that $\gamma \in \Delta$, $\gamma \in 2G$, and $\gamma \neq v(4)$. Then $V_r^\gamma(F)/\mathring{V}_r^\gamma(F)$ is isomorphic to*

$$\left(\bigoplus_{1 \leq i_1 < \cdots < i_{r-1} \leq n} D(B(i_1, \ldots, i_{r-1}))\right) \oplus \left(\bigoplus_{1 \leq i_1 < \cdots < i_{r-2} \leq n} D(B(i_1, \ldots, i_{r-2}))\right)$$

via the map which sends the multi-indexed element $f_{i_1,\dots,i_{r-1}} \in D(B(i_1,\dots,i_{r-1}))$ (*resp.* $f_{i_1,\dots,i_{r-2}} \in D(B(i_1,\dots,i_{r-2}))$) *to the r-fold*

$$\left\langle\left\langle -\left(1-\pi^{\gamma}\left(f_{i_1},\dots,i_{r-1}\right)^{*}\right), -t_{i_1},\dots,-t_{i_{r-1}}\right\rangle\right\rangle$$

$$\left(\textit{resp.}\ \left\langle\left\langle -\left(1-\pi^{\gamma}\left(f_{i_1,\dots,i_{r-2}}\right)^{*}\right), -t_{i_1},\dots,-t_{i_{r-2}},-\pi\right\rangle\right\rangle\right).$$

Proof. This follows immediately from Theorem 2.8 and the list of generators given in List 2.8 once we se that the defined map really is a group homomorphism. This follows as in Corollary 2.9. □

COROLLARY 2.11. *Suppose* $\gamma = v(4)$. *Then the ideal quotient* $V_r^{v(4)}(F)/I^{r+1}(F)$ *is isomorphic to*

$$\bigoplus_{1\le i_1<\cdots<i_{r-1}\le n}\left[\mathscr{F}/\left(\wp(\mathscr{F})+D\left(C(i_1,\dots,i_{r-1})\right)\right)\right]$$

$$\oplus \bigoplus_{1\le i_1<\cdots<i_{r-2}\le n}\left[\mathscr{F}/\left(\wp(\mathscr{F})+D\left(C(i_1,\dots,i_{r-2})\right)\right)\right]$$

via the map which sends the multi-indexed $f_{i_1,\dots,i_{r-1}}$ (*resp.* $f_{i_1},\dots,i_{r-2}$) *to the r-fold Pfister form*

$$\left\langle\left\langle -\left(1-4\left(f_{i_1,\dots,i_{r-1}}\right)^{*}\right), -t_{i_1},\dots,-t_{i_{r-1}}\right\rangle\right\rangle$$

$$\left(\textit{resp.}\ \left\langle\left\langle -\left(1-4\left(f_{i_1,\dots,i_{r-2}}\right)^{*}\right), -t_{i_1},\dots,-t_{i_{r-2}},-\pi\right\rangle\right\rangle\right).$$

Proof. Again, this follows immediately from Theorem 2.7, and the list of generators given in List 2.8. □

To close this section we shall define and compute one more ideal quotient. In what follows, for any field K (regardless of characteristic) we shall let $W(K)$ denote the ring $\mathbf{Z}[K^{\cdot}/K^{\cdot 2}]/\mathscr{I}(K)$ where $\mathbf{Z}[K^{\cdot}/K^{\cdot 2}]$ is the group algebra of square classes over $\mathbf{Z}$, and $\mathscr{I}(K)$ is the ideal generated by elements of the form $[1]+[-1]$, $[a]+[b]-[a+b]-[ab(a+b)]$ whenever $a,b\in K$; $a\ne -b$. Let $\langle a_1,\dots,a_n\rangle\mathrm{Z}$ denote the image of $[a_1]+\cdots+[a_n]$ in $W(K)$. If the characteristic of K is not 2, then $W(K)$ is isomorphic to the usual Witt ring of K (cf. **[L]**), and if the characteristic of K is 2, then $W(K)$ is isomorphic to the Witt ring of symmetric bilinear forms of K (**[K2]**, **[M2]**) as defined by Milnor in **[M2]**. By $I^n(K)$

we shall mean the ideal in $W(K)$ generated by the 'Pfister forms' $([1] - [a_1]) \cdots ([1] - [a_n])$ in $\mathbf{Z}[K^{\cdot}/K^{\cdot 2}]$. As usual we denote $I^n(K)/I^{n+1}(K)$ by $\bar{I}^n(K)$.

DEFINITION 2.12. Let F be a valued field with residue field $\mathscr{F}$. We define $\mathring{I}^r(F)$ to be the subideal of $I^r(F)$ generated by $I^{r+1}(F)$ and all r-folds of the form $\langle\langle -u, -x_1, \ldots, -x_{r-1}\rangle\rangle$ where $x_1, \ldots, x_{r-1} \in F$, and $u \in U$ with $\bar{u} = 1 \in \mathscr{F}$. We define $T(F)$ to be the ideal of $W(F)$ generated by all 1-folds $\langle\langle -u\rangle\rangle$ where u is a unit of F satisfying $\bar{u} = 1 \in \mathscr{F}$.

We now assume (for simplicity) that F is a discretely valued field with residue class field $\mathscr{F}$. We construct a ring homomorphism ρ_π: $W(F) \to W(\mathscr{F})$ and a group homomorphism δ_π: $W(F) \to W(\mathscr{F})$ using the group algebra definition of $W(K)$. These constructions are well-known (cf. [**L**, p. 145]), but are included here for completeness.

To construct ρ_π, choose $\pi \in F$ with $v(\pi) = 1$. If U denotes the units of F, then every element $[f] \in F^{\cdot}/F^{\cdot 2}$ has a unique representation $\mathrm{mod}(F^{\cdot 2})$ $[f] = [u][\pi^i]$ for some $u \in U$ and some $i = 0, 1$. The mapping $[f] \mapsto [\bar{u}]$ gives a group homomorphism $F^{\cdot}/F^{\cdot 2} \to \mathscr{F}^{\cdot}/\mathscr{F}^{\cdot 2}$, which induces a ring homomorphism ρ'_π: $\mathbf{Z}[F^{\cdot}/F^{\cdot 2}] \to \mathbf{Z}[\mathscr{F}^{\cdot}/\mathscr{F}^{\cdot 2}]$. One easily verifies that $\rho'_\pi(\mathscr{I}(F)) \subseteq \mathscr{I}(\mathscr{F})$, and hence induces the desired map ρ_π. By checking the behavior of ρ_π on r-fold Pfister forms one obtains that $\rho_\pi(I^n(F)) \subseteq I^n(\mathscr{F})$, and hence one obtains surjections $\bar{\rho}_\pi$: $\bar{I}^r(F) \to \bar{I}^r(\mathscr{F})$.

To construct δ_π (often called the 'second residue homomorphism') we consider the group homomorphism δ'_π: $\mathbf{Z}[F^{\cdot}/F^{\cdot 2}] \to W(\mathscr{F})$ defined by $[u] \mapsto 0$ and $[\pi u] \mapsto \langle \bar{u}\rangle$ for all $u \in U$. One checks that $\mathscr{I}(F) \subseteq \ker(\delta'_\pi)$ so that δ'_π induces a group homomorphism δ_π: $W(F) \to W(\mathscr{F})$. Further, for all $r \geq 1$, one has that $\delta_\pi(I^r(F)) \subseteq I^{r-1}(\mathscr{F})$, so that one obtains surjections $\bar{\delta}_\pi$: $\bar{I}^r(F) \to \bar{I}^{r-1}(\mathscr{F})$. Lastly we observe that $T(F) \subseteq \ker(\delta_\pi) \cap \ker(\rho_\pi)$.

THEOREM 2.13. *For all $r \geq 1$, the mappings ρ_π and δ_π induce isomorphisms*

$$\bar{\rho}_\pi \oplus \bar{\delta}_\pi\colon I^r(F)/[(I^r(F) \cap T(F)) + \mathring{I}^r(F)] \to \bar{I}^r(\mathscr{F}) \oplus \bar{I}^{r-1}(\mathscr{F}).$$

Proof. We define a map χ: $W(\mathscr{F}) \to W(F)/T(F)$ as follows: Let $(\ )^*$: $\mathscr{F}^{\cdot} \to F^{\cdot}$ by any lifting, and consider the map χ_1: $\mathbf{Z}[\mathscr{F}^{\cdot}/\mathscr{F}^{\cdot 2}] \to W(F)/T(F)$ defined by $\chi_1([f]) = \langle (f)^*\rangle$. This makes sense, and does not depend upon the choice of $(\ )^*$ in view of the definition of $T(F)$. Also, one easily obtains that $\chi_1(\mathscr{I}(\mathscr{F})) = 0$, and thus χ_1 induces the desired map χ. Analogously we construct a map $\chi_\pi\colon W(\mathscr{F}) \to I(F)/T(F)$ which is induced by $\chi_{\pi 1}\colon \mathbf{Z}[\mathscr{F}^{\cdot}/\mathscr{F}^{\cdot 2}] \to I(F)/T(F)$ defined by $\chi_{\pi 1}([f]) = \langle (f)^* \rangle \langle\langle -\pi \rangle\rangle$.

These two maps just described induce homomorphisms

$$\bar{\chi}\colon \bar{I}^r(\mathscr{F}) \to I^r(F)/\big(I^{r+1}(F) + (T(F) \cap I^r(F))\big)$$

and

$$\bar{\chi}_\pi\colon \bar{I}^{r-1}(\mathscr{F}) \to I^r(F)/\big((I^r(F) \cap T(F)) + I^{r-1}(F)\big).$$

According to the remarks preceding the Theorem, the map

$$\bar{\rho}_\pi \oplus \bar{\delta}_\pi\colon I^r(F)/\big((I^r(F) \cap T(F)) + \mathring{I}^r(F)\big) \to \bar{I}^r(\mathscr{F}) \oplus \bar{I}^{r-1}(\mathscr{F})$$

is surjective, since $\langle\langle -\pi \rangle\rangle W(F) \subseteq \ker(\rho_\pi)$ and $\delta_\pi(\langle\langle -\pi \rangle\rangle I^{r-1}(F)) = I^{n-1}(\mathscr{F})$. Hence, to prove the Theorem it suffices to prove that $(\rho_\pi + \delta_\pi)(\chi + \chi_\pi)$ is the identity on $I^r(F)/((I^r(F) \cap T(F)) + \mathring{I}^r(F))$. As $I^r(F)$ is generated by r-folds of the form $\langle\langle -u_1, \ldots, -u_r \rangle\rangle$ and $\langle\langle -u_1, \ldots, -u_{r-1}, -\pi \rangle\rangle \bmod \mathring{I}^r(F)$ where the u_j are units of F with $\bar{u}_j \neq 1 \in \mathscr{F}$, it suffices to see what happens to these elements. Evidently $\rho_\pi(\langle\langle -u_1, \ldots, -u_r \rangle\rangle) = \langle\langle -\bar{u}_1, \ldots, -\bar{u}_r \rangle\rangle$ while $\rho_\pi(\langle\langle -u_1, \ldots, -u_{r-1}, -\pi \rangle\rangle) = 0$, and $\delta_\pi(\langle\langle -u_1, \ldots, -u_r \rangle\rangle) = 0$ while $\delta_\pi(\langle\langle -u_1, \ldots, -u_{r-1}, -\pi \rangle\rangle) = \langle\langle -\bar{u}_1, \ldots, -\bar{u}_{r-1} \rangle\rangle$. As

$$\begin{aligned} \chi(\langle\langle -\bar{u}_1, \ldots, -\bar{u}_r \rangle\rangle) &= \big\langle\big\langle -(\bar{u}_1)^*, \ldots, -(\bar{u}_r)^* \big\rangle\big\rangle \\ &\equiv \langle\langle -u_1, \ldots, -u_r \rangle\rangle \bmod T(F), \end{aligned}$$

and as

$$\begin{aligned} \chi_\pi \langle\langle -\bar{u}_1, \ldots, -\bar{u}_{r-1} \rangle\rangle &= \big\langle\big\langle -(\bar{u}_1)^*, \ldots, -(\bar{u}_{r-1})^*, -\pi \big\rangle\big\rangle \\ &\equiv \langle\langle -u_1, \ldots, -u_{r-1}, -\pi \rangle\rangle \bmod T(F), \end{aligned}$$

the result follows. □

3. The graded analogue of Springer's theorem. We continue to assume that F is a discretely valued dyadic valued field of characteristic 0 with residue class field $\mathscr{F}$ of characteristic 2. In this section we accomplish two things. First we relate the ideals $V_r^\gamma(F)$ defined in §2 to some more natural ideals $I^{r,\gamma}(F)$. Second, we use these latter ideals to prove results that relate $GW(F)$ to $GW(\mathscr{F})$ and the value $v(2) \in \mathbf{Z}$. These results are the dyadic analogues of the graded version of Springer's

Theorem (cf. [S] and [W]). We recall that n is the 2-dimension of $\mathscr{F}$. The results of this section in the special case where $n = 1$ are crucial to the examples studied in §4.

DEFINITION 3.1. For all $r \geq 1$ and $\gamma > 0$ the ideal $T_r^\gamma(F)$ is the ideal of $W(F)$ generated by all r-folds of the form $\langle\langle -u, -x_1, \ldots, -x_{r-1}\rangle\rangle$, where $u \in U^\gamma$. We define $I^{r,\gamma}(F)$ by $I^{r,\gamma}(F) := T_r^\gamma(F) + I^{r+1}(F)$.

It is clear from the definitions that $V_r^\gamma(F) \subseteq I^{r,\gamma}(F)$. In some sense one hopes that these two ideals should be 'almost equal', but this is hard to make precise. Using List 2.8, we do however have the following:

LEMMA 3.2. *If $r \geq 2$ and $\tau \in T_r^\gamma(F)$, then there exist elements $\sigma_1, \ldots, \sigma_s$ from List 2.8 such that $\tau \equiv \sigma_1 + \cdots + \sigma_s \bmod(T_r^{\gamma+1}(F) + T_{r+1}^1(F))$. Consequently the inclusions $V_r^\gamma(F) \subseteq I^{r,\gamma}(F)$ induce surjections $i_{r,\gamma}$: $V_r^\gamma(F)/V_r^{\gamma+1}(F) \to I^{r,\gamma}(F)/I^{r,\gamma+1}(F)$ for all $\gamma \in \Delta$. In particular, if $r > n + 2$, then $I^{r,\gamma}(F) = I^{r,\gamma+1}(F)$ for all γ.*

The proof of Lemma 3.2 uses a number of computational facts, which we now list separately for future reference.

Facts 3.3. Let $x, y, \pi, \rho \in F$ with $v(\pi), v(\rho) > 0$. Then:

(i) $\langle\langle -x\rangle\rangle + \langle\langle -y\rangle\rangle = \langle\langle -xy\rangle\rangle + \langle\langle -x, -y\rangle\rangle$ in $W(F)$.

(ii) $\langle\langle -(1-y), -x\rangle\rangle = \langle\langle -(1-y), -xy\rangle\rangle$

(iii) $\langle\langle -(1+x), -(1+y)\rangle\rangle = \langle\langle -(1-xy), -x(1+x)(1+y)\rangle\rangle$

(iv) $\langle\langle -(1-(\pi+\rho)), -x\rangle\rangle \equiv \langle\langle -(1-\pi), -x\rangle\rangle + \langle\langle -(1-\rho), -x\rangle\rangle$ $\bmod(T_2^{v(\pi)+v(\rho)}(F) + T_3^1(F))$

(v) If $v(x) = v(y) = v(x+y) = 0$ then

$$\begin{aligned}\langle\langle -(1-\pi), -(x+y)\rangle\rangle &\equiv \langle\langle -(1-(x/(x+y))\pi), -x\rangle\rangle \\ &\quad + \langle\langle -(1-(y/(x+y))\pi), -y\rangle\rangle \bmod\big(T_2^{2v(\pi)}(F) + T_3^1(F)\big).\end{aligned}$$

Proof. (i) and (ii) are easy and well-known. For (iii) note that $\langle\langle -(1+x), -(1+y)\rangle\rangle = \langle\langle -(1+x), x(1+y)\rangle\rangle$ by (ii) and that

$$\begin{aligned}&\langle 1, -(1+x), x+xy, -x(1+x)(1+y)\rangle \\ &\quad = \langle 1, -(1-xy), (1+x)(x+xy)(1-xy), -x(1+x)(1+y)\rangle\end{aligned}$$

since $-(1+x)+(x+xy) = -(1-xy)$. Thus (iii) follows. For (iv) note that

$$\langle\langle -(1-\pi), -x\rangle\rangle + \langle\langle -(1-\rho), -x\rangle\rangle \equiv \langle\langle -(1-\pi)(1-\rho), -x\rangle\rangle \bmod(T_3^1(F))$$

in view of (i), and that

$$\begin{aligned}\langle\langle -(1-\pi)(1-\rho), -x\rangle\rangle &= \langle\langle -(1-(\pi+\rho)+\pi\rho), -x\rangle\rangle \\ &\equiv \langle\langle -(1-(\pi+\rho)), -x\rangle\rangle \bmod(T_2^{v(\pi)+v(\rho)}(F) + T_3^1(F))\end{aligned}$$

since $(1-(\pi+\rho)+\pi\rho) = (1-(\pi+\rho))u$ for some $u \in U^{v(\pi)+v(\rho)}$. This proves (iv). Finally for (v) note that by (iv)

$$\begin{aligned}&\langle\langle -(1-\pi), -(x+y)\rangle\rangle \\ &\quad \equiv \langle\langle -(1-(x/(x+y))\pi), -(x+y)\rangle\rangle \\ &\qquad + \langle\langle -(1-(y/(x+y))\pi), -(x+y)\rangle\rangle \bmod(T_2^{2v(\pi)}(F) + T_3^1(F)),\end{aligned}$$

the latter sum being equal to

$$\langle\langle -(1-(x/(x+y))\pi), -x\pi\rangle\rangle + \langle\langle -(1-(y/(x+y))\pi), -y\pi\rangle\rangle$$

by (ii). Mod$(T_3^1(F))$ this is

$$\begin{aligned}&\equiv \langle\langle -(1-(x/(x+y))\pi), -x\rangle\rangle + \langle\langle -(1-(x/(x+y))\pi), -\pi\rangle\rangle \\ &\quad + \langle\langle -(1-(y/(x+y))\pi), -y\rangle\rangle + \langle\langle -(1-(y/(x+y))\pi), -\pi\rangle\rangle.\end{aligned}$$

By (iv), then (ii) we find mod$(T_2^{2v(\pi)}(F) + T_3^1(F))$ that

$$\begin{aligned}&\langle\langle -(1-(x/(x+y))\pi), -\pi\rangle\rangle + \langle\langle -(1-(y/(x+y))\pi), -\pi\rangle\rangle \\ &\qquad \equiv \langle\langle -(1-\pi), -\pi\rangle\rangle \equiv \langle\langle -(1-\pi), -1\rangle\rangle = 0.\end{aligned}$$

This proves (v). □

Proof of Lemma 3.2. By definition $T_r^\gamma(F)$ is generated by all r-folds of the form $\langle\langle -u, -x_1, \dots, -x_{r-1}\rangle\rangle$ where $u \in U^\gamma$. Thus it suffices to prove the Lemma for such r-folds. If $u \in U^{\gamma+1}$, there is nothing to prove. If some x_j is a unit with $\bar{x}_j = 1 \in \mathscr{F}$ then by 3.3 (iii) we see that this r-fold lies in $T_r^{\gamma+1}(F)$, so again there is nothing to prove. Applying (i), the preceding precisely means that it suffices to study the r-folds of the form $\langle\langle -u, -x_1, \dots, -x_{r-1}\rangle\rangle$ and $\langle\langle -u, -x_1, \dots, -x_{r-2}, -\pi\rangle\rangle$ where $u \in (U^\gamma - U^{\gamma+1})$, and the x_i are units with $\bar{x}_i \neq 1 \in \mathscr{F}$.

Next observe that any such r-fold can be written $\mathrm{mod}(T_r^{\gamma+1}(F) + T_{r+1}^1(F))$ as a sum of r-folds of the form $\langle\langle -u, -t_{i_1}, \ldots, -t_{i_{r-1}}\rangle\rangle$ or $\langle\langle -u, -t_{i_1}, \ldots, -t_{i_{r-2}}, -\pi\rangle\rangle$ where $u \in U^\gamma$ and $1 \le i_1 < \cdots < i_{r-1} \le n$. This follows from 3.3 (iii) and (v), together with the fact that as $\mathscr{F} = \mathscr{F}^2(\bar{t}_1, \ldots, \bar{t}_n)$ one can express

$$x_i \equiv \sum_{j \in 2^n} x_{ij}^2 t^j \ \mathrm{mod}(\pi) \quad \text{in } F.$$

Thus using the multilinearity $\mathrm{mod}(T_{r+1}^1(F))$ (this is 3.3(i)) one can expand the r-folds $\langle\langle -u, -x_1, \ldots, -x_{r-1}\rangle\rangle$ and $\langle\langle -u, -x_1, \ldots, -x_{r-2}, -\pi\rangle\rangle$ into a sum of r-folds of the desired shape. Finally, taking 3.3 (iii) and (iv) together, we observe that $\mathrm{mod}(T_r^{\gamma+1}(F) + T_{r+1}^1(F))$ it is only necessary to consider those r-folds of the form $\langle\langle -u, -t_{i_1}, \ldots, -t_{i_{r-1}}\rangle\rangle$ and $\langle\langle -u, -t_{i_1}, \ldots, -t_{i_{r-2}}, -\pi\rangle\rangle$ where the $\overline{\pi^{-\gamma}(1-u)}$'s range over a $\mathbf{Z}/2\mathbf{Z}$-basis of $\mathscr{F}^+$. In case $\gamma \notin 2G$, the result is now clear since the Type A generators given in List 2.8 include such u's with the $\overline{\pi^{-\gamma}(1-u)}$'s ranging over a $\mathbf{Z}/2\mathbf{Z}$-basis of $\mathscr{F}^+$.

In case $\gamma \in 2G$ and $\gamma \ne v(4)$, according to the description of the Type B generators listed in 2.8, we must show that r-folds of the form $\langle\langle -(1 - \pi^\gamma(\bar{\tau}x_i^2)^*), \ -t_{i_1}, \ldots, -t_{i_{r-1}}\rangle\rangle$ (resp. $\langle\langle -(1 - \pi^\gamma(\bar{\tau}x_i^2)^*), -t_{i_1}, \ldots, -t_{i_{r-2}}, -\pi\rangle\rangle$) where $\tau \in A(T) - B(i_1, \ldots, i_{r-1})$ (resp. $\tau \in A(T) - B(i_1, \ldots, i_{r-2})$) can be represented $\mathrm{mod}(T_r^{\gamma+1}(F) + T_{r+1}^1(F))$. If $\tau = 1$, then as $\gamma \in 2G$, $0 < \gamma < v(4)$, there is some $z \in F$ with $z^2(1 - \pi^\gamma(x_i^2)^*) \in U^{\gamma+1}$. Thus there is no problem in this case. The case of $\tau \ne 1$ is treated by inducting backwards on the order of the $r-1$-tuple $(i_1, \ldots, i_{r-1})$ (resp. $(i_1, \ldots, i_{r-2})$) in the lexicographic order in n^{r-1} (resp., n^{r-2}). Suppose such τ has the form $\tau = t_{i_j}\tau'$ where $\tau' \in A(t_{i_j+1}, \ldots, t_n)$ and $i_j \in \{i_1, \ldots, i_{r-1}\}$ (resp. $i_j \in \{i_1, \ldots, i_{r-2}\}$). Since $\langle\langle -(1 - \pi^\gamma(\bar{\tau}x_i^2)^*), \ -t_{i_j}\tau'\rangle\rangle = 0$ (remember that $\gamma \in 2G$ implies $\pi^\gamma \in F^2$), we find by 3.3 (i) that $\langle\langle -(1 - \pi^\gamma(\bar{\tau}x_i^2)^*), \ -t_{i_j}\rangle\rangle \equiv \langle\langle -(1 - \pi^\gamma(\bar{\tau}x_i^2)^*), \ -\tau'\rangle\rangle \ \mathrm{mod}\, T_3^1(F)$. This shows that the original r-fold can be written mod $(T_r^{\gamma+1}(F) + T_{r+1}^1(F))$ as a sum of r-folds where the $(r-1)$-tuples involved (resp. $(r-2)$-tuples) are larger than $(i_1, \ldots, i_{r-1})$ (resp. $(i_1, \ldots, i_{r-2})$) in the lexicographic order. This completes the case where $\gamma \in 2G$ and $\gamma \ne v(4)$.

Finally we treat the case where $\gamma = v(4)$. We must see that r-folds of the form $\langle\langle -(1 - 4g_i), -t_{i_1}, \ldots, -t_{i_{r-1}}\rangle\rangle$ can be represented where the $\bar{g}_i$ form a basis of $\wp(\mathscr{F}) + D(A_0 - B(i_1, \ldots, i_{r-1}))$. Note however that if $\bar{g}_i \in \wp(\mathscr{F})$, then for some $z \in F$ one has that $z^2(1 - 4g_i) \in U^{v(4)+1}$, so that such r-folds lie in $T_r^{v(4)+1}$. We are thus reduced to treating g_i's where the $\bar{g}_i$'s are assumed to lie in $\bar{\tau}\mathscr{F}^2$ for some $\tau \in A_0 - B(i_1, \ldots, i_{r-1})$. The arguments applied above for the case $\gamma \in 2G$ apply verbatim in this case. This completes the proof of the first statement of Lemma 3.2. The second statement is an immediate consequence of the first, given the definitions. This completes the proof of 3.2. □

Recall [D], that a valued field F is called 2-Henselian if Hensel's Lemma holds for quadratic polynomials over F, i.e. if $a, b \in O_F$ and if $x^2 + \bar{a}x + \bar{b}$ is separable with a root in $\mathscr{F}$, then $x^2 + ax + b$ has a root in F. In our case this implies that if $u \in U^{v(4)}$ satisfies $\overline{(1-u)/4} \in \wp(\mathscr{F})$, then $u \in F^2$. One particularly important property of 2-Henselian valuations is that if F is 2-Henselian, then so is any quadratic extension of F (in the necessarily unique extended valuation). For some details see [D].

Our next result related the ideals $T(F)$ and $\mathring{I}^r(F)$ defined in 2.12 to the ideal $V_r^\gamma(F)$.

THEOREM 3.4. *If F is a discretely valued, dyadic valued field, and if F is 2-Henselian, then for all r we have:*

(i) $V_r^\gamma(F) = I^{r,\gamma}(F)$ *for all* $\gamma \in \Delta$

(ii) $I^r(F) \cap T(F) \subseteq \mathring{I}^r(F) = I^{r,1}(F)$. *In particular* $I^r(F)/I^{r,1}(F) \cong \bar{I}^r(\mathscr{F}) \oplus \bar{I}^{r-1}(\mathscr{F})$.

Proof. First note that Δ is a finite set. Next note that $U^{v(4)+1} \subseteq F^2$ by the 2-Henselian property, and thus we may conclude that $\langle\langle -u, -x_1, \ldots, -x_{r-1}\rangle\rangle = 0 \in W(F)$ whenever $u \in U^{v(4)+1}(F)$. In particular, if $\gamma > v(4)$ this means that $V_r^\gamma(F) = I^{r,\gamma}(F) = I^{r+1}(F)$. Now let $\rho \in I^{r,\gamma}(F)$ for $\gamma \in \Delta$. By the surjectivity of $i_{r,\gamma}$ we can find some $\chi_1 \in V_r^\gamma(F)$ with $\chi_1 - \rho \in I^{r,\gamma+1}(F)$. Repeating this process (noting that $V_r^{\gamma+1} \subseteq V_r^\gamma(F)$) a sufficient number of times, we find some $\chi \in V_r^\gamma(F)$ with $\chi - \rho \in I^{r,v(4)+1}(F) = I^{r+1}(F)$. As $I^{r+1}(F) \subseteq V_r^\gamma(F)$ we conclude that $I^{r,\gamma}(F) \subseteq V_r^\gamma(F)$, which proves (i).

To prove (ii) we apply the first statement of Lemma 3.2 and proceed by induction on r to prove the stronger statement that $I^r(F) \cap T(F) \subseteq T_r^1(F)$. As $T_r^1(F) \subseteq I^{r,1}(F)$, (ii) will follow. If $r = 1$ the result is trivial, as $T(F) = T_1^1(F)$. For $r = 2$ the result holds since $T(F)$ is generated by 1-fold Pfister forms and hence $I^2(F) \cap T(F) = T(F) \cdot IF = T_2^1(F)$. For $r > 2$, if $\rho \in I^r(F) \cap T(F)$, then by induction we have that $\rho \in T_{r-1}^1(F)$, so choose γ maximal so that $\rho \in T_{r-1}^\gamma(F) + T_r^1(F)$. Applying 3.2 we can find σ_i in List 2.8 such that $\rho \equiv \sigma_1 + \cdots + \sigma_s \bmod(T_{r-1}^{\gamma+1}(F) + T_r^1(F))$. In particular, $\rho \equiv \sigma_1 + \cdots + \sigma_s \bmod(V_{r-1}^{\gamma+1}(F))$, using the first part of this Theorem. However, $\rho \in I^r(F)$, so according to Theorem 2.8 each σ_i must occur with even multiplicity. But $2\sigma_i \in T_r^1(F)$, so $\rho \in T_{r-1}^{\gamma+1}(F) + T_r^1(F)$, and thus no maximal γ exists. Finally as $T_{r-1}^{\gamma}(F) = 0$ for $\gamma > v(4)$ we have $I^r(F) \cap T(F) \subseteq T_r^1(F)$. The final statement follows from this and from Theorem 2.13. This proves Theorem 3.4. □

THEOREM 3.5. *If F is discretely valued, then for all $r \geq 2$ and for all $\gamma \in \Delta$,*

$$\rho_{r,\gamma}\colon V_r^{\gamma}(F)/\mathring{V}_r^{\gamma}(F) \to I^{r,\gamma}(F)/I^{r,\gamma+1}(F)$$

is an isomorphism.

Proof. By the preceding Lemma, the Theorem is true if F is 2-Henselian. If F is not 2-Henselian, let F_h be a Henselization of F inside the algebraic closure of F. Consider the commutative diagram:

$$\begin{array}{ccc} V_r^{\gamma}(F)/\mathring{V}_r^{\gamma}(F) & \to & I^{r,\gamma}(F)/I^{r,\gamma+1}(F) \\ \downarrow & & \downarrow \\ V_r^{\gamma}(F_h)/\mathring{V}_r^{\gamma}(F_h) & \to & I^{r,\gamma}(F_h)/I^{r,\gamma+1}(F_h) \end{array}$$

The left-hand vertical map is an isomorphism in view of Corollaries 2.9, 2.10, 2.11 and the lower horizontal map is an isomorphism as just mentioned. This shows that $\rho_{r,\gamma}$ is injective and proves the Theorem in view of Lemma 3.2. □

We are now ready to describe the 'graded analogue' of Springer's Theorem in the discrete case. We refer the reader to the paper of A. Wadsworth [W, Prop. 4.7] for the same result in the non-dyadic case. The result, as stated, is slightly unsatisfactory in that only a graded group homomorphism is described, no a graded ring homomorphism. We do remark however, that it is reasonably clear from the computations that one could describe a ring structure (although extremely messy) on $A(\mathscr{F}, v(4))$ below. This ring structure would depend upon the 'arithmetic' of the particular uniformizing parameter π, as well as $\mathscr{F}$ and $v(4)$. To do this would take us too far afield, and is not necessary for the current applications of this paper.

In the following we let $\Delta = \{1, 2, \ldots, v(4)\}$ and for each $r \geq 2$ we set $A_r(\mathscr{F}, v(4)) = \bar{I}^r(\mathscr{F}) \oplus \bar{I}^{r-1}(\mathscr{F}) \oplus \bigoplus_{\gamma \in \Delta} V_r^{\gamma}(\mathscr{F})$ where by $V_r^{\gamma}(\mathscr{F})$ we mean $(\mathscr{F})^{\binom{n}{r-1}}$ if $\gamma \notin 2G$, we mean

$$\left(\bigoplus_{1 \leq i_1 < \cdots < i_{r-1} \leq n} \left(\bigoplus_{\tau \in B(i_1, \ldots, i_{r-1})} \bar{\tau}\mathscr{F}^2 \right) \right)$$

$$\oplus \left(\bigoplus_{1 \leq i_1 < \cdots < i_{r-2} \leq n} \left(\bigoplus_{\tau \in B(i_1, \ldots, i_{r-2})} \bar{\tau}\mathscr{F}^2 \right) \right)$$

when $\gamma \in 2G$, $\gamma \neq v(4)$, and we mean

$$\left(\bigoplus_{1\le i_1<\cdots<i_{r-1}\le n}(\mathscr{F})/[\wp(\mathscr{F})+D(C(i_1,\ldots,i_{r-1}))]\right)$$
$$\oplus\left(\bigoplus_{1\le i_1<\cdots<i_{r-2}\le n}(\mathscr{F})/[\wp(\mathscr{F})+D(C(i_1,\ldots,i_{r-2}))]\right)$$

whenever $\gamma = v(4)$. By A_0 we simply mean $\mathbf{Z}/2\mathbf{Z}$ and $A_1(\mathscr{F}, v(4)) :=$ $(\mathscr{F}^{\cdot}/\mathscr{F}^{\cdot 2}) \oplus \mathbf{Z}/2\mathbf{Z} \oplus \bigoplus_{\gamma\in\Delta} V_1^{\gamma}(F)$ where in this case $V_1^{\gamma}(F) \cong \mathscr{F}$ when $\gamma \notin 2G$, $V_1^{\gamma}(F) \cong (\bigoplus_{\tau\in A_0(T)} \bar{\tau}\mathscr{F}^2)$ when $\gamma \in 2G$, $\gamma \neq v(4)$, and $V_1^{v(4)}(F) \cong \mathscr{F}/\wp(\mathscr{F})$. We then denote by $A(\mathscr{F}, v(4))$ the graded group $\bigoplus_{r=0}^{n+2} A_r(\mathscr{F}, v(4))$.

According to the preceding Theorems 3.4, 3.5 and Corollaries 2.9, 2.10, 2.11 we have that in case F is 2-Henselian, whenever $r \geq 2$ that

$$I^r(F)/I^{r,1}(F)\oplus\cdots\oplus I^{r,\gamma}(F)/I^{r,\gamma+1}(F)$$
$$\oplus\cdots\oplus I^{r,v(4)}(F)/I^{r+1}(F)=A_r(\mathscr{F},v(4)),$$

with all the identifications mentioned in these previous results. Further, one is able to identity the elementary 2-group $I^r(F)/I^{r+1}(F)$ with the direct sum $I^r(F)/I^{r,1}(F) \oplus \cdots \oplus I^{r,v(4)}(F)/I^{r+1}(F)$ in this case by using the List 2.8 elements as "representatives" for the elements of the quotients $I^{r,s}(F)/I^{r,s+1}(F)$ inside $I^r(F)/I^{r+1}(F)$. Now, in case F is not 2-Henselian let F_h be a Henselian of F. The composition of the functorial maps $I^r(F)/I^{r+1}(F) \to I^r(F_h)/I^{r+1}(F_h)$ together with all the above mentioned identifications give rise to a map ω_r: $I^r(F)/I^{r+1}(F) \to A_r(\mathscr{F}, v(4))$.

For $r = 1$ the map ω_1: $I(F)/I^2(F) \to A_1(\mathscr{F}, v(4))$ arises from the identification $I(F_h)/I^2(F_h) \cong F_h^{\cdot}/F_h^{\cdot 2}$. By valuation theory it follows that $F_h^{\cdot}/F_h^{\cdot 2} \cong \mathscr{F}^{\cdot}/\mathscr{F}^{\cdot 2} \oplus \mathbf{Z}/2\mathbf{Z} \oplus U^1/(U^1)^2$. Since F_h is Henselian, $U^{v(4)+1}(F) \subseteq F^{\cdot 2}$, so we can decompose

$$U^1/(U^1)^2\cong U^1/U^2(U^1)^2\oplus U^2(U^1)^2/U^3(U^1)^2$$
$$\oplus\cdots\oplus U^{v(4)}(U^1)^2/(U^1)^2.$$

Again, by valuation theory and the Henselian condition, one readily checks that the maps r_i: $\mathscr{F} \to U^i(U^1)^2/U^{i+1}(U^1)^2$ given by $r_i(f) =$ $[1 + \pi^i\{f\}]$ for $1 \leq i < v(4)$, and $r_{v(4)}(f) = [1 + 4\{f\}]$ are well-defined surjective homomorphisms. Such r_i are injective if i is odd, have kernel $\mathscr{F}^2$ if i is even $< v(4)$, and has kernel $\wp(\mathscr{F})$ if $i = v(4)$. From this

$F_h^{\cdot} / F_h^{\cdot 2} \cong A_1(\mathscr{F}, v(4))$ follows. We may now give:

ANALOGUE OF SPRINGER'S THEOREM 3.6. *Suppose F is a discretely valued field with residue field $\mathscr{F}$ of characteristic 2. Then the previously described maps ω_r induce a surjective graded group homomorphism*

$$\omega\colon GW(F) \to A(\mathscr{F}, v(4)).$$

Moreover, such ω is an isomorphism if and only if F is 2-Henselian.

Proof. The surjectivity of ω is a consequence of the surjectivity of $F^{\cdot}/F^{\cdot 2} \to F_h^{\cdot}/F_h^{\cdot 2}$, which gives the surjectivity of $I^r(F)/I^{r+1}(F) \to I^r(F_h)/I^{r+1}(F_h)$. If F is 2-Henselian, then by earlier remarks, ω is an isomorphism. Conversely, if ω is an isomorphism, then necessarily $F^{\cdot}/F^{\cdot 2} \cong I(F)/I^2(F) \cong I(F_h)/I^2(F_h) \cong F_h^{\cdot}/(F_h)^{\cdot 2}$. Since F as characteristic 0, this implies that F must be 2-Henselian. □

COROLLARY 3.7. If F is 2-Henselian and discretely valued, then $I^{n+3}(F) = 0$ and $I^{n+2}(F) \cong \mathscr{F}/\wp(\mathscr{F}) + \mathscr{F}_0^2(t_1, \dots, t_n)$. In particular, $I^{n+2}(F) = 0$ if $\wp(\mathscr{F}) = \mathscr{F}$.

4. Applications to the amenability problem. As mentioned in the introduction, one reason for looking for a generalization of Springer's Theorem was to study in greater detail certain fields which are known to provide counterexamples to the phenomenon of '1-amenability'. The problem of 1-amenability was first studied in detail in **[ELW1]**, and subsequently counterexamples were found in **[ELTW]**. In this section we first look closely at some such counterexamples, both to illustrate how one may apply the results of §3, and to answer some questions about such counterexamples not resolved in **[ELTW]**. We then shall describe a counterexample to the property known as 'strong 1-amenability' introduced in **[ELW2]**, the first such counterexample found. For the most part we shall follow the notation of **[ELTW]**.

Throughout this section we will assume that F is 2-Henselian with $G = \mathbf{Z}$, $v(2) = 1$ or 2, $n = 1$ i.e. $\mathscr{F} = \mathscr{F}^2 + \bar{t}\mathscr{F}^2$, and that $\mathscr{F} = \wp(\mathscr{F}) + \bar{t}\mathscr{F}^2$ where $t \in U$ with $\bar{t} \notin \mathscr{F}^2$. We fix a uniformizing parameter $\pi \in F$ (i.e. $v(\pi) = 1 \in G = \mathbf{Z}$). Much of this section can be generalized without these restrictions, but we have chosen these conditions to keep the computations reasonable in length. In order to facilitate our computations we shall fix a lifting $(\ )^*\colon \mathscr{F} \to F$ which is constructed as follows: Let $(\ )_0^*\colon \mathscr{F} \to F$ be any fixed lifting with $(1)_0^* = 1$, and then for $f \in \mathscr{F}$ with $f = a^2 + \bar{t}b^2$ we set $(f)^* = (a)_0^{*2} + t(b)_0^{*2}$. We note that $(\bar{t})^* = t$, and it is easily seen that whenever $f, g \in \mathscr{F}$ then $(f)^*(g)^* \equiv (fg)^* \bmod(2, \pi^2)$ and $(f)^* + (g)^* \equiv (f+g)^* \bmod(2, \pi^2)$.

We begin by studying quadratic extensions of F. If M is a quadratic extension of F, the valuation of F extends uniquely to a valuation of M (as F is 2-Henselian), whose residue class field is denoted $\overline{M}$ and value group is denoted G_M. As F is 2-Henselian, and since G is discrete, it follows that F has no immediate quadratic extensions (not necessarily true if G is not discrete). Thus in case $v(2) = 1$ there are four types of quadratic extensions of F, which we name and describe as follows:

(i) *inseparably unramified*: Here $M = F(t^{1/2})$ where $t \in U$ and $\bar{t} \notin \mathscr{F}^{\cdot 2}$.

(ii) *tamely ramified*: Here $M = F(\pi^{1/2})$ where $v(\pi) = 1$ in $G = \mathbf{Z}$.

(iii) *separably unramified*: Here $M = F((1 + 4g)^{1/2})$ where $g \in O_F$ and $\bar{g} \notin \wp(\mathscr{F})$. In this case one easily obtains that $\overline{M} = \mathscr{F}(\bar{\alpha})$ where $\alpha = (1 + (1 + 4g)^{1/2})/2$ satisfies $\bar{\alpha}^2 + \bar{\alpha} = \bar{g}$ in $\mathscr{F}$.

(iv) *wildly ramified*: Here $M = F((1 + \pi f)^{1/2})$ where $v(\pi) = 1$, and $f \in O_F$ with $\bar{f} \neq 0 \in \mathscr{F}$. In this case one can check that

$$v\left(1 + (1 + \pi f)^{1/2}\right) = v(\pi)/2 = 1/2 \in v(M).$$

In case $v(2) > 1$, then one obtains an additional case:

(v) *wildly unramified*. Here $M = F((1 + \pi^2 f)^{1/2})$ where $0 < v(\pi) < v(2)$ and $\bar{f} \notin \mathscr{F}^2$. In this case one can check that

$$\gamma = \pi^{-1}\left(1 + (1 + \pi^2 f)^{1/2}\right) \quad \text{satisfies } \bar{\gamma}^2 = \bar{f} \text{ in } \mathscr{F}$$

so that $\overline{M} = \mathscr{F}(\bar{f}^{1/2})$.

Cases (i) and (v) may strike the reader as odd at first glance, but this is because they do not occur in number theory where residue fields are always perfect. It is also customary, when defining unramified extensions, to include the condition that the degree be prime to the characteristic of $\mathscr{F}$. We ignore this convention since we are exclusively interested in the case where $[\mathbf{M} : \mathbf{F}]$ is a power of char($\mathscr{F}$).

Before proceeding to a sequence of computational Lemmas involving these quadratic extensions of F, we record here a Lemma that provides us with some conventient technical devices for proving these Lemmas.

LEMMA 4.1. *Suppose the residue field of F satisfies $\mathscr{F} = \mathscr{F}^2 + \bar{t}\mathscr{F}^2$. Then for any $m \in F^{\cdot}$ with $v(m) > 0$:*

(i) *The map $\mathscr{F} \to I(F)/I^{1, v(m)+1}(F)$ given by $f \mapsto \langle\langle -(1 + m(f)^*)\rangle\rangle$ is an additive group homomorphism that is independent of the choice of $(\ )^*$.*

(ii) *The map $\mathscr{F}^{\cdot} \to \mathscr{F}^+$ given by $a^2 + \bar{t}b^2 \mapsto b^2/(a^2 + \bar{t}b^2)$ is a group homomorphism with kernel $\mathscr{F}^{\cdot 2}$.*

(iii) *The map* $\mathscr{F}^{\bullet} \to \mathscr{F}^{+}$ *given by* $a^2 + \bar{t}b^2 \mapsto ab/(a^2 + \bar{t}b^2)$ *is a group homomorphism with kernel* $\mathscr{F}^{\bullet 2} \cup \bar{t}\mathscr{F}^{\bullet 2}$.

Proof. (i) and (ii) follow by straightforward computations. So does (iii), but also note that

$$b^2/(a^2 + \bar{t}b^2) = \big(ab/(a^2 + \bar{t}b^2)\big)^2 + \bar{t}\big(b^2/(a^2 + \bar{t}b^2)\big)^2.$$

Thus the map in (iii) is really the composite of the map in (ii) with the map $a^2 + \bar{t}b^2 \mapsto a$. □

Until further notice we shall assume that $v(2) = 1$. For each of the four cases listed above we shall compute generators for the kernels $I^2(M/F) := \ker(I^2(F) \to I^2(M))$. Our hypothesis that $\mathscr{F} = \wp(\mathscr{F}) + \bar{t}\mathscr{F}^2$, together with Corollary 3.7 gives that $I^3(F) = 0$. Thus according to 3.6, we can list a $\mathbf{Z}/2\mathbf{Z}$-basis for $I^2(F)$ as follows:

(i) $\langle\langle -(x_i)^*, -\pi\rangle\rangle$ where the x_i's range over a basis of $\mathscr{F}^{\bullet}/\mathscr{F}^{\bullet 2}$,

(ii) $\langle\langle -(1 + \pi(f_i)^*), -t\rangle\rangle$ where the f_i's range over a basis of $\mathscr{F}^{+}$,

(iii) $\langle\langle -(1 + 4(g_i)^*), -\pi\rangle\rangle$ where the g_i's range over a basis of $\mathscr{F}^{+} \bmod \wp(\mathscr{F})$.

We now give:

LEMMA 4.2. (*Inseparably Unramified Case.*) *Suppose that* $v(2) = 1$ *and* $M = F(t^{1/2})$. *Then* $I^2(M/F)$ *has as a* $\mathbf{Z}/2\mathbf{Z}$*-basis the following* 2*-folds*:

(i) $\langle\langle -t, -\pi\rangle\rangle$

(ii) $\langle\langle -(1 + \pi(f_i)^*), -t\rangle\rangle$ *where the* f_i*'s range over a basis of* $\mathscr{F}^{+}$.

Proof. Clearly each of the listed 2-folds vanishes in $W(M)$. To see that these form a basis for $I^2(M/F)$ we must see that they generate $I^2(M/F)$. For this we show that the remaining generators for $I^2(F)$ remain independent inside $I^2(M)$. These remaining generators can be listed as:

(i) $\langle\langle -(x_i)^*, -\pi\rangle\rangle$ where the x_i's range over a basis of $\mathscr{F}^{\bullet}/(\mathscr{F}^{\bullet 2} \cup \bar{t}\mathscr{F}^{\bullet 2})$.

(ii) $\langle\langle -(1 + 4(g_i)^*), -\pi\rangle\rangle$ where the g_i's range over a basis of $\mathscr{F}^{+} \bmod \wp(\mathscr{F})$.

Evidently, the x_i's in (i) can all be expressed in the form $x_i = y_i^2 + \bar{t}$ ($y_i \neq 0$) since $\mathscr{F} = \mathscr{F}^2 + \bar{t}\mathscr{F}^2$. For such x_i we have that $(x_i)^* = (y_i)_0^* + t$. Thus as

$$\big((y_i)_0^{*2} + t\big)/\big((y_i)_0^* + t^{1/2}\big)^2 = 1 - 2\big(t^{1/2}(y_i)_0^*\big)/\big((y_i)_0^* + t^{1/2}\big)^2$$

in M, setting $z_i = (y_i)_0^*/((y_i)_0^* + t^{1/2})^2$ gives

$$\left\langle\left\langle -(x_i)^*, -\pi\right\rangle\right\rangle \cong_M \left\langle\left\langle -\left(1 - 2t^{1/2}z_i\right), -\pi\right\rangle\right\rangle = \left\langle\left\langle -\left(1 - 2t^{1/2}z_i\right), 2\pi t^{1/2}z_i\right\rangle\right\rangle$$
$$\equiv \left\langle\left\langle -\left(1 + 2t^{1/2}z_i\right), -t^{1/2}\right\rangle\right\rangle \bmod I^{2,2}(M).$$

Note that $\bar{z}_i = y_i/y_i^2 + \bar{t} \in \mathscr{F} \subseteq \mathscr{F}(t^{1/2})^2$, which is the image of x_i under the map of part (iii) of Lemma 4.1. Since the x_i's are all independent mod the kernel of this map, we see by part (i) of Lemma 4.1 and by Corollary 2.9 that the $\langle\langle -(1 + 2t^{1/2}z_i), -\pi\rangle\rangle$ are necessarily independent in $I^{2,1}(M)/I^{2,2}(M)$.

According to Lemma 1.12 $\mathscr{F} \cap \wp(\mathscr{F}(t^{1/2})) = \wp(\mathscr{F})$. Thus the g_i's remain independent in $\mathscr{F}(t^{1/2}) \bmod \wp(\mathscr{F}(t^{1/2}))$, so that the type (iii) generators listed above remain independent in $I^{2,2}(M)$. This proves the Lemma. □

LEMMA 4.3. (*Tamely Ramified Case.*) *Suppose* $v(2) = 1$ *and* $M = F(\pi^{1/2})$. *Then* $I^2(M/F)$ *has as a* $\mathbf{Z}/2\mathbf{Z}$*-basis the following* 2*-folds*:

(i) $\langle\langle -(x_i)^*, -\pi\rangle\rangle$ *where the* x_i *range over a basis of* $\mathscr{F}^{\boldsymbol{\cdot}}/\mathscr{F}^{\boldsymbol{\cdot}2}$.

(ii) $\langle\langle -(1 - \pi(f_i)^*), -t\rangle\rangle$ *where the* f_i *range over a basis of* $\bar{t}\mathscr{F}^2$.

(iii) $\langle\langle -(1 + 4(g_i)^*), -\pi\rangle\rangle$ *where the* g_i *range over a basis of* $\mathscr{F}^+ \bmod \wp(\mathscr{F})$.

Proof. The fact that the type (i) or (iii) 2-folds vanish in $W(M)$ is clear. For the type (ii) 2-folds note that as $(f_i)^* \in tF^2$, for such f_i we have that $\langle\langle -(1 - \pi(f_i)^*), -t\rangle\rangle = \langle\langle -(1 - \pi(f_i)^*), -\pi\rangle\rangle$ by Fact 3.3 (ii), so likewise these 2-folds vanish in $W(M)$.

The remaining basis elements for $I^2(F)$ can be listed as follows:

(ii) $\langle\langle -(1 - \pi(f_i)^*), -t\rangle\rangle$ where the f_i range over a basis of $\mathscr{F}^2$. Using the fact that $(f_i)^* \in F^2$ for these f_i's we find that in $M^{\boldsymbol{\cdot}}/M^{\boldsymbol{\cdot}2}$ we have

$$\left[1 - \pi(f_i)^*\right] = \left[\left(1 - \pi(f_i)^*\right)/\left(1 - \left(\pi(f_i)^*\right)^{1/2}\right)^2\right]$$
$$= \left[1 - 2\left(\pi(f_i)^*\right)^{1/2} + w_i\right]$$

for some $w_i \in M$ with $v(w_i) \geq v(4)$. Since $v(2\pi^{1/2}) \notin 2G_M$, and as $\overline{M} = \mathscr{F}$, we observe that the 2-folds

$$\left\langle\left\langle -\left(1 - \pi(f_i)^*\right), -t\right\rangle\right\rangle = \left\langle\left\langle -\left(1 - 2\left(\pi(f_i)^*\right)^{1/2} + w_i\right), -t\right\rangle\right\rangle$$
$$\equiv \left\langle\left\langle -\left(1 - 2\left(\pi(f_i)^*\right)^{1/2}\right), -t\right\rangle\right\rangle \bmod I^{2,v(4)}(M)$$

by 3.3. Thus they are independent in $I^{2,v(2\pi^{1/2})}(M)$ by Corollary 2.9. This proves the Lemma. □

LEMMA 4.4. (*Separably Unramified Case.*) *Suppose* $v(2)=1$ *and* $M = F((1+4g)^{1/2})$ *where* $g \in O_F$ *and* $\bar{g} \notin \wp(\mathscr{F})$. *Then* $I^2(M/F)$ *is generated by the* 2-*fold* $\langle\langle -(1+4g), -\pi\rangle\rangle$.

Proof. Clearly the stated form vanishes in $W(M)$. We note that $G_M = G_F$, and that $\overline{M} = \mathscr{F}(\bar{\alpha})$ where $\bar{\alpha}^2 + \bar{\alpha} = \bar{g}$. Thus $\mathscr{F} \cap \overline{M}^2 = \mathscr{F}^2$, and by a direct calculation $\mathscr{F} \cap \wp(\overline{M}) = \wp(\mathscr{F}) \cup (g + \wp(\mathscr{F}))$. From these observations it is clear by Theorem 3.6 that the remaining generators of $I^2(F)$ remain linearly independent in $I^2(M)$. This proves the Lemma. □

One last Lemma of this type is:

LEMMA 4.5. (*Wildly Ramified Case.*) *Suppose* $v(2)=1$ *and* $M = F((1+\pi)^{1/2})$. *Then* $I^2(M/F)$ *has as a* $\mathbf{Z}/2\mathbf{Z}$-*basis the following* 2-*folds*:
(ii) $\langle\langle -(1+\pi(f_i)^*), t\rangle\rangle$ *where* $\wp(f_i) \in \bar{t}\mathscr{F}^2$.
(iii) $\langle\langle -(1+4(g_i)^*), -\pi\rangle\rangle$ *where the* g_i *range over a basis of* $\mathscr{F} \bmod \wp(\mathscr{F})$.

Proof. We set $\pi' = 1 + (1+\pi)^{1/2}$ inside M and note that $\pi'^2 = \pi + 2\pi'$ so that $v(\pi'^2) = v(\pi) = v(2) = 1$, and $[\pi] = [1 - 2\pi'^{-1}]$ inside $M^\bullet/M^{\bullet 2}$. For $f = a^2 + \bar{t}b^2 \in \mathscr{F}^\bullet$ we find mod $I^{2,v(2)}(M)$ that

$$\begin{aligned}
\left\langle\left\langle -(f)^*, -\pi\right\rangle\right\rangle &= \left\langle\left\langle -(f)^*, -(1-2\pi'^{-1})\right\rangle\right\rangle \\
&= \left\langle\left\langle -\left((a)_0^{*2} + t(b)_0^{*2}\right), -(1-2\pi'^{-1})\right\rangle\right\rangle \\
&\equiv \left\langle\left\langle -\left(1 - \left(t(b)_0^{*2}/\left((a)_0^{*2} + t(b)_0^{*2}\right)\right)2\pi'^{-1}\right), -t(b)_0^{*2}\right\rangle\right\rangle \\
&\quad + \left\langle\left\langle -\left(1 - \left((a)_0^{*2}/\left((a)_0^{*2} + t(b)_0^{*2}\right)\right)2\pi'^{-1}\right), -(a)_0^{*2}\right\rangle\right\rangle \\
&\qquad\qquad \text{(by Fact 3.3(v))} \\
&= \left\langle\left\langle -\left(1 - \left(t(b)_0^{*2}/\left((a)_0^{*2} - t(b)_0^{*2}\right)\right)2\pi'^{-1}\right), -t\right\rangle\right\rangle \\
&\equiv \left\langle\left\langle -\left(1 - \left(tb/(a^2+tb^2)\right)^* 2\pi'^{-1}\right), -t\right\rangle\right\rangle.
\end{aligned}$$

Thus by Lemma 4.1 (ii), and by Corollary 2.9, we see that for f_i chosen independent mod $\mathscr{F}^2$, the 2-folds $\langle\langle -(f_i)^*, -\pi\rangle\rangle$ are independent in $I^{2,v(\pi')}(M) \bmod I^{2,v(2)}(M)$.

Suppose now that $\wp(f) \in \bar{t}\mathscr{F}^2$. Then $f^2 + f = \bar{t}b^2$ implies that $(f)^* = (f)_0^{*2} + t(b)_0^{*2} \equiv (f)^{*2} + t(b)_0^{*2} \bmod(4)$ (as $a \equiv b \bmod(2)$ implies $a^2 \equiv b^2 \bmod(4)$ inside M). In particular, recalling that $\pi'^2 - 2\pi' = \pi$, we find that

$$\left(\left(1 - \pi'(f)^*\right)^2 + t\left(\pi'(b)_0^*\right)^2\right) = 1 - 2\pi'(f)^* + \pi'^2\left((f)^{*2} + t(b)_0^{*2}\right)$$

$$\equiv 1 + \pi(f)^* \bmod(8) \textit{ in } M.$$

It follows that $\langle\langle -(1 + \pi(f)^*), t\rangle\rangle = 0 \in W(M)$. This shows that the listed elements of type (ii) vanish in $W(M)$. In view of Lemma 3.3 (iii) together with the fact that M is 2-Henselian, since $\pi \equiv 1 - 2\pi'^{-1} \bmod M^{\cdot 2}$ is a 1-unit in M, the type (iii) generators all vanish in M.

For a type (ii) generator of $I^2(F)$ with $f = a^2 + \bar{t}b^2$ we have using that $I^3(F) = 0$ and that $U^{v(4\pi')} \subseteq M^2$ that

$$\left\langle\left\langle -\left(1 + \pi(f)^*\right), t\right\rangle\right\rangle = \left\langle\left\langle -\left(1 + (\pi'^2 - 2\pi')(f)^*\right), t\right\rangle\right\rangle$$

$$= \left\langle\left\langle -\left(1 + \pi'^2(f)^*\right), t\right\rangle\right\rangle + \left\langle\left\langle -\left(1 - 2\pi'(f)^*\right), t\right\rangle\right\rangle$$

(by Fact 3.3 (iv))

$$= \left\langle\left\langle -\left(1 + \pi'^2(a)_0^{*2}\right), t\right\rangle\right\rangle + \left\langle\left\langle -\left(1 + \pi'^2(b)_0^{*2}t\right), t\right\rangle\right\rangle$$

$$+ \left\langle\left\langle 1 - \pi'^4(a)_0^{*2}(b)_0^{*2}t/\left(1 + \pi'^2(a)_0^{*2}\right)\left(1 + \pi'(b)_0^{*2}\right), t\right\rangle\right\rangle$$

$$+ \left\langle\left\langle -\left(1 - 2\pi'(f)^*\right), t\right\rangle\right\rangle$$

$$= \left\langle\left\langle -\left(1 + \pi'^2(a)_0^{*2}\right)/\left(1 - \pi'(a)_0^*\right)^2, t\right\rangle\right\rangle$$

$$+ \left\langle\left\langle -\left(1 + \pi'^4(a)_0^{*2}(b)_0^{*2}t\right), t\right\rangle\right\rangle + \left\langle\left\langle -\left(1 - 2\pi'(f)^*\right), t\right\rangle\right\rangle$$

$$= \left\langle\left\langle -\left(1 + \left(2\pi'(a)_0^*/\left(1 - \pi'(a)_0^*\right)^2\right)\right), t\right\rangle\right\rangle$$

$$+ \left\langle\left\langle -\left(1 - 2\pi'(f)^*\right), t\right\rangle\right\rangle$$

$$= \left\langle\left\langle -\left(1 + 2\pi'\left((a)_0^* + t(f)^*\right)\right), t\right\rangle\right\rangle.$$

Next we observe that the homomorphism $\mathscr{F} \to \mathscr{F}$ defined by $a^2 + \bar{t}b^2 \mapsto a + a^2 + \bar{t}b^2$ has kernel precisely those $f \in \mathscr{F}$ such that $\wp(f) \in \bar{t}\mathscr{F}^2$. For, $\wp(f) \in \bar{t}\mathscr{F}^2$ if and only if $f + f^2 = \bar{t}b^2$, which is equivalent to $f = f^2 + \bar{t}b^2$. Thus for f_i chosen to be independent in $\mathscr{F}$ $\mathrm{mod}\{f\colon \wp(f) \in \bar{t}\mathscr{F}^2\}$, Corollary 2.9 shows the 2-folds of the form $\langle\langle -(1 + \pi(f)^*), t\rangle\rangle$ are independent in $I^{2,v(2\pi')}(M)$ $\mathrm{mod}(I^{2,v(4)}(M))$. The Lemma is now proved. □

We now turn to the study of multiquadratic extensions $M = F(x_1^{1/2}, \ldots, x_n^{1/2})$ of F. Following **[ELTW]**, for such an extension we define $ID(M/F) = (\langle\langle -x_1\rangle\rangle, \ldots, \langle\langle -x_n\rangle\rangle) \subseteq W(F)$. Clearly $ID(M/F) \subseteq I(M/F)$. The quotient $I(M/F)/ID(M/F)$ is denoted by $h_2(M/F)$, and in case $h_2(M/F)$ is trivial M is said to be a 1-amenable extension of F. In **[ELW1]** the question was raised if all multiquadratic extensions were 1-amenable, (and it was proved there in case $n = 1$ or 2). However in **[ELTW]** a counterexample with $n = 3$ was discovered. In Theorem 4.8 below we study a version of the counterexample of **[ELTW]** and show that for all $n \geq 3$ there exist M with $h_2(M/F) \cong \mathbf{Z}/2\mathbf{Z}$. From this it follows (see Remark 4.10) that counterexamples exist for all $n \geq 3$ with $h_2(M/F)$ any finite elementary 2-group.

COROLLARY 4.6. *If $v(2) = 1$, $\mathscr{F} = \mathscr{F}^2 + \bar{t}\mathscr{F}^2 = \wp(\mathscr{F}) + \bar{t}\mathscr{F}^2$, and if M is a multiquadratic extension of F which either contains both an inseparable unramified quadratic subextension and a tamely ramified quadratic subextension, or does not contain any wildly ramified subquadratic extensions, then M is a 1-amenable extension of F.*

Proof. In the first case, if M contains both an inseparable unramified and a tamely ramified quadratic subextension, then without any loss of generality we may assume that $t^{1/2}$ and $\pi^{1/2}$ lie in M. According to Lemmas 4.2 and 4.3 we find that $I^2(F) \subseteq ID(M/F)$. Recalling from **[ELTW]** that $h_2(M/F) = I^2(M/F)/I^2D(M/F)$ where $I^2D(M/F) = ID(M/F) \cap I^2(M/F)$, we see that $h_2(M/F) = 0$ in this case.

In the second case, in view of the case just treated we can assume that $M = F(x_1^{1/2}, \ldots, x_n^{1/2})$ where $x_1 = 1 + 4g_1, \ldots, x_{n-1} = 1 + 4g_{n-1}$ and where x_n is either t, π, or $1 + 4g_n$. In case x_n is t, then $I^2(F)$ is generated over $ID(M/F) \cap I^2(F)$ by the following 2-folds:

(i) $\langle\langle -(x)^*, -\pi\rangle\rangle$ where the x range over a basis of $\mathscr{F}^{\bullet}/(\mathscr{F}^{\bullet 2}, \bar{t}\mathscr{F}^{\bullet 2})$.

(ii) $\langle\langle -(1 + 4h_i), -\pi\rangle\rangle$ where the h_i's range over a basis of $\mathscr{F}/\wp(\mathscr{F}) + G$, where G is generated by $\bar{g}_1, \ldots, \bar{g}_{n-1}$.

Setting $F_1 = F(x_1^{1/2}, \ldots, x_{n-1}^{1/2})$ one sees by Lemma 4.1 that both the (i) and (iii) type 2-folds remain independent in $I^2(F_1(t^{1/2})) = I^2(M)$. Thus $W(M/F) = ID(M/F)$ in this case. In case $x_n = \pi$, the proof is the same, only that Lemma 4.2 is used. Finally, in case $x_n = 1 + 4g_n$ then by repeated applications of Lemma 4.3 one sees that $I^2(M/F)$ is generated precisely by $\langle\langle -(1 + 4g_1), -\pi\rangle\rangle, \ldots, \langle\langle -(1 + 4g_n), -\pi\rangle\rangle$ from which $I^2(M/F) = I^2D(M/F)$ immediately follows. This proves the Corollary. □

For the next Theorem we shall assume that $\mathscr{F}/\wp(\mathscr{F})$ is infinite as well as $\mathscr{F} = \wp(\mathscr{F}) + t\mathscr{F}^2$. We show in the following Lemma that indeed such fields do exist, and that these properties are inherited under quadratic extensions (separable or inseparable). $\mathscr{F}((u))$ denotes the field of formal Laurent series over $\mathscr{F}$. The proof of the Lemma, being entirely straightforward is omitted.

LEMMA 4.7. *If $\mathscr{F}$ is a field of characteristic 2, then*:

(i) *If $\mathscr{F}$ is perfect then $\mathscr{F}((u))/[\wp(\mathscr{F}((u))) + t(\mathscr{F}((u)))^2] \cong \mathscr{F}/\wp(\mathscr{F})$, and $\mathscr{F}((u))/\wp(\mathscr{F}((u)))$ is infinite.*

(ii) *If $\mathscr{F} = \wp(\mathscr{F}) + v\mathscr{F}^2 = \mathscr{F}^2 + v\mathscr{F}^2$ then $\mathscr{F}(v^{1/2}) = \wp(\mathscr{F}(v^{1/2})) + v^{1/2}(\mathscr{F}(v^{1/2}))^2$ and $\mathscr{F}(\alpha) = \wp(\mathscr{F}(\alpha)) + v\mathscr{F}(\alpha)^2$ whenever $\alpha^2 + \alpha \in \mathscr{F}$.*

In what follows we take $\mathscr{F}$ to be $\mathscr{F}'((\bar{t}))$ where $\mathscr{F}'$ is perfect and $\wp(\mathscr{F}') = \mathscr{F}'$. We shall take F to be a discretely valued 2-Henselian valued field with residue class field $\mathscr{F}$ and with $v(2) = 1$. For the existence of such fields see [G, p. 70]. We now fix $x_1 = t$, $x_2 = 1 + \pi$, $x_3 = 1 + \pi g$, $x_4 = 1 + 4g_4, \ldots, x_n = 1 + 4g_n$, where $n \geq 3$ $g_4, \ldots, g_n \in O_F$ and $\bar{g}, \bar{g}_4, \ldots, \bar{g}_n$ are independent $\operatorname{mod}(\wp(\mathscr{F}))$. For convenience we express $\bar{g} = a^2 + \bar{t}b^2$ in $\mathscr{F}$ and set $\alpha = \overline{2\pi^{-1}} \in \mathscr{F}$. In case $M = F(x_1^{1/2}, \ldots, x_n^{1/2})$ we have:

THEOREM 4.8. *For such M and F, $h_2(M/F)$ is trivial in case $b = 0$ or $\bar{t}b^2/\alpha^2 \notin \wp(\mathscr{F})$. If $\gamma^2 + \gamma = \bar{t}b^2/\alpha^2$ in $\mathscr{F}$ then $h_2(M/F)$ has two elements, the non-zero class given by the 2-fold $\langle\langle -(\alpha^2\gamma^2 + tb^2)^*, -\pi\rangle\rangle$.*

Proof. The idea behind this calculation is the following: Let $F_1 = F(x_1^{1/2}, x_4^{1/2}, x_5^{1/2}, \ldots, x_n^{1/2})$. Note that $I^3(F) = 0$, and as previously remarked $h_2(M/F) = I^2(M/F)/I^2D(M/F)$. We shall find a basis for

$I^2(F) \bmod I^2D(M/F)$ and compute the image of this basis inside $I^2(F_1)$. Using the fact that biquadratic extensions are 1-amenable (cf. **[ELW1]**) we find that $I^2(M/F_1) = (\langle\langle -x_2\rangle\rangle, \langle\langle -x_3\rangle\rangle) \cdot I(F_1) \subseteq I^2(F_1)$. Comparing the images of our basis with this kernel will give the result.

First, as all the type (ii), (iii) generators of $I^2(F)$ vanish in $I^2(M)$, by Lemmas 4.2, 4.4, 4.5 we see that every element of $I^2(F)$ can be represented mod $I^2D(M/F)$ by a 2-fold of the form $\langle\langle -(x^2 + \bar{t})^*, -\pi\rangle\rangle$ where $x \in \mathscr{F}$. For as x ranges over $\mathscr{F}$, $(x^2 + \bar{t})$ ranges over all classes of $\mathscr{F}^{\cdot}/\mathscr{F}^{\cdot 2}$ except [1]. Next we note $\bar{F}_1 = \mathscr{F}(\bar{t}^{1/2}, \alpha_4, \dots, \alpha_n)$ where α_i satisfies $\alpha_i^2 + \alpha_i = \bar{g}_i$. Thus we know by Lemma 4.7 that $\bar{F}_1 = \wp(\bar{F}_1) + \bar{t}^{1/2}\bar{F}_1$. We fix $(\,)_1^*$: $\bar{F}_1 \to F_1$ by setting $(a^2 + \bar{t}^{1/2}b^2)_1^* = (a)_2^{*2} + t^{1/2}(b)_2^{*2}$ where we assume that $(\ )_2^*$: $\bar{F}_1 \to F_1$ is a lifting which extends the mapping $a + b\bar{t}^{1/2} \mapsto (a)_0^* + t^{1/2}(b)_0^*$ whenever $a, b \in \mathscr{F}$. We remark that we are now able to calculate the ideals $(\langle\langle -x_2\rangle\rangle)$, and $(\langle\langle -x_3\rangle\rangle)$ inside $W(F_1)$ using Lemma 4.5, as long as we use the lifting $(\,)_1^*$.

We calculate as $(x^2 + \bar{t})^* = (x)_0^{*2} + t$ and as

$$(x^2 + \bar{t})_1^* = \left((x + \bar{t}^{1/2})^2\right)_1^* = (x)_0^{*2} + t + 2(x)_0^* t^{1/2}$$

that $(x^2 + \bar{t})^* = (x^2 + \bar{t})_1^* - 2(x)_0^* t^{1/2}$. In particular, $[(x^2 + \bar{t})^*] = [1 - 2(x)_0^* t^{1/2}/(x^2 + \bar{t})_1^*]$ inside $F_1^{\cdot}/F_1^{\cdot 2}$, so that our generators

$$\begin{aligned}&\left\langle\left\langle -(x^2 + \bar{t})^*, -\pi\right\rangle\right\rangle\\ &\qquad \equiv \left\langle\left\langle -\left(1 + \pi\left(\alpha x\bar{t}^{1/2}/(x^2 + \bar{t})\right)_1^*\right), -\pi\right\rangle\right\rangle \bmod V_2^2(F_1),\end{aligned}$$

where $\alpha = \overline{2\pi^{-1}} \in \mathscr{F}$. From this we see that as $\alpha x\bar{t}^{1/2}/(x^2 + \bar{t}) \in \bar{t}^{1/2}\bar{F}_1^2$ our generators are in fact $\equiv \langle\langle -(1 + \pi(\alpha x\bar{t}^{1/2}/(x^2 + \bar{t}))_1^*), -t^{1/2}\rangle\rangle$ mod $V_2^2(F_1)$.

Next, noting that $\bar{F}_1 = \wp(\bar{F}_1) + \bar{t}^{1/2}\bar{F}_1^2$, we find that $V_2^2(F_1)$ is generated only by the 2-folds of the form $\langle\langle -(1 + 4(h_i)^*), -\pi\rangle\rangle$ where the h_i range over a basis of $\bar{F}_1 \bmod \wp(\bar{F}_1)$. Thus $V_2^2(F_1) \subseteq (\langle\langle -x_2\rangle\rangle)$ inside $W(F_1)$ by Lemma 4.5. Further, again according to Lemma 4.5 we find that $I^2(F_1) \cap (\langle\langle -x_2\rangle\rangle, \langle\langle -x_3\rangle\rangle)$ is generated mod $V_2^2(F_1)$ by the 2-folds of the form $\langle\langle -(1 + \pi(f_i)_1^*), t^{1/2}\rangle\rangle$ where $\wp(f_i) \in \bar{t}^{1/2}\bar{F}_1^2$ or $\wp(f_i/\bar{g}) \in \bar{t}^{1/2}\bar{F}_1^2$. Thus working mod $V_2^2(F_1)$ it follows that the 2-fold $\langle\langle -(x^2 + \bar{t})^*, -\pi\rangle\rangle$ lies in $I^2(M/F)$ if and only if the equation $\alpha x\bar{t}^{1/2}/(x^2 + \bar{t}) = f_1 + f_2$ can be solved for $f_1, f_2 \in \bar{F}_1$ where $\wp(f_1)$, $\wp(f_2/\bar{g}) \in \bar{t}^{1/2}\bar{F}_1^2$.

A straightforward calculation shows that $\wp(\bar{t}^{1/2}/(w^2 + \bar{t}^{1/2}z^2)) \in \bar{t}^{1/2}\bar{F}_1^2$ if and only if $z = 1$. Thus we must solve the equation

$$(1) \qquad \alpha x\bar{t}^{1/2}/(x^2 + \bar{t}) = \bar{t}^{1/2}/\left(w_1^2 + \bar{t}^{1/2}\right) + \bar{g}\bar{t}^{1/2}/\left(w_2^2 + \bar{t}^{1/2}\right)$$

for $w_1, w_2 \in \bar{F}_1$. As $\alpha x/(x^2 + \bar{t}) \in \mathscr{F} \subseteq \bar{F}_1^2$ we have (multiplying by $\bar{t}^{1/2}$) that

$$\bar{t}^{1/2}/\left(w_1^2 + \bar{t}^{1/2}\right) + \bar{g}\bar{t}^{1/2}/\left(w_2^2 + \bar{t}^{1/2}\right)$$
$$= \left(\bar{t} + \bar{t}^{1/2}w_1^2\right)/\left(w_1^2 + \bar{t}^{1/2}\right)^2 + \left(\bar{g}\bar{t} + \bar{g}\bar{t}^{1/2}w_2^2\right)/\left(w_2^2 + \bar{t}^{1/2}\right)^2 \in \bar{t}^{1/2}\bar{F}_1^2.$$

This implies that $\bar{t}(w_2^2 + \bar{t}^{1/2})^2 = \bar{t}\bar{g}(w_1^2 + \bar{t}^{1/2})^2$. In particular, $w_2^2 + \bar{t}^{1/2} = \bar{g}^{1/2}(w_1^2 + \bar{t}^{1/2})$ (note that as $\bar{g} \in \mathscr{F}$, $\bar{g}^{-1/2} \in \bar{F}_1^2$) so that

$$(2) \qquad \alpha x/(x^2 + \bar{t}) = \left(1 + \bar{g}^{1/2}\right)/\left(w_1^2 + \bar{t}^{1/2}\right)$$

must be solved for w_1. In case $\bar{g} \in \mathscr{F}^2$, then $\bar{g}^{1/2} \in \bar{F}_1^2$, so eq. (2) cannot be solved for w_1. Thus M is 1-amenable over F in this case. If $\bar{g} = a^2 + \bar{t}b^2$, $b \neq 0$, then $\bar{g}^{1/2} = a + b\bar{t}^{1/2}$ and $(1 + a + b\bar{t}^{1/2})/(w_1^2 + \bar{t}^{1/2}) \in \bar{F}_1^2$ implies that $1 + a + bw_1^2 = 0$, i.e. $w_1^2 = (1 + a)/b$. Substituting this into (2) gives $\alpha x/(x^2 + \bar{t}) = b$. Thus we must solve $x^2 + (\alpha/b)x + \bar{t} = 0$, which setting $(\alpha/b)y = x$ gives $\wp(y) = \bar{t}(b^2/\alpha^2)$. The conclusion stated in the Theorem now follows. □

REMARK 4.9. Setting $\pi = 2$ and $g = t/(1 + t^2)$ in Theorem 4.8 gives a version of the example where $n = 3$ studied in **[ELTW]**. Evidently, $\alpha = 1$, $b = 1/(1 + t)$, $\wp(1/(1 + \bar{t})) = \bar{t}(1/(1 + \bar{t}))^2$ so that $\langle\langle -(1 + t)^*, -2\rangle\rangle$ gives the non-trivial class of $h_2(M/F)$ in this case. It can be easily checked that this generator corresponds with the class of "q" computed in Remark 5.8 (iv) of **[ELTW]**.

REMARK 4.10. Using the notion of direct sumf of Witt rings (cf. **[M]**), and the techniques of constructing fields with these Witt rings (cf. **[M]**, **[Ku]**) one can use Theorem 4.8 to construct fields F with multiquadratic extensions M ($n \geq 3$) with $h_2(M/F)$ any prescribed finite elementary 2-group. By applying the Theorem of Merkurjev **[Me]** one can conclude that the group $N_2(M/F)$ of **[ELTW]** is this same elementary 2-group. Putting this together with Remark 3.10 of **[ELTW]** we find that the following relationships between the possible values of $h_2(M/F)$ and $N_2(M/F)$ may occur: For any $n \geq 3$, $1 < m_1 \leq m_2$ there exists fields F, $M = F(x_1^{1/2}, \ldots, x_n^{1/2})$ with $|N_2(M/F)| = 2^{m_1}$ and $|h_2(M/F)| = 2^{m_2}$.

Our final result concerns the homology group $h_3(M/F)$ (as defined in **[ELTW]** for a certain triquadratic extension M of F. Whenever M is a multiquadratic extension of F, $h_3(M/F)$ is the homology of the following zero sequence:

$$W(F) \to W(M) \xrightarrow{s} \bigoplus_{[M:L_i]=2} W(L_i)$$

where s is the sum of all the transfers s^*_{M/L_i}: $W(M) \to W(L_i)$, where the L_i range over all the subfields of codimension 2 in M which contain F. The example below is the first constructed example where $h_3(M/F)$ is non-trivial, although their existence has been suspected for some time.

For our final example we consider F where $2^{1/2} \in F$, $v(2^{1/2}) = 1$, and further where $\wp(\mathscr{F}) = \mathscr{F}$. (For example $\mathscr{F}$ could be the separable closure of $\mathbf{Z}/2\mathbf{Z}(t)$, where t is transcendental over $\mathbf{Z}/2\mathbf{Z}$.) Again $I^3(F) = 0$ and we note that $1 + 4O_F \subseteq F^{\cdot 2}$. From this we see that $I^2(F)$ has the following four types of generators:

(i) $\langle\langle -(x_i^2 + \bar{t})^*, -2^{1/2}\rangle\rangle$ where $x_i \in \mathscr{F}$.
(ii) $\langle\langle -(1 + 2^{1/2}(f_i)^*), -t\rangle\rangle$ where $f_i \in \mathscr{F}$.
(iii) $\langle\langle -(1 + 2(\bar{t}y_i^2)^*), -2^{1/2}\rangle\rangle$ where $y_i \in \mathscr{F}$.
(iv) $\langle\langle 1 + 2^{3/2}(f_i)^*, -t\rangle\rangle$ where $f_i \in \mathscr{F}$.

We begin with a version of Lemma 4.5:

LEMMA 4.11. (*Wildly ramified case.*) *Suppose that*

$$M = F\left(\left(1 + 2^{1/2} + 2^{3/2}(h)^*\right)^{1/2}\right).$$

Then $I^2(M/F)$ *is generated by*:

(ii) $\langle\langle -[1 + 2^{1/2}(f_i)^* + 2^{3/2}(f_ih)^* + 2^{1/2}((f_i)^* - (f_i)_0^*)^2], t\rangle\rangle$ *where* $\wp(f_i) \in t\mathscr{F}^2$.

(iii) $\langle\langle -(1 + 2(\bar{t}w^2)^*), -2^{1/2}\rangle\rangle + \langle\langle -(1 + 2^{3/2}(\bar{t}^2w^4)^*), -t\rangle\rangle$ *where* $w \in \mathscr{F}$.

(iv) $\langle\langle -(1 + 2^{3/2}(\bar{t}y^2)^*), -t\rangle\rangle$ *where* $y \in \mathscr{F}$.

Proof. We set $\pi = 1 + (1 + 2^{1/2} + 2^{3/2}(h)^*)^{1/2}$ in M and observe that $\pi^2 = 2^{1/2} + 2^{3/2}(h)^* + 2\pi$, so that π can be taken as a uniformizing parameter for M. It follows that $v(\pi) = v(2)/4$. We now investigate the type (ii) elements listed above. We set $f = a^2 + \bar{t}b^2$ where $\wp(f) \in \bar{t}\mathscr{F}^2$ (so $a = f$). Setting $r = 1 + 2^{1/2}(f)^* + 2^{3/2}(fh)^* + 2^{1/2}((f)^* - (f)_0^*)^2$, we compute in M/M^2 that

$$[r] = \left[1 + (f)^*\left(2^{1/2} + 2^{3/2}(h)^*\right) + 2^{1/2}\left((f)^* - (f)_0^*\right)^2\right]$$

$$\left(\text{as } (fh)^* \equiv (f)^*(h)^* \bmod 2\right)$$

$$= \left[1 + (f)^*(\pi^2 - 2\pi) + 2^{1/2}\left((f)^* - (f)_0^*\right)^2\right]$$

$$\left(\text{as } \pi^2 = 2^{1/2} + 2^{3/2}(h)^* + 2\pi\right)$$

$$= \left[\left(1 - \pi(f)^*\right)^2 + \pi^2\left((f)^* - (f)^{*2}\right) + 2^{1/2}\left((f)^* - (f)_0^*\right)^2\right]$$

$$= \left[\left(1 - \pi(f)^*\right)^2 + \pi^2\left(\left(f_0^{*2} + t(b)_0^{*2} - (f)^{*2}\right)\right.\right.$$

$$\left.\left. + 2^{1/2}\left((f)^* - (f)_0^*\right)^2\right)\right]$$

$$\left(\text{as } f = f^2 + \bar{t}b^2\right)$$

$$= \left[\left(1 - \pi(f)^*\right)^2 + t\left(\pi(f)_0^*\right)^2\right.$$

$$\left. + \left((f)_0^* - (f)^*\right)\left((\pi^2 - 2^{1/2})\left((f)^* - (f)_0^*\right) + 2\pi^2(f)_0^*\right)\right]$$

$$= \left[\left(1 - \pi(f)^*\right)^2 + t\left(\pi(f)_0^*\right)^2\right]$$

as $v((f)_0^* - (f)^*) \geq v(2^{1/2})$ and as $v(\pi^2 - 2^{1/2}) \geq 5v(2)/4$ in M. From this we see that $\langle\langle -r, t\rangle\rangle = 0$ in $W(M)$.

Next we consider the type (iii) sums of 2-folds. Since $[2^{1/2}] = [1 - 2\pi^{-1} - 2^{3/2}\pi^{-2}(h)^*]$ in $M^{\cdot}/M^{\cdot 2}$ we find, for $w \in \mathscr{F}$,

$$\left\langle\left\langle -\left(1 + 2(\bar{t}w^2)^*\right), -2^{1/2}\right\rangle\right\rangle$$

$$= \left\langle\left\langle -\left(1 + 2(\bar{t}w^2)^*\right), -\left(1 - 2\pi^{-1} - 2^{3/2}\pi^{-2}(h)^*\right)\right\rangle\right\rangle$$

$$= \left\langle\left\langle -\left(1 + 4\pi^{-1}(\bar{t}w^2)^*\right), -\left(2\pi^{-1} + 2^{3/2}\pi^{-2}(h)^*\right)\right\rangle\right\rangle$$

$$\times \left\langle\left\langle \left(1 + 2(\bar{t}w^2)^*\right)\left(1 - 2\pi^{-1} - 2^{3/2}\pi^{-2}(h)^*\right)\right\rangle\right\rangle$$

$$= \left\langle\left\langle -\left(1 + 4\pi^{-1}(\bar{t}w^2)^*\right), -\pi\right\rangle\right\rangle = \left\langle\left\langle -\left(1 + 4\pi^{-1}(\bar{t}w^2)^*\right), -t\right\rangle\right\rangle$$

inside $W(M)$. Also $\langle\langle -(1+2^{3/2}(\bar{t}^2w^4)^*), -t\rangle\rangle \cong \langle\langle -(1+4\pi^{-1}(\bar{t}w^2)^*), -t\rangle\rangle$ over M since

$$\left(1+2^{3/2}(\bar{t}w^2)_0^{*2}\right)/\left(1+2^{1/2}\pi(\bar{t}w^2)_0^*\right)^2 \equiv 1+2^{3/2}\pi(\bar{t}w^2)_0^*$$

$$\equiv 1-4\pi^{-1}(tw^2)^* \bmod(4)$$

as $\overline{2^{1/2}\pi^{-2}} = 1 \in \mathscr{F}$. Thus the type (iii) elements listed vanish.

Finally,

$$\left\langle\left\langle -\left(1+2^{3/2}(\bar{t}y^2)^*\right), -t\right\rangle\right\rangle = \left\langle\left\langle -\left(1+2^{3/2}(\bar{t}y^2)^*\right), 2^{1/2}\right\rangle\right\rangle$$

$$= \left\langle\left\langle -\left(1+2^{3/2}(\bar{t}y^2)^*\right), \left(1-2\pi^{-1}-2^{3/2}(h)^*\right)\right\rangle\right\rangle = 0 \in W(M)$$

as $v(2^{3/2}2\pi^{-1}) > v(4)$ in M, so the type (iv) elements listed vanish.

To see that the above elements actually generate $I^2(M/F)$ we now show that the remaining generators for $I^2(F)$ remain independent inside $I^2(M)$. These remaining generators may be listed as:

(i) $\langle\langle -(x)^*, -2^{1/2}\rangle\rangle$ where the x's range over a base of $\mathscr{F}^{\bullet}$ mod$(\mathscr{F}^{\bullet 2})$.

(ii) $\langle\langle -(1+2^{1/2}(f_i)^*), -t\rangle\rangle$ where the f_i range over a basis of $\mathscr{F}$ mod$\{f\colon \wp(f) \in \bar{t}\mathscr{F}^2\}$.

(iv) $\langle\langle -(1+2^{3/2}(w_i^2)^*), -t\rangle\rangle$ where the w_i form a basis of $\mathscr{F}$.

For the type (i) generators we have for $x = a^2+\bar{t}b^2 \in \mathscr{F}^{\bullet}$ that mod $I^{2,v(2)}(M)$:

$$\left\langle\left\langle -(x)^*, -2^{1/2}\right\rangle\right\rangle = \left\langle\left\langle -(x)^*, -\left(1-2\pi^{-1}\left(1+2^{1/2}\pi^{-1}(h)^*\right)\right)\right\rangle\right\rangle$$

$$= \left\langle\left\langle -(x)^*, -(1-\pi^3u)\right\rangle\right\rangle$$

$$\left(\text{where } u = 2\pi^{-4}\left(1+2^{1/2}\pi^{-1}(h)^*\right) \text{ is a 1-unit}\right)$$

$$= \left\langle\left\langle -(1-\pi^3u), -\left((a)_0^{*2}+t(b)_0^{*2}\right)\right\rangle\right\rangle$$

$$\equiv \left\langle\left\langle -\left(1-\pi^3u\left((a)_0^{*2}/(x)^*\right)\right), -(a)_0^{*2}\right\rangle\right\rangle$$

$$+\left\langle\left\langle -\left(1-\pi^3u\left((b)_0^{*2}t/(x)^*\right)\right), -(b)_0^{*2}t\right\rangle\right\rangle \quad \text{(by 3.3(v))}$$

$$\equiv 0+\left\langle\left\langle -\left(1-\pi^3ut\left((b)_0^{*2}/\left((a)_0^{*2}+(b)_0^{*2}t\right)\right)\right), -t\right\rangle\right\rangle.$$

Since the x's were chosen independent $\operatorname{mod}(\overline{\mathscr{F}^{\cdot 2}})$, Lemma 4.1 (ii) shows that the residues $ut\big((b)_0^{*2}/\big((a)_0^{*2}+(b)_0^{*2}t\big)\big)$ are linearly independent inside $\mathscr{F}^{+}$. This, together with Lemma 4.1 (i) and Corollary 2.9 shows that the 2-folds $\langle\langle -(x)^*, -2^{1/2}\rangle\rangle$ are all independent mod $I^{2,4v(\pi)}(M)$.

Next, for the type (ii) generators above, we express $f_i = x_i^2 + \bar{t}y_i^2$. Since $\pi^2 \equiv 2^{1/2} + 2\pi \operatorname{mod}(2^{3/2})$ we obtain that:

$$\begin{aligned} 1 + 2^{1/2}(f_i)^* &\equiv 1 + (\pi^2 - 2\pi)(f_i)^* \\ &\equiv \big(1 - \pi^2(f_i)^*\big)\big(1 + 2\pi(f_i)^*\big) \quad \operatorname{mod}(2^{3/2}). \end{aligned}$$

By direct calculation:

$$\begin{aligned} \big(1 - \pi^2(f_i)^*\big) &= \big(1 - \pi^2\big((x_i)_0^{*2} + t(y_i)_0^{*2}\big)\big) \\ &\equiv \big(1 - t\pi^2(y_i)_0^{*2}\big)\big(1 - t\pi^4(x_i)_0^{*2}(y_i)_0^{*2}\big)\big(1 - \pi^2(x_i)_0^{*2}\big) \quad \operatorname{mod}(2^{3/2}) \end{aligned}$$

Multiplying by the square $(1 - \pi(x_i)_0^*)^2$ we find that:

$$\big(1 - \pi^2(x_i)_0^{*2}\big) \equiv \big(1 - 2\pi(x_i)_0^*\big) \operatorname{mod} M^{\cdot 2}\big(1 + 2^{3/2}O_M\big).$$

Putting all this information together, and using the fact that $\langle\langle -(1 - tz^2), -t\rangle\rangle = 0$, we finally obtain that:

$$\begin{aligned} &\Big\langle\Big\langle -\big(1 + 2^{1/2}(f_i)^*\big), -t\Big\rangle\Big\rangle \\ &\qquad \equiv \Big\langle\Big\langle -\big(1 + 2\pi(f_i + x_i)^*\big), -t\Big\rangle\Big\rangle \quad \operatorname{mod} I^{2,3v(2)/2}(M). \end{aligned}$$

However the mapping $f \mapsto f + x$ whenever $f = x^2 + \bar{t}y^2$ in $\mathscr{F}$ is an additive homomorphism with kernel $\{f\colon \wp(f) \in \bar{t}\mathscr{F}^2\}$, and thus since the f_i's were chosen independent mod $\{f\colon \wp(f) \in \bar{t}\mathscr{F}^2\}$ these 2-folds remain independent mod $I^{2,3v(2)/2}(M)$ (using 4.1 (i)).

For the type (iv) generators, set $u = 2^{3/2}/\pi^6$ and note:

$$\begin{aligned} (1 - u) &= \big(\pi^6 - \big(\pi^2 - 2^{3/2}(h)^* - 2\pi\big)^3\big)/\pi^6 \\ &= \big[\pi^6 - \big(\pi^6 - 6\pi^5 + \text{higher order terms}\big)\big]/\pi^6 \\ &= \pi^3\big(6\pi^{-4} + \text{higher order terms}\big) \\ &= \pi^3 w \quad \text{for some unit } w \in U_M. \end{aligned}$$

Consider $f = a^2 + \bar{t}b^2$ in $\mathscr{F}$. Then in $I^2(M)$:

$$\langle\langle -(1 + 2^{3/2}(f)^*), -t\rangle\rangle = \langle\langle -(1 + \pi^6 u((a)_0^{*2} + t(b)_0^{*2})), -t\rangle\rangle$$

$$= \langle\langle -(1 + \pi^6 u(a)_0^{*2}), -t\rangle\rangle + \langle\langle -(1 + \pi^6 ut(b)_0^{*2}), -t\rangle\rangle$$

$$\left(\text{as } I^{2,3v(2)}(M) = I^3(M) = 0\right)$$

$$= \left\langle\left\langle \frac{-(1 + \pi^6 u(a)_0^{*2})}{(1 - \pi^3(a)_0^*)^2}, -t\right\rangle\right\rangle + \langle\langle -(1 + \pi^6 ut(b)_0^{*2}), u\rangle\rangle$$

$$= \left\langle\left\langle -\left(1 + \left((\pi^6(u-1)(a)_0^* + 2\pi^3(a)_0^*)/(1 - \pi^3(a)_0^*)^2\right)\right), -t\right\rangle\right\rangle$$

$$+ \langle\langle -(1 + \pi^6 ut(b)_0^{*2}), 1 - \pi^3 w\rangle\rangle \quad \text{(using 3.3 (ii))}$$

$$= \left\langle\left\langle -\left(1 + (2\pi^3(a)_0^* - \pi^9 w(a)_0^{*2})/(1 - \pi^3(a_0^*)^2)\right), -t\right\rangle\right\rangle$$

$$+ \langle\langle (1 + \pi^6 ut(b)_0^{*2}), -1(1-2)(1 - \pi^3 w)\rangle\rangle$$

$$= \langle\langle -(1 + 2\pi^3(a)_0^*), -t\rangle\rangle + 0$$

as $I^{2,9v(2)/4}(M) = 0$ and using Fact 3.3 (iii) for the second term. Since the map $f \mapsto a$ where $f = a^2 + \bar{t}b^2$ is an additive homomorphism with kernel $\bar{t}\mathscr{F}^2$, we see using Lemma 4.1 (i) together with Corollary 2.9 that the images of the type (iv) generators are independent mod $I^{2,7v(2)/4}(M)$. These independence statements conclude the proof of the Lemma. □

We also need a similar result in the "wildly unramified" situation:

LEMMA 4.12. (*Wildly Unramified Case.*) *Suppose that*

$$M = F\left((1 + 2t)^{1/2}\right).$$

Then $I^2(M/F)$ is generated by the 2-folds:

(iii) $\langle\langle -(1 + 2t), -2^{1/2}\rangle\rangle$

(iv) $\langle\langle -(1 + 2^{3/2}(f)^*), -t\rangle\rangle$ *for all* $f \in \mathscr{F}$.

Proof. We set $\pi = 1 + (1 + 2t)^{1/2}$ and note that $\pi^2 = 2t + 2\pi$. Thus $v(\pi) = v(2^{1/2})$ and $\overline{\pi/2^{1/2}} = \bar{t}^{1/2} \in \overline{M} = \mathscr{F}(t^{1/2})$. Clearly the Type (iii) 2-fold listed above vanishes in $W(M)$. Since $[t] = [1 - 2\pi^{-1}]$ in $M^{\cdot}/M^{\cdot 2}$ we find: $\langle\langle -(1 + 2^{3/2}(f)^*), -t\rangle\rangle = \langle\langle -(1 + 2^{3/2}(f)^*), -(1 - 2\pi^{-1})\rangle\rangle = 0 \in W(M)$ by 3.3 (iii) since $v(2^{3/2}2\pi^{-1}) = v(4)$ and since $\wp(\overline{M}) = \overline{M}$.

Next we note that $I^2(F)$ is generated over the above 2-folds by:

(i) $\langle\langle -(x^*), -2^{1/2}\rangle\rangle$ where $x \in \mathscr{F}^{\boldsymbol{\cdot}}$ range over a basis of $\mathscr{F}^{\boldsymbol{\cdot}}/\mathscr{F}^{\boldsymbol{\cdot}2}$.

(ii) $\langle\langle -(1 + 2^{1/2}(f)^*), -t\rangle\rangle$ where $f \in \mathscr{F}^+$ range over a basis of $\mathscr{F}^+$.

(iii) $\langle\langle -(1 + 2(\bar{t}y_i^2)^*), -2^{1/2}\rangle\rangle$ where y_i range over a basis of $\mathscr{F}^+$ mod $\{0,1\}$.

We compute the images of these 2-folds inside $I^2(M)$. In case (i) consider $\langle\langle -(x)^*, -2^{1/2}\rangle\rangle$ for any $x \in \mathscr{F}^{\boldsymbol{\cdot}}$. Write $x = a^2 + \bar{t}b^2$, with a or $b \neq 0$. Then $(x)^* = \alpha^2 + t\beta^2$, where $\alpha = (a)_0^*$ and $\beta = (b)_0^*$. Computing in $M^{\boldsymbol{\cdot}}/M^{\boldsymbol{\cdot}2}$ we find that:

$$\begin{aligned}[(x)^*] &= \left[(\alpha^2 + t\beta^2)/(\alpha + \beta\pi/2^{1/2})^2\right]\\ &= \left[(\alpha^2 + t\beta^2)/(\alpha^2 + 2^{1/2}\pi\beta + (t+\pi)\beta^2)\right]\\ &= \left[1 - (\pi\beta^2 + 2^{1/2}\pi\beta)/(\alpha^2 + t\beta^2 + \pi\beta^2 + 2^{1/2}\pi\beta)\right].\end{aligned}$$

Since

$$\begin{aligned}&\left(1 - (\pi\beta + 2^{1/2}\pi\beta)/(\alpha^2 + t\beta^2 + \pi\beta^2 + 2^{1/2}\pi\beta)\right)\\ &\qquad\equiv \left(1 - \pi\left(\beta^2/(\alpha^2 + t\beta^2)\right)\right) \operatorname{mod}(\pi^2),\end{aligned}$$

we find:

$$\begin{aligned}&\left\langle\left\langle -(x)^*, -2^{1/2}\right\rangle\right\rangle\\ &\qquad\equiv \left\langle\left\langle -\left(1 - \pi\left(\beta^2/(\alpha^2 + t\beta^2)\right)\right), -2^{1/2}\right\rangle\right\rangle \bmod I^{2,2v(\pi)}(M).\end{aligned}$$

From this it is easy to see that the mapping $\theta\colon \mathscr{F} \to I^2(M)/I^{2,2v(\pi)}(M)$ given by $x \mapsto \langle\langle -(x)^*, -2^{1/2}\rangle\rangle \bmod I^{2,2v(\pi)}(M)$ is the composition of the homomorphism $\mathscr{F}^{\boldsymbol{\cdot}} \to \mathscr{F}^+$ given by $a^2 + \bar{t}b^2 \mapsto b^2/(a^2 + \bar{t}b^2)$ (Lemma 4.1(ii)) with the homomorphism $\mathscr{F}^+ \to I^2(M)/I^{2,2v(\pi)}(M)$ given by $y \mapsto \langle\langle -(1 - \pi(y)^*), -2^{1/2}\rangle\rangle$ (Lemma 4.1(i)). (Recall that the second map is independent of the choice of the lift $(y)^*$ of y.) Since the first map in this composition has kernel $\mathscr{F}^{\boldsymbol{\cdot}2}$ and the second is injective by Corollary 2.9, θ is a homomorphism with kernel $\mathscr{F}^{\boldsymbol{\cdot}2}$. Hence, the type (i) elements listed above as generators remain linearly independent in $I^2(M)/I^{2,2v(\pi)}(M)$.

For the type (ii) generators observe that as $[t] = [1 - 2\pi^{-1}] = [1 - \pi(2^{1/2}\pi^{-1})^2]$ in $M^{\boldsymbol{\cdot}}/M^{\boldsymbol{\cdot}2}$ we find that

$$\begin{aligned}\left\langle\left\langle -\left(1 - 2^{1/2}(f)^*\right), -t\right\rangle\right\rangle &= \left\langle\left\langle -\left(1 - 2^{1/2}(f)^*\right), -\left(1 - \pi(2^{1/2}\pi^{-1})^2\right)\right\rangle\right\rangle\\ &= \left\langle\left\langle -\left(1 - 2^{1/2}(f)^*\pi(2^{1/2}\pi^{-1})^2\right), -\pi u\right\rangle\right\rangle\end{aligned}$$

by Fact 3.3 (iii) where $u = (1 - 2^{1/2}(f)^*)(1 - \pi(2^{1/2}\pi^{-1})^2) \in U^1(M)$. As $\langle\langle -(1 - 2^{1/2}(f)^*\pi(2^{1/2}\pi^{-1})^2), -u\rangle\rangle \in I^{2,3v(\pi)}(M)$, this shows that

$$\left\langle\left\langle -\left(1 - 2^{1/2}(f)^*\right), -t\right\rangle\right\rangle$$
$$\equiv \left\langle\left\langle -\left(1 - 2^{1/2}(f)^*\pi(2^{1/2}\pi^{-1})^2\right), -\pi\right\rangle\right\rangle \bmod I^{2,3v(\pi)}(M).$$

Note that $(2^{1/2}\pi)/2$ has residue $\bar{t}^{1/2}$ inside $\overline{M}$, and that each $f \in \overline{M}^{\cdot 2}$. Thus by Corollary 2.10 the type (ii) generators remain linearly independent inside $I^{2,2v(\pi)}(M)/I^{2,3v(\pi)}(M)$.

Finally for the type (iii) generators we note that

$$\left(1 - 2t(y)_0^{*2}\right)/\left(1 - 2^{1/2}(y)_0^*\pi 2^{-1/2}\right)^2 \equiv 1 - 2\pi\left(y^2 + y\right)^* \bmod(4)$$

in M as $(\pi 2^{-1/2})^2 = t + \pi$. Thus

$$\left\langle\left\langle -\left(1 + 2(ty^2)^*\right), -2^{1/2}\right\rangle\right\rangle$$
$$\equiv \left\langle\left\langle -\left(1 - 2\pi\left(y^2 + y\right)^*\right), -2^{1/2}\right\rangle\right\rangle \bmod I^{2,v(4)}(M).$$

Further,

$$\left\langle\left\langle -\left(1 - 2\pi\left(y^2 + y\right)^*\right), -2^{1/2}\right\rangle\right\rangle$$
$$= \left\langle\left\langle -\left(1 - 2\pi\left(y^2 + y\right)^*\right), -2^{3/2}\pi\left(y^2 + y\right)^*\right\rangle\right\rangle$$

by Fact 3.3 (ii). Since $y \in \mathscr{F}$, and $\mathscr{F} \subseteq \overline{M}^{\cdot 2}$, $(y^2 + y)^* \in U^1(M)M^{\cdot 2}$. Thus $[2^{3/2}\pi(y^2 + y)^*] = [\pi 2^{-1/2}][u] \in M^{\cdot}/M^{\cdot 2}$ for some $u \in U^1(M)$. From this, Fact 3.3 (ii), and the multilinearity of 2-folds, we obtain

$$\left\langle\left\langle -\left(1 + 2(ty^2)^*\right), -2^{1/2}\right\rangle\right\rangle$$
$$\equiv \left\langle\left\langle -\left(1 - 2\pi\left(y^2 + y\right)^*\right), -\pi 2^{-1/2}\right\rangle\right\rangle \bmod I^{2,4v(\pi)}(M).$$

Since $\overline{\pi 2^{-1/2}} = \bar{t}^{1/2} \in M$, and as the y's were chosen to be independent mod $\{0, 1\}$, (so the $y^2 + y$'s are linearly independent in $\mathscr{F}^+$), the independence of these type (iii) 2-folds follows from Corollary 2.9. Thus the 2-folds of types (i), (ii), (iii) are all independent in $I^2(M)$. This proves the Lemma. □

Before proving our final result we need a technical result that enables us to compute $h_3(M/F)$ for a triquadratic extension. This result and its proof given below were communicated to the author by Adrian Wadsworth. In what follows $M = F(a^{1/2}, b^{1/2}, c^{1/2})$ where we assume that $[M: F] = 8$. We define $\alpha = \langle\langle -a\rangle\rangle W(F)$, $\beta = \langle\langle -b\rangle\rangle W(F)$, $\gamma = \langle\langle -c\rangle\rangle W(F)$, $\delta = \langle\langle -ac\rangle\rangle W(F)$ and $\varepsilon = \langle\langle -ab\rangle\rangle W(F)$.

PROPOSITION 4.13. *$h_3(M/F) \cong N/D$ where*
$$N = (\beta + \gamma) \cap (\beta + \delta) \cap (\gamma + \varepsilon) \cap (\delta + \varepsilon) \cap \operatorname{ann}(\alpha)$$
and where
$$D = (\operatorname{ann}(\alpha)) \cdot (\beta + \gamma) = (\beta \cap \varepsilon) + (\gamma \cap \delta).$$

Proof. Let $K = F(a^{1/2})$, $L_1 = F(b^{1/2}, c^{1/2})$, $L_2 = F(b^{1/2}, (ac)^{1/2})$, $L_3 = F(c^{1/2}, (ab)^{1/2})$ and $L_4 = F((ab)^{1/2}, (ac)^{1/2})$. The L_i are the four fields such that $F \subseteq L_i \subseteq M$, $[M: L_i] = 2$, but $K \not\subseteq L_i$. We set $N_1 = \ker(s)$ (s the map described above) and we set $D_1 = \operatorname{im}(r_{M/F})$ where for any two fields $F \subseteq L$ the map $r_{F/L}$ is the map $W(F) \to W(L)$ induced by field inclusion. Then $D_1 \subseteq N_1$ and by definition $h_3(M/F) = N_1/D_1$. According to [**ELW**, Th. 2.10] one has $h_3(M/K) = 0$. Therefore, $N_1 = \operatorname{im}(r_{M/K}) \cap \bigcap_{i=1}^4 \ker(s^*_{M/L_i})$, and $\ker(r_{M/K}) = \beta W(K) + \gamma W(K)$. Let $N_2 = r^{-1}_{M/K}(N_1)$ and $D_2 = r^{-1}_{M/K}(D_1)$ inside $W(K)$. Evidently, $D_2 = \operatorname{im}(r_{K/F}) + \ker(r_{M/K}) = \ker(s^*_{K/F}) + \beta W(K) + \gamma W(K)$. Let $N_3 = s^*_{K/F}(N_2)$ and $D_3 = s^*_{K/F}(D_2)$. Then because $N_1 \subseteq \operatorname{im}(r_{M/K})$ and $\ker(s^*_{K/F}) \subseteq D_2$ we have that $h_3(M/F) \cong N_1/D_1 \cong N_2/D_2 \cong N_3/D_3$. To complete the proof we show that $N = N_3$ and $D = D_3$.

Since $a^{1/2} \notin L_i$ we may assume that s^*_{M/L_i} is derived from the L_i-linear map given by $s_{M/L_i}(1) = 0$ and $s_{M/L_i}(a^{1/2}) = 1$. Likewise we may assume that $s^*_{K/F}$ arises from $s_{K/L}(1) = 0$ and $s_{K/L}(a^{1/2}) = 1$. Then $s^*_{M/L_i} \circ r_{M/K} = r_{L_i/F} \circ s^*_{K/F}$. Hence for $q \in W(K)$ we have that $q \in N_2$ if and only if $0 = s^*_{M/L_i}(r_{M/K}(q)) = r_{L_i/F}(s^*_{K/F}(q))$ for $i = 1,2,3,4$ if and only if $s^*_{K/F}(q) \in \bigcap_{i=1}^4 \ker(r_{L_i/F})$. Since $L_1 = F(b^{1/2}, c^{1/2})$ is a biquadratic extension of F, $\ker(r_{L_1/F}) = \langle\langle -b \rangle\rangle W(F) + \langle\langle -c \rangle\rangle W(F) = \beta + \gamma$. Analogous formulas hold for $\ker(r_{L_i/F})$ where $i = 2,3,4$. Recall that $\operatorname{im}(s^*_{K/F}) = \operatorname{ann}_{W(F)}(\alpha)$. Therefore,
$$\begin{aligned} N_3 &= s^*_{K/F}(N_2) = \operatorname{im}\left(s^*_{K/F}\right) \cap \bigcap_{i=1}^{4} \ker\left(r_{L_i/F}\right) \\ &= \operatorname{ann}_{W(F)}(\alpha) \cap (\beta + \gamma) \cap (\beta + \delta) \cap (\gamma + \varepsilon) \cap (\delta + \varepsilon) = N. \end{aligned}$$
For D_3 we have, using the Frobenius reciprocity of the transfer,
$$\begin{aligned} D_3 &= s^*_{K/F}(D_2) = s^*_{K/F}\left(\ker\left(s^*_{K/F}\right) + \beta W(K) + \gamma W(K)\right) \\ &= s^*_{K/F}(\beta W(K)) + s^*_{K/F}(\gamma W(K)) = \beta \operatorname{im}\left(s^*_{K/F}\right) + \gamma \operatorname{im}\left(s^*_{K/F}\right) \\ &= \beta \operatorname{ann}_{W(F)}(\alpha) + \gamma \operatorname{ann}_{W(F)}(\alpha) = (\beta + \gamma) \operatorname{ann}_{W(F)}(\alpha) = D. \end{aligned}$$
The second formula for D follows because $\beta \operatorname{ann}_{W(F)}(\alpha) = \beta \cap \varepsilon$ and $\gamma \operatorname{ann}_{W(F)}(\alpha) = \gamma \cap \delta$ by [**ELW**, Lemma 2.9]. This proves the proposition. □

THEOREM 4.14. *For F satisfying the standing hypotheses of this section with $2^{1/2} \in F$, $v(2^{1/2}) = 1$, $\wp(\mathscr{F}) = \mathscr{F}$ and*

$$M = F\left((1 + 2^{3/2})^{1/2}, (1 + 2^{1/2})^{1/2}, (1 + 2t)^{1/2}\right)$$

we have that $h_3(M/F)$ is infinite.

Proof. We set $a = 1 + 2^{3/2}$, $b = 1 + 2^{1/2}$ and $c = 1 + 2t$ and we use the notation of Proposition 4.13. We now directly compute N and D using Lemmas 4.11 and 4.12.

(1) $\beta \cap I^2(F)$ has as generators:

(ii) $\langle\langle -(1 + 2^{1/2}(f)^* + 2^{1/2}((f)^* - (f)_0^*)^2), t\rangle\rangle$, where $\wp(f) \in \bar{t}\mathscr{F}^2$,

(iii) $\langle\langle -(1 + 2(\bar{t}w^2)^*), -2^{1/2}\rangle\rangle + \langle\langle -(1 + 2^{3/2}(\bar{t}^2w^4)^*), -t\rangle\rangle$, where $w \in \mathscr{F}$,

(iv) $\langle\langle -(1 + 2^{3/2}(\bar{t}w^2)^*), -t\rangle\rangle$ for $w \in \mathscr{F}$.

(2) $\gamma \cap I^2(F)$ has as generators

(iii) $\langle\langle -(1 + 2t), -2^{1/2}\rangle\rangle$,

(iv) $\langle\langle -(1 + 2^{3/2}(f)^*), -t\rangle\rangle$, where $f \in \mathscr{F}$.

(3) $\delta \cap I^2(F)$ has as generators

(iii) $\langle\langle -(1 + 2t)(1 + 2^{3/2}), -2^{1/2}\rangle\rangle$,

(iv) $\langle\langle -(1 + 2^{3/2}(f)^*), -t\rangle\rangle$, where $f \in \mathscr{F}$.

(4) $\varepsilon \cap I^2(F)$ has as generators

(ii) $\langle\langle -(1 + 2^{1/2}(f)^* + 2^{3/2}(f)^* + 2^{1/2}((f)^* - (f)_0^*)^2), t\rangle\rangle$ where $\wp(f) \in \bar{t}\mathscr{F}^2$.

(iii) $\langle\langle -(1 + 2(\bar{t}w^2)^*), -2^{1/2}\rangle\rangle + \langle\langle -(1 + 2^{3/2}(\bar{t}^2w^4)^*), -t\rangle\rangle$ where $w \in \mathscr{F}$.

(iv) $\langle\langle -(1 + 2^{3/2}(\bar{t}w^2)^*), -t\rangle\rangle$ where $w \in \mathscr{F}$.

From this we shall show that $\beta \cap \varepsilon$ is generated by the forms:

(iii) $\langle\langle -(1 + 2(\bar{t}w^2)^*), -2^{1/2}\rangle\rangle + \langle\langle -(1 + 2^{3/2}(\bar{t}^2w^4)^*), -t\rangle\rangle$ where $w \in \mathscr{F}$.

(iv) $\langle\langle -(1 + 2^{3/2}(\bar{t}w^2)^*), -t\rangle\rangle$ where $w \in \mathscr{F}$.

Let J be the ideal generated by these forms. Then from Lists (1) and (4), $J = \beta \cap I^{2,v(2)}(F) = \varepsilon \cap I^{2,v(2)}(F) \subseteq \beta \cap \varepsilon$. We show that $(\beta \cap \varepsilon)/J = 0$. Note that $\beta \cap \varepsilon \subseteq I^2(F)$.

The map $f \mapsto \langle\langle -(1 + 2^{1/2}(f)^* + 2^{1/2}((f)^* - (f)_0^*)^2), t\rangle\rangle$ induces a group homomorphism $\mathscr{F} \to I^2(F)/I^{2,v(2)}(F)$ by 4.1, so it induces a surjective homomorphism θ: $\{f \in \mathscr{F}\colon \wp(f) \in \bar{t}\mathscr{F}^2\} \to (\beta \cap I^2(F))/J$.

Take any $f \in \theta^{-1}((\beta \cap \varepsilon)/J)$. Then ε contains

$$\left\langle\left\langle -\left(1 + 2^{1/2}(f)^* + 2^{1/2}\left((f)^* - (f)_0^*\right)^2\right), t\right\rangle\right\rangle$$

$$= \left\langle\left\langle -\left(1 + 2^{1/2}(f)^* + 2^{1/2}\left((f)^* - (f)_0^*\right)^2 + 2^{3/2}(f)^*\right), t\right\rangle\right\rangle$$

$$+ \left\langle\left\langle -\left(1 + 2^{3/2}(f)^*\right), t\right\rangle\right\rangle$$

(using 3.3 (iv) and $I^{2,v(4)}(F) = 0$). Since the first summand lies in ε, so must the second. Hence $f \in \bar{t}\mathscr{F}^2$ from the description of $\varepsilon \cap I^2(F)$. But also $\wp(f) \in \bar{t}\mathscr{F}^2$, as f lies in the domain of θ. Together these imply that $f = 0$, i.e. $\beta \cap \varepsilon = J$.

From this we obtain that $D = (\beta \cap \varepsilon) + (\gamma \cap \delta) = I^{2,v(2)}(F)$. Next, since $I^3(F) = 0$, $I^2(F) \subseteq \operatorname{ann}(\alpha)$. Also by inspection it is easy to see that $\langle\langle -(1 + 2^{1/2}(f)^*), t\rangle\rangle$ lies in $(\beta + \gamma), (\beta + \delta), (\gamma + \varepsilon), (\delta + \varepsilon)$ whenever $\wp(f) \in \bar{t}\mathscr{F}^2$. This shows that these latter 2-folds are all non-zero in N/D and concludes the proof of Theorem 4.15. □

Remark 4.15. The preceding counterexample was obtained by replacing the unramified quadratic extension $F(t^{1/2})$ of the counterexample of **[ELTW]** by the wildly unramified quadratic extension $F((1 + 2t)^{1/2})$ (where $2^{1/2} \in F$). We also remark that using the calculations of Theorem 4.14 one may also show that $h_3(F) \neq 0$ for $F = \mathbf{Q}(2^{1/2})(t)$ and the triquadratic extension M/F as described above. Thus strong 1-amenability fails for function fields over global fields.

Remark 4.16. It follows from the Theorem of Merkurjev that since $I^3(F) = 0$ that $I^2(F) \cong H^2(F,2)$, and similarly for all 2-extensions of F as well. Thus in this situation we find that $h_3(M/F)$ is isomorphic to the homology of:

$$H^2(F,2) \overset{\text{res}}{\to} H^2(M,2) \overset{\text{cor}}{\to} \bigoplus_{[M:L]=2} H^2(L,2).$$

Thus in the terminology of **[STW]** we have that $N_3(M/F)$ is infinite as well.

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