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STRONG NEGLIGIBILITY OF σ-COMPACTA DOES NOT CHARACTERIZE HILBERT SPACE

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It is proved that the complement X of a σZ-set in a Q-manifold is an l^2-manifold if every finite dimensional compactum is strongly negligible in X. Moreover, we show that this statement is false in the general setting: there exists a complete AR in which every σ-compactum is strongly negligible but which does not satisfy the discrete 2-cells property.

1. Introduction. One of the most interesting topological properties of Hilbert space l^2 is the strong negligibility of σZ-sets (and hence of σ-compacta), see Anderson [**2**]. (The terminology can be found in §2.) In this paper we investigate whether it is possible to characterize l^2 with the help of the concept of strong negligibility. In [**11, 12**] a sequence X_0, X_1, $X_2, \ldots$ of fake topological Hilbert spaces was constructed, which satisfied, among many other properties, the following: (a) each X_k is the complement of a σZ-set in the Hilbert cube Q, (b) every compactum in X_k is a Z-set and (c) a σ-compactum in X_k is strongly negligible iff its dimension is at most k. The fact that $(X_k)_{k=0}^{\infty}$ forms an inverse sequence whose limit is l^2 suggests that one cannot use the method in [**11**] for the construction of fake Hilbert spaces with stronger negligibility properties. Indeed, we show in this paper that a complement of a σZ-set in a Q-manifold with the property that every finite-dimensional compactum is strongly negligible, must be an l^2-manifold.

In the general setting, however, the situation is more complicated. We construct a complete absolute retract in which every σ-compactum is strongly negligible but which is not homeomorphic to Hilbert space. We also obtain a positive result in this context: if X is a complete ANR in which every Z-set is strongly negligible and moreover every compactum is a strong Z-set, then X is an l^2-manifold. Toruńczyk's [**17**] celebrated characterization of l^2-manifolds reduces our problem to establishing connections between strong negligibility and discrete approximation properties. This is the approach for this article.

2. Preliminaries. In this section we introduce and discuss the key concepts: negligibility, the discrete approximation property and (strong) Z-sets. All topological spaces are assumed to be separable and metrizable.

If X is a space then $\mathscr{H}(X)$ denotes the group of autohomeomorphisms of X. The identity mapping on X is denoted by 1_X or simply by 1. We say that $h \in \mathscr{H}(X)$ is *supported on* $V \subset X$ if h restricts to the identity on $X \setminus V$. Let $\mathscr{U}$ be a collection of subsets of X. Mappings $f, g\colon Y \to X$ are called *$\mathscr{U}$-close* if for each $y \in Y$ with $f(y) \neq g(y)$ there is a $U \in \mathscr{U}$ containing both $f(y)$ and $g(y)$. Note that if $h \in \mathscr{H}(X)$ is $\mathscr{U}$-close to 1 then h is supported on $\bigcup\mathscr{U}$.

DEFINITION 1. Let X be a space and S a subset. S is called *negligible* in X if X is homeomorphic to $X \setminus S$. The set S is called *strongly negligible* in X if for every collection $\mathscr{U}$ of open subsets of X (not necessarily a covering of X) there is a homeomorphism h from X onto $X \setminus (S \cap \bigcup \mathscr{U})$ that is $\mathscr{U}$-close to 1_X. S is called *almost strongly negligible* if for every open covering $\mathscr{U}$ of X there is a homeomorphism h from X onto $X \setminus S$ that is $\mathscr{U}$-close to 1_X.

For a discussion of the concepts negligible and strongly negligible and their relation with pseudo-boundaries see Dijkstra [**11**, §1.2]. The following result has been taken from Dijkstra [**11**, §1.2].

PROPOSITION 1. *Strong negligibility is open hereditary, closed hereditary and in complete spaces σ-additive.*

The concept almost strongly negligible is introduced mainly for technical reasons. It is weaker than strongly negligible as follows from the observation that it is neither closed nor open hereditary. Consider the space $I \times \mathbf{P}$ where I is the interval $[0,1]$ and $\mathbf{P}$ is the space of irrational numbers. It is an immediate consequence of the Alexandroff and Urysohn [**1**] characterization of $\mathbf{P}$ that singletons are strongly negligible in $\mathbf{P}$. Consequently, every component $I \times \{p\}$ of $I \times \mathbf{P}$ is almost strongly negligible. However, the sets $\{(0,p)\}$ and $(0,1] \times \{p\}$ are not negligible in $I \times \mathbf{P}$ since deleting them would result in spaces in which not every component is homeomorphic to I.

DEFINITION 2. Let X be a space and let S be a closed subset of X. The set S is called a *Z-set in X* if for every continuous $f\colon Q \to X$ and every open covering $\mathscr{U}$ of X there is a continuous $g\colon Q \to X \setminus S$ that is $\mathscr{U}$-close to f. The set S is called a *strong Z-set in X* if for every open covering $\mathcal{U}$ of X there is a continuous $h\colon X \to X$ that is $\mathcal{U}$-close to 1_X and that satisfies $\mathrm{Cl}_X(h(X)) \cap S = \varnothing$. A countable union of (strong) Z-sets is called a (strong) σZ-set.

The concept of a strong Z-set was recently introduced by Bestvina et al. [**5**]. The examples constructed in that paper to show that Z-set and strong Z-set are different concepts play an important role in §4.

Since in a complete space the complement of a negligible set is complete and hence a G_δ-set, we trivially have that (in complete spaces) every strongly negligible set is a σZ-set. Generally for incomplete spaces this is false since a negligible set need not be an F_σ-set. Consider for instance the space $C \times \mathbf{Q}$, where C is a Cantor set and $\mathbf{Q}$ the space of rational numbers. Let A be a countable dense subset of C and consider $P = (C \setminus A) \times \{0\}$. The set P is homeomorphic to $\mathbf{P}$ and hence not an F_σ-set in the σ-compact space $C \times \mathbf{Q}$. It is a consequence of the Alexandroff and Urysohn [**1**] characterization of $C \times \mathbf{Q}$ that P is strongly negligible in $C \times \mathbf{Q}$.

DEFINITION 3. Let $C(Y, X)$ denote the set of continuous functions from Y into X. A space X is said to satisfy the *discrete approximation property* if for every sequence $(f_i)_{i=1}^{\infty}$ in $C(Q, X)$ and every open covering $\mathcal{U}$ of X there exists a sequence $(g_i)_{i=1}^{\infty}$ in $C(Q, X)$ such that each g_i is $\mathcal{U}$-close to f_i and the sequence $(g_i(Q))_{i=1}^{\infty}$ has no cluster points, i.e.

$$\bigcap_{i=1}^{\infty} \mathrm{Cl}_X\left(\bigcup_{j=i}^{\infty} g_j(Q)\right) = \varnothing .$$

A space is said to satisfy the *discrete n-cells property* if the same condition holds for sequences in $C(I^n, X)$.

Toruńczyk's theorem [**17**] states that a complete ANR is an l^2-manifold iff it satisfies the discrete approximation property. Bowers [**7**] has shown that the complement of a σZ-set in a Q-manifold has the discrete approximation property if it satisfies the discrete n-cells property for all $n \in \{0\} \cup \mathbf{N}$.

DEFINITION 4. Let n be an element of $\{-1, 0, 1, 2, \dots\}$ and identify the n-sphere S^n with the geometric boundary of the $(n+1)$-ball B^{n+1}. A subset A of a space X is called *locally n-connected rel X* if for every $x \in X$ and neighbourhood U of x there is a neighbourhood V of x such that each element of $C(S^n, V \cap A)$ is extendable to an element of $C(B^{n+1}, U \cap A)$.

The set A is called LC^n rel X if it is locally i-connected rel X for $i = -1,0,\ldots, n$.

In the degenerate case $n = -1$ we have S^n is empty and B^{n+1} is a singleton, whence locally (-1)-connected rel X simply means dense in X. If a σZ-set is LC^{n-1} rel a Q-manifold then its complement has the discrete n-cells property, Bowers [**8**].

3. Boundary sets. A *boundary set* is a σZ-set in a Q-manifold such that its complement is an l^2-manifold. In this section we shall deal with our problem in the class of spaces that are homeomorphic to the complement of a σZ-set in a Q-manifold.

LEMMA. *Let X be the complement of a σZ-set A in a Q-manifold M and let n be a non-negative integer. If every n-dimensional σ-compactum is negligible in X then X has the discrete n-cells property.*

Proof. In Dijkstra [**10**] it is shown that Q contains an n-capset. It is easily seen that this result holds for any Q-manifold. An n-capset A_n in M is characterized by (a) A_n is an n-dimensional σZ-set in M and (b) for every $\leq n$-dimensional σZ-set S and every collection $\mathscr{U}$ of open subsets of M there is an $h \in \mathscr{H}(M)$ that is $\mathscr{U}$-close to 1_M with $h(S \cap \bigcup \mathscr{U}) \subset A_n$. Any set B that contains A_n is LC^{n-1}rel M. This can be seen as follows. Let $\mathscr{U}$ be an open AR-set in M and let f: $S^i \to U \cap B$ be continuous, where $i < n$. Extend f to a g: $B^{i+1} \to U$ and put $K = f(S^i)$. Consider the space $C = B^{i+1} \setminus g^{-1}(K)$. Let $\mathscr{U}$ be a canonical covering of $U \setminus K$ with respect to M (Borsuk [**6**, III. 1.4]) and select a Z-imbedding h: $C \to U \setminus K$ that is $\mathscr{U}$-close to $g|C$ (Chapman [**9**, 18.2]). Since $\mathscr{U}$ is canonical we have $\bar{h} = h \cup (g | g^{-1}(K))$ is a continuous map from B^{i+1} into U. The set $h(C)$ is homeomorphic to C and hence $\leq n$-dimensional. So we can find an $\alpha \in \mathscr{H}(M)$ that is $\{U \setminus K\}$-close to 1_M with $\alpha(h(C)) \subset A_n$. This implies that $\alpha \circ \bar{h}$: $B^{i+1} \to U \cap B$ is the required extension of f, so B is LC^{n-1} rel M.

According to Bessaga and Pełczyński [**4**, V. 3.1] we may assume that A_n is disjoint from A. We have that $A \cup A_n$ is LC^{n-1} rel M and hence $M \setminus (A \cup A_n)$ has the discrete n-cells property, Bowers [**8**]. By assumption A_n is negligible in $M \setminus A = X$ and hence X has the discrete n-cells property.

Note that this implies that the fake Hilbert space X_k in Dijkstra [**11**] has the discrete k-cells property. An easy adaptation of the proof of

Dijkstra [**11**, 5.4.2] shows that X_k does not have the discrete $(k+1)$-cells property.

THEOREM 1. *If X is the complement of a σZ-set A in a Q-manifold M then the following statements are equivalent.*

(a) *X is an l^2-manifold (A is a boundary set).*

(b) *Every finite dimensional σ-compactum is negligible in X.*

(c) *Every finite dimensional compactum is strongly negligible in X.*

(d) *Every σ-compactum is strongly negligible in X.*

Proof. (a) $\to$ (d) Anderson [**2**].

(d) $\to$ (c) Trivial.

(c) $\to$ (b) Proposition 1.

(b) $\to$ (a) The Lemma and the fact that if a complement of a σZ-set in a Q-manifold has the discrete n-cells property for every n then it is an l^2-manifold, Bowers [**7**].

The implication (c) $\to$ (a) can be proved directly without using Bowers [**7**] or Toruńczyk [**17**]. Let B be an *fd*-capset in M. According to Anderson [**3**] we have $M \setminus B$ is an l^2-manifold. Since Z-sets are thin in M we may assume that B and A are disjoint, Bessaga and Pełczyński [**4**, V. 3.1]. So A is a σ-compactum in the l^2-manifold $M \setminus B$ and hence negligible, Anderson [**2**]. On the other hand, B is a countable union of finite-dimensional compacta in $X = M \setminus A$ and hence negligible, Proposition 1. So we have $X = M \setminus A \approx M \setminus (A \cup B) \approx M \setminus B$ which means that X is an l^2-manifold.

Comparing the Lemma with Theorem 1 it is natural to ask whether in this setting (complements of σZ-sets) the discrete n-cells property implies certain negligibility properties.

PROPOSITION 2. *For every $n \in \mathbf{N}$ there is a space X which is the complement of a σZ-set in Q such that X satisfies the discrete n-cells property and contains a non-negligible singleton.*

Proof. Let A_n be an n-capset in Q, see the proof of the Lemma. Let C_i be a Z-imbedded $(i+1)$-cell in $Q \setminus A_n$ and assume moreover that the C_i's are disjoint and that $(C_i)_{i=1}^{\infty}$ converges to a point $x \in Q \setminus (A_n \cup \bigcup_{i=1}^{\infty} C_i)$. Let F_i be the geometric interior of an i-face of C_i. Our example is given by

$$X = Q \setminus \left(A_n \cup \bigcup_{i=1}^{\infty} F_i \right).$$

Then X is the complement of a σZ-set in Q that satisfies the discrete n-cells property. De Groot and Nishiura [**14**] have shown that the space $C_i \setminus F_i$ has defect i, i.e. $C_i \setminus F_i$ has a $\geq i$-dimensional remainder in any compactification. Since $\dim A_n = n$ it follows that x is the only point in X every neighbourhood of which contains X-closed copies of $C_i \setminus F_i$ for every i. This implies that $\{x\}$ is non-negligible in X.

4. A counterexample in the general setting. We shall see that in general strong negligibility of compacta does not characterize l^2-manifolds among the complete ANR's.

PROPOSITION 3. *There exists a topologically complete absolute retract X with the properties* (a) *every σ-compactum is strongly negligible in X and* (b) *X does not have the discrete 2-cells property.*

The basis for our construction is formed by the examples in Bestvina et al. [**5**]. These spaces are AR's that are very similar to l^2 but have one "bad" point, i.e. there is a singleton that is not a strong Z-set and whose complement is Hilbert space. Obviously, we cannot use these spaces directly since the bad point is non-negligible. The idea is to modify the spaces in [**5**] in such a way that a space is formed with many bad points so that deleting a few of them will not make a difference.

Proof. Consider the set

$$A = \left(\left\{\frac{1}{n} \mid n \in \mathbf{N}\right\} \times I\right) \cup (I \times \{0\}) \subset \mathbf{R}^2$$

with the Euclidean topology. Put $\alpha = (0,0) \in A$ and define

$$\hat{A} = \left((A \setminus \{\alpha\}) \times l^2\right) \cup \{\alpha\}.$$

If π is the "projection" from $\hat{A}$ onto A then basic neighbourhoods of α in $\hat{A}$ are preimages of neighbourhoods of α in A. Furthermore, the set $(A \setminus \{\alpha\}) \times l^2$ is an open subset of $\hat{A}$ that carries the product topology. Bestvina et al. [**5**] constructed the spaces A and $\hat{A}$ and proved the following: A and $\hat{A}$ are topologically complete AR's and $\{\alpha\}$ is a Z-set but not a strong Z-set in both A and $\hat{A}$.

Let S be a universal pseudo-boundary in $\mathbf{R}$, see Geoghegan and Summerhill [**13**]. Then S is a zero-dimensional σ-compactum in $\mathbf{R}$ such that for every zero-dimensional σ-compactum K in $\mathbf{R}$ and every collection $\mathscr{U}$ of open subsets of $\mathbf{R}$, there is an $h \in \mathscr{H}(\mathbf{R})$ with h and 1 $\mathscr{U}$-close and $h(K \cup S) \cap \bigcup \mathscr{U} = S \cap \bigcup \mathscr{U}$. The set S is homeomorphic to $C \times \mathbf{Q}$, in fact any dense copy of $C \times \mathbf{Q}$ in $\mathbf{R}$ meets the requirements. Now, let B be the product $\hat{A} \times \mathbf{R}$ and define our example X by

$$X = B \setminus (\{\alpha\} \times S).$$

We first make a few simple observations. Since $\{\alpha\}$ is a Z-set in $\hat{A}$, the set $\{\alpha\} \times \mathbf{R}$ is a Z-set in B. Consequently, $\{\alpha\} \times S$ is a σZ-set in the complete AR B and hence X is a complete AR, see Toruńczyk **[16]**. The set $P = \{\alpha\} \times (\mathbf{R} \setminus S)$ is a Z-set in X that is homeomorphic to $\mathbf{P}$. The complement of P in X is $(A \setminus \{\alpha\}) \times l^2 \times \mathbf{R}$ and according to Toruńczyk **[15]** homeomorphic to l^2. Let $\xi = \pi \times 1_{\mathbf{R}}$ and observe that basic neighbourhoods of (α, r) in B are preimages under the mapping ξ of basic neighbourhoods of (α, r) in $A \times \mathbf{R}$.

Claim 1. If $p \in P$ then there is an open covering $\mathcal{U}$ of X and a sequence $(g_i)_{i=1}^{\infty}$ in $C(I^2, X)$ such that for every sequence $(h_i)_{i=1}^{\infty}$ in $C(I^2, X)$ that is $\mathcal{U}$-close to $(g_i)_{i=1}^{\infty}$, $(h\{-_i(I^2))_{i=1}^{\infty}$ has p as a cluster point.

This means that X fails to satisfy the discrete 2-cells property at points of P. It implies that no non-empty subset of P is a strong Z-set in X.

Proof. Let (α, r) be an arbitrary point in P. Construct for every $i \in \mathbf{N}$ a homeomorphism f_i: $I \to J_i$, where J_i is the arc

$$\left(\left\{\frac{1}{i+1}, \frac{1}{i}\right\} \times I\right) \cup \left(\left[\frac{1}{i+1}, \frac{1}{i}\right] \times \{0\}\right) \subset A.$$

Let $\mathcal{V}$ be an open covering of $\mathbf{R}^3$ with sets of diameter less than $1/2$ with respect to the standard metric. Put $\mathcal{U} = \{\xi^{-1}(V) \mid V \in \mathbf{V}\}$. For technical reasons the g_i's will be functions from $I \times [-1, 1]$ into X:

$$g_i(s, t) = ((f_i(s), 0), t + r) \quad \text{for } s \in I \text{ and } t \in [-1, 1],$$

where 0 is the zero vector in l^2. Suppose that h_i is $\mathcal{U}$-close to g_i. Then $\xi \circ h_i$ and $\xi \circ g_i$ are $\mathcal{V}$-close and hence $d(\xi \circ h_i, \xi \circ g_i) < 1/2$. Note that $\xi \circ g_i$ is a homeomorphism from $I \times [-1, 1]$ onto $J_1 \times [r - 1, r + 1]$. Since $\xi \circ h_i$ is close to $\xi \circ g_i$ we have that the image of $\xi \circ h_i$ must contain the set

$$\left[\frac{1}{i+1}, \frac{1}{i}\right] \times \{0\} \times \left[r - \frac{1}{2}, r + \frac{1}{2}\right],$$

which is a central region in the disk $J_1 \times [r - 1, r + 1]$. Consequently, (α, r) is a cluster point of the sequence $(\xi \circ h_i(I \times [-1, 1]))_{i=1}^{\infty}$. Since basic neighbourhoods of (α, r) in B are preimages under ξ of neighbourhoods of (α, r) in $A \times \mathbf{R}$ we have (α, r) is a cluster point of $(h_i(I \times [-1,1]))_{i=1}^{\infty}$ in B and X.

Claim 2. Every σ-compactum in X is strongly negligible.

Proof. If L is a σ-compact subset of X then $L \cap P$ and $L \setminus P$ are also σ-compacta. Since strong negligibility is σ-additive it suffices to show that every compact set in P and in $X \setminus P$ is strongly negligible in X. We have seen that $X \setminus P$ is Hilbert space and hence every compactum K in $X \setminus P$ is strongly negligible in $X \setminus P$. Since we may assume that the associated homeomorphisms are supported on a set whose closure in X misses P, we may extend them with 1_P and conclude that K is strongly negligible in X.

Consider now the case that $\{\alpha\} \times K$ is a compact subset of P. Let $\mathscr{U}$ be a collection of open subsets of B. Since K is a zero-dimensional subset of $\mathbf{R}$ it is possible to select a sequence $O_1, O_2, O_3, \ldots$ of bounded, disjoint, open intervals in $\mathbf{R}$ and positive real numbers $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ such that

$$\mathscr{V} = \left\{\xi^{-1}\left(U^2_{\varepsilon_i}(\alpha) \times O_i\right) \mid i \in \mathbf{N}\right\}$$

is a refinement of $\mathscr{U}$ and $\bigcup\mathscr{V} \cap (\{\alpha\} \times K) = \bigcup\mathscr{U} \cap (\{\alpha\} \times K)$ (U^2_ε denotes the ε-ball in $\mathbf{R}^2$). We shall construct a homeomorphism h: $X \to X \setminus ((\{\alpha\} \times K) \cap \bigcup \mathscr{V})$ such that h and 1 are $\mathscr{V}$-close.

Since S is a capset for zero-dimensional compacta there is a homeomorphism f: $\mathbf{R} \to \mathbf{R}$ that is $\{O_i \mid i \in \mathbf{N}\}$-close to 1 and that satisfies

$$f(S) \cap \bigcup_{i=1}^{\infty} O_i = (S \cup K) \cap \bigcup_{i=1}^{\infty} O_i.$$

Define the isotopy H: $\mathbf{R} \times I \to \mathbf{R}$ by $H(r,t) = \phi(r,t)(r - f(r)) + f(r)$, where

$$\phi(r,t) = \begin{cases} \min\{1, t/\varepsilon_i\} & \text{if } r \in O_i, \\ 1 & \text{if } r \notin \bigcup_{i=1}^{\infty} O_i. \end{cases}$$

Since H is supported on $\bigcup_{i=1}^{\infty} O_i$ it suffices to show that H is continuous and each H_t one-to-one. Since f is strictly increasing it is easily seen that each $H_t \mid O_i$ is strictly increasing. This means that H_t is one-to-one for every $t \in I$. Since it is obvious that $H \mid (O_i \times I)$ is continuous it suffices to verify the continuity in points $(r,t) \in (\mathbf{R} \setminus \bigcup_{i=1}^{\infty} O_i) \times I$. Note that

$H(r,t) = f(r) = r$, so for an arbitrary $(r', t') \in \mathbf{R} \times I$ we have

$$\begin{aligned}|H(r',t') - H(r,t)| &\le |\phi(r',t')(r' - f(r')) + f(r') - f(r)| \\ &\le |r' - r + f(r) - f(r')| + |f(r') - f(r)| \\ &\le |r' - r| + 2|f(r') - f(r)|.\end{aligned}$$

Since f is continuous this yields the continuity of H. Define the homeomorphism $h \in \mathscr{H}(B)$ by

$$h(x,r) = \big(x, H\big(r, \min\{1, d_2(\pi(x), \alpha)\}\big)\big) \quad \text{for } x \in \hat{A} \text{ and } r \in \mathbf{R},$$

where d_2 is the standard metric on $\mathbf{R}^2$. For $r \in \mathbf{R}$ we have $H(r,0) = f(r)$ and hence $h(\alpha, r) = (\alpha, f(r))$. This means that

$$h(\{\alpha\} \times S) \cap \bigcup \mathscr{V} = (\{\alpha\} \times (S \cup K)) \cap \bigcup \mathscr{V}.$$

Consequently, $h \mid X$ is a homeomorphism from X onto

$$X \setminus ((\{\alpha\} \times K) \cap \bigcup \mathscr{V}).$$

If $(x,r) \notin \bigcup \mathscr{V}$ then either $r \notin \bigcup_{i=1}^{\infty} O_i$ or $r \in O_i$ and $d_2(\pi(x), \alpha) \ge \varepsilon_i$ for some i. In the first case we have $\phi(r,t) = 1$ and hence $h(x,r) = (x,r)$. In the second case we also find $h(x,r) = (x,r)$ since $\phi(r,t) = 1$ if $t \ge \varepsilon_i$. Furthermore, it is obvious that $h(V) = V$ for each $V \in \mathscr{V}$. To sum it up $h \mid X$: $X \to X \setminus ((\{\alpha\} \times K) \cap \bigcup \mathscr{V})$ is a homeomorphism that is $\mathscr{V}$-close to 1. Since $\mathscr{V}$ refines $\mathscr{U}$ we have that $\{\alpha\} \times K$ is strongly negligible in X. This proves Claim 2 and the Proposition.

REMARKS. This example also answers two natural questions concerning strong negligibility. We trivially have that in complete spaces every strongly negligible set is a σZ-set. The example shows that strong negligibility of a set does not imply that the set is a strong σZ-set.

Strong negligibility implies the existence of many autohomeomorphisms of the space. For example, a space with a strongly negligible singleton cannot be rigid. One might conjecture a relation between strong negligibility and homogeneity in connected spaces. Since the example contains two kinds of points (in $X \setminus P$ every singleton is a strong Z-set in X and in P no singleton is strong Z-set) it follows that a connected space need not be homogeneous even if every compactum is strongly negligible.

5. A positive result in the general setting. In §4 we found that strong negligibility of compacta does not characterize l^2-manifolds among the complete ANR's. In an l^2-manifold, however, not just compacta but all Z-sets are strongly negligible, Anderson **[2]**. In fact, in l^2-manifolds the concepts σZ-set and strongly negligible set coincide. Strong negligibility of Z-sets alone does not characterize l^2-manifolds. Consider the spaces $\mathbf{R}^n$, which satisfy this condition, simply because they have no Z-sets. Obviously, we need some additional condition that guarantees that there are enough Z-sets. It is natural to ask the following

Question. Let X be a complete ANR in which every Z-set is strongly negligible and moreover with the property that every compactum is a Z-set. Is X necessarily an l^2-manifold?

Note that in the space X of §4 every compactum is a Z-set and every strong Z-set is contained in $X \setminus P \approx l^2$ and hence strongly negligible. This example does not settle the problem above since P is a Z-set in X whose complement is l^2. We do not know the answer to the aforementioned question but we have the following result.

THEOREM 2. *If X is a space in which every Z-set is almost strongly negligible and moreover every compactum in X is a strong Z-set then X has the discrete approximation property.*

Proof. Let $C(Q, X)$ denote the space of continuous mappings from Q into X equipped with the compact-open topology. Select a sequence h_1, h_2, $h_3, \ldots$ in $C(Q, X)$ that is dense. Assume that we have an arbitrary sequence f_1, f_2, $f_3, \ldots$ in $C(Q, X)$. We shall construct inductively a sequence g_1, g_2, $g_3, \ldots$ in $C(Q, X)$ such that each g_i is close to f_i and the set of cluster points of $g_1(Q)$, $g_2(Q)$, $g_3(Q), \ldots$ is a Z-set in X. Let ρ be a metric on X and let $\varepsilon > 0$.

The compactum $h_1(Q)$ is by assumption a strong Z-set. Consequently, there exists a continuous α_1: $X \to X$ with $\rho(\alpha_1, 1_X) < (\varepsilon/2)2^{-1}$ and $\rho(\alpha_1(X), h_1(Q)) = \delta_1 > 0$. Put $g_1 = \alpha_1 \circ f_1$. Assume now that α_n, g_n and δ_n have been determined. Since $g_n(Q) \cup h_{n+1}(Q)$ is a strong Z-set there is an α_{n+1}: $X \to X$ that satisfies

$$\rho(\alpha_{n+1}, 1_X) < \min\{\varepsilon/2, \delta_1, \ldots, \delta_n\}2^{-n-1}$$

and

$$\rho(\alpha_{n+1}(X), g_n(Q) \cup h_{n+1}(Q)) = \delta_{n+1} > 0.$$

Put $g_{n+1} = \alpha_{n+1} \circ \cdots \circ \alpha_1 \circ f_{n+1}$ and note that $\rho(g_{n+1}, f_{n+1}) < \varepsilon/2$.

Let Z be the set of cluster points of the sequence $g_1(Q)$, $g_2(Q)$, $g_3(Q), \ldots$, i.e.

$$Z = \bigcap_{n=1}^{\infty} \operatorname{Cl}\left(\bigcup_{k=n}^{\infty} g_k(Q) \right).$$

If $k > n$ then $\rho(\alpha_k \circ \cdots \circ \alpha_{n+1}, 1_X) < \delta_n/2$ and hence

$$\rho(\alpha_k \circ \cdots \circ \alpha_n(X), g_{n-1}(Q) \cup h_n(Q)) > \delta_n/2.$$

Since $\alpha_k \circ \cdots \circ \alpha_n(X)$ contains $g_k(Q)$ this implies that Z is disjoint from $g_{n-1}(Q)$ and $h_n(Q)$. So every g_i and h_i is a mapping from Q into $X \setminus Z$. Since $\{h_i \mid i \in \mathbf{N}\}$ is dense in $C(Q, X)$ this means that Z is a Z-set. By assumption Z is almost strongly negligible and there is a homeomorphism β: $X \setminus Z \to X$ with $\rho(\beta, 1) < \varepsilon/2$. The sequence $(\beta \circ g_i)_{i=1}^{\infty}$ obviously has the following properties: $\rho(\beta \circ g_i, f_i) < \varepsilon$ and $(\beta \circ g_i(Q))_{i=1}^{\infty}$ has no cluster points in X. This proves that X satisfies the discrete approximation property.

COROLLARY 2. *A complete* ANR *is an l^2-manifold iff every Z-set is strongly negligible and every compactum is a strong Z-set.*

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