PACIFIC JOURNAL OF MATHEMATICS
Vol. 128, No. 2, 1987

UNIQUENESS OF STRONG SOLUTIONS TO STOCHASTIC DIFFERENTIAL EQUATIONS IN THE PLANE WITH DETERMINISTIC BOUNDARY PROCESS

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Under the assumption of the existence of a weak solution and the pathwise uniqueness of solutions, existence and uniqueness of a strong solution to the stochastic differential system of non Markovian type in the plane

$$dX_z = \alpha(z, X)\,dB_z + \beta(z, X)\,dz \quad \text{for } z \in \mathbf{R}^2_+,$$
$$\partial X = x$$

is obtained where x is a continuous real valued function on $\partial \mathbf{R}^2_+$.

1. Introduction. Consider a stochastic differential equation of non-Markovian type in the plane

$$dX_z = \alpha(z, X)\,dB_z + \beta(z, X)\,dz \tag{1.0}$$

i.e.,

$$X_{s,t} - X_{0,t} - X_{s,0} + X_{0,0} = \int_{R_z} \alpha(\zeta, X)\,dB_\zeta + \int_{R_z} \beta(\zeta, X)\,d\zeta \tag{1.1}$$

for $z = (s, t) \in \mathbf{R}^2_+$ and $R_z = [0, s] \times [0, t]$ where B is an $\{\mathfrak{F}_z\}$-Brownian motion on an equipped probability space $(\Omega, \mathfrak{F}, P;\ \mathfrak{F}_z)$ with $\partial B = 0$, ∂B being the restriction of B to the boundary $\partial \mathbf{R}^2_+$ of $\mathbf{R}^2_+$, and consider the boundary condition

$$\partial X = x \tag{1.2}$$

where x is a fixed element in the space ∂W of all continuos real valued functions on $\partial \mathbf{R}^2_+$. Let W be the space of all continuous real valued functions on $\mathbf{R}^2_+$. The coefficients α and β are real valued functions on $\mathbf{R}^2_+ \times W$ satisfying certain measurability conditions that imply that for each $\omega \in \Omega$, $\alpha(z, X(\cdot, \omega))$ and $\beta(z, X(\cdot, \omega))$ depend only on that part of the sample function $X(\cdot, \omega)$ which precedes z in the sense of the partial ordering of $\mathbf{R}^2_+$. We refer to **[8]** or **[10]** for these measurability conditions.

In this article, by an equipped probability space we mean a complete probability measure space $(\Omega, \mathfrak{F}, P)$ with an increasing and right continuous family $\{\mathfrak{F}_z, z \in \mathbf{R}^2_+\}$ of sub-σ-fields of $\mathfrak{F}$, each containing all the null sets in $(\Omega, \mathfrak{F}, P)$. We do not assume the conditional independence of $\mathfrak{F}_z^1 = \sigma(\bigcup_{v \in \mathbf{R}_+} \mathfrak{F}_{s,v})$ and $\mathfrak{F}_z^2 = \sigma(\bigcup_{u \in \mathbf{R}_+} \mathfrak{F}_{u,t})$ relative to $\mathfrak{F}_z$ for $z = (s, t) \in \mathbf{R}_z^2$ since this condition is not needed for the existence of our stochastic integrals with respect to an $\{\mathfrak{F}_z\}$-Brownian motion.

DEFINITION 1. By a solution of the stochastic differential equation (1.1) we mean a pair of 2-parameter stochastic processes (X, B) on an equipped probability space $(\Omega, \mathfrak{F}, P;\ \mathfrak{F}_z)$ such that B is an $\{\mathfrak{F}_z\}$-Brownian motion with $\partial B = 0$, X is an $\{\mathfrak{F}_z\}$-adapted process whose sample functions are all continuous on $\mathbf{R}_+^2$ and the stochastic integrals in (1.1) exist and satisfy (1.1) with probability 1.

DEFINITION 2. We say that the stochastic differential equation (1.1) satisfies the pathwise uniqueness condition if whenever (X, B) and (X', B) with the same B are two solutions of (1.1) on the same equipped probability space and $\partial X = \partial X'$ then $X = X'$.

Let $\mathfrak{B}(W)$ be the σ-field generated by the cylinder sets in W. With respect to the metric of uniform convergence on the compact subsets of $\mathbf{R}_+^2$, W is a complete separable metric space and the σ-field of the Borel sets in W is equal to $\mathfrak{B}(W)$. Let m_W be the Wiener measure on $(W, \mathfrak{B}(W))$ concentrated on those elements of W which vanish on $\partial \mathbf{R}_+^2$. For $z \in \mathbf{R}_+^2$, let $\mathfrak{B}_z(W)$ be the σ-field generated by the cylinder sets $\{w \in W;\ w(\zeta) \in E\}$ where $E \in \mathfrak{B}(\mathbf{R})$ and $\zeta \leq z$. We write $\mathfrak{B}_z(W)^*$ for the σ-field generated by $\mathfrak{B}_z(W)$ and the subsets of the null sets in $(W, \mathfrak{B}(W), m_W)$.

DEFINITION 3. A solution (X, B) of (1.1) on an equipped probability space $(\Omega, \mathfrak{F}, P;\ \mathfrak{F}_z)$ is called a strong solution of the boundary value problem (1.1) and (1.2) if there exists a transformation F of W into W such that

1° for every $z \in \mathbf{R}_+^2$, F is $\mathfrak{B}_z(W)^*/\mathfrak{B}_z(W)$ measurable,

2° $X(\cdot, \omega) = F[B(\cdot, \omega)]$ for a.e. $\omega \in \Omega$.

In [**8**] we showed that if the coefficients α and β in (1.1) satisfy a certain Lipschitz condition then (1.1) satisfies the pathwise uniqueness condition. There we also showed that under the Lipschitz condition and an order of growth condition on α and β a strong solution exists for (1.1) with a nondeterministic boundary condition. In the present paper we study the independence of the transformation F in Definition 3 from the equipped probability space $(\Omega, \mathfrak{F}, P;\ \mathfrak{F}_z)$. The main result is the following theorem.

THEOREM. *Let $x \in \partial W$ be fixed. Suppose the stochastic differential system* (1.1) *and* (1.2) *has a solution on some equipped probability space and* (1.1) *satisfies the pathwise uniqueness condition. Then there exists a transformation F of W into W, unique up to a null set in $(W, \mathfrak{B}(W), m_W)$, such that*

1° *for every $z \in \mathbf{R}^2_+$, F is $\mathfrak{B}_z(W)^*/\mathfrak{B}_z(W)$ measurable,*
2° *if $(\Omega, \mathfrak{F}, P;\ \mathfrak{F}_z)$ is an equipped probability space on which there exists an $\{\mathfrak{F}_z\}$-Brownian motion B with $\partial B = 0$, then $X \equiv F[B]$ is a solution of the stochastic differential system* (1.1) *and* (1.2) *on the equipped probability space $(\Omega, \mathfrak{F}, P;\ \mathfrak{F}_z)$.*
3° *any solution (X, B) of the differential system* (1.1) *and* (1.2) *satisfies $X = F[B]$.*

The proof of this theorem is given in §3. In constructing a unique strong solution we adopt Ikeda and Watanabe's approach in [7].

2. Some lemmas for the construction of a unique strong solution. In what follows we write W_i, $i = 0, 1$ and 2 for copies of W. Let (X, B) be a solution to the stochastic differential system (1.1) and (1.2) on an equipped probability space $(\Omega, \mathfrak{F}, P;\ \mathfrak{F}_z)$ and let Q be the probability distribution of (X, B) on the measurable space $(W_1 \times W_0,\ \mathfrak{B}(W_1 \times W_0))$ where $\mathfrak{B}(W_1 \times W_0)$ is the σ-field of the Borel sets in $W_1 \times W_0$ in its product topology.

Let π be the projection of $W_1 \times W_0$ onto W_0. The probability distribution on $(W_0, \mathfrak{B}(W_0))$ of the transformation π defined on the probability space $(W_1 \times W_0,\ \mathfrak{B}(W_1 \times W_0), Q)$ is then the Wiener measure m_W.

Let $Q^{(\cdot)}$ with $Q^{w_0}(A_1)$ for $(A_1, w_0) \in \mathfrak{B}(W_1) \times W_0$ be a regular conditional probability of Q under π, i.e.,

(C.1) for every $w_0 \in W_0$, Q^{w_0} is a probability measure on $(W_1, \mathfrak{B}(W_1))$.
(C.2) for every $A_1 \in \mathfrak{B}(W_1)$, $Q^{(\cdot)}(A_1)$ is $\mathfrak{B}(W_0)$ measurable,
(C.3) for every $A_1 \in \mathfrak{B}(W_1)$ and $A_0 \in \mathfrak{B}(W_0)$

$$Q(A_1 \times A_0) = \int_{A_0} Q^{w_0}(A_1) m_W(dw_0).$$

From these defining properties of the regular conditional probability follows that if $\mathfrak{C}_1 = \{W_1, \phi\}$ and $A_1 \in \mathfrak{B}(W_1)$ then

$$Q(A_1 \times W_0 \mid \mathfrak{C}_1 \otimes \mathfrak{B}(W_0))(w_1, w_0) = Q^{w_0}(A_1) \tag{2.0}$$

for all $w_1 \in W_1$ for a.e. w_0 in $(W_0, \mathfrak{B}(W_0), m_W)$.

The existence of a regular conditional probability $Q^{(\cdot)}$ is ensured by the fact that both the domain $W_1 \times W_0$ and the image space W_0 of the transformation π are complete separable metric spaces (see Parthasarathy [4]). The following lemma is an extension of Neveu's proof in [3] for a lemma by Yamada and Watanabe [7].

LEMMA 1. *For* $z = (s,t) \in \mathbf{R}^2_+$, *let*

$$\mathfrak{B}^0_z(W) = \mathfrak{B}_z(W),$$

$$\mathfrak{B}^1_z(W) = \sigma\{w(u,v),\ u \in [0,s],\ v \in [0,\infty),\ w \in W\},$$

$$\mathfrak{B}^2_z(W) = \sigma\{w(u,v),\ u \in [0,\infty),\ v \in [0,t],\ w \in W\},$$

$$\mathfrak{B}^3_z(W) = \sigma\{w(u,v),\ u \in [0,s] \text{ or } v \in [0,t],\ w \in W\}.$$

Then for every $A_1 \in \mathfrak{B}^j_z(W_1)$, $Q^{(\cdot)}(A_1)$ *is* $\mathfrak{B}^j_z(W_0)^*$ *measurable for* $j = 0$, 1, 2 *or* 3.

Proof. Let

$$\mathfrak{B}^4_z(W) = \sigma\{w(u,v) - w(0,v) - w(u,t) + w(0,t),\quad u < s,\ t < v,\ w \in W\},$$

$$\mathfrak{B}^5_z(W) = \sigma\{w(u,v) - w(s,v) - w(u,0) + w(s,0),\quad s < u,\ v < t,\ w \in W\},$$

$$\mathfrak{B}^6_z(W) = \sigma\{w(u,v) - w(s,v) - w(u,t) + w(s,t),\quad s < u,\ < v,\ w \in W\}.$$

Consider the case where $A_1 \in \mathfrak{B}^3_z(W_1)$. Let us show that $\mathfrak{B}^3_z(W_1) \otimes \mathfrak{B}^3_z(W_0)$ and $\mathfrak{C}_1 \otimes \mathfrak{B}^6_z(W_0)$ are independent with respect to Q. Now for a transformation ψ of Ω into $W_1 \times W_0$ defined by

$$\psi(\omega) = (X(\cdot,\omega), B(\cdot,\omega)) \in W_1 \times W_0 \quad \text{for } \omega \in \Omega,$$

we have

$$\psi^{-1}(\mathfrak{B}^3_z(W_1) \otimes \mathfrak{B}^3_z(W_0)) \subset \mathfrak{F}^1_z \vee \mathfrak{F}^2_z$$

and, denoting $z = (s,t)$,

$$\psi^{-1}(\mathfrak{C}_1 \otimes \mathfrak{B}^6_z(W_0)) = \sigma\{B(u,v) - B(s,v) - B(u,t) + B(s,t),\ s \le u,\ t \le v\}.$$

The two σ-fields on the right sides of the last two expressions are independent with respect to P since B is an $\{\mathfrak{B}_z\}$-Brownian motion on $(\Omega, \mathfrak{F}, P; \mathfrak{F}_z)$. This then implies the independence of $\mathfrak{B}_z^3(W_1) \otimes \mathfrak{B}_z^3(W_0)$ and $\mathfrak{C}_1 \otimes \mathfrak{B}_z^6(W_0)$ with respect to Q.

According to a well known theorem in probability theory, if $\mathfrak{A}_1$, $\mathfrak{A}_2$ and $\mathfrak{A}_3$ are sub-σ-fields of $\mathfrak{A}$ in a probability space $(S, \mathfrak{A}, \mu)$ such that $\mathfrak{A}_1 \vee \mathfrak{A}_2$ and $\mathfrak{A}_3$ are independent, then

$$\mu(A_1 | \mathfrak{A}_2) = \mu(A_1 | \mathfrak{A}_2 \vee \mathfrak{A}_3) \quad \text{for any } A_1 \in \mathfrak{A}_1.$$

With $\mathfrak{A}_1 = \mathfrak{B}_z^3(W_1) \otimes \mathfrak{B}_z^3(W_0)$, $\mathfrak{A}_2 = \mathfrak{C}_1 \otimes \mathfrak{B}_z^3(W_0)$ and $\mathfrak{A}_3 = \mathfrak{C}_1 \otimes \mathfrak{B}_z^6(W_0)$ and noting $\mathfrak{B}_z^3(W_0) \vee \mathfrak{B}_z^6(W_0) = \mathfrak{B}(W_0)$, we have for our $A_1 \in \mathfrak{B}_z^3(W_1)$

$$\begin{aligned} &Q\big(A_1 \times W_0 \,|\, \mathfrak{C}_1 \otimes \mathfrak{B}_z^3(W_0)\big)(w_1, w_0) \\ &\qquad = Q\big(A_1 \times W_0 \,|\, \mathfrak{C}_1 \otimes \mathfrak{B}(W_0)\big)(w_1, w_0) \\ &\text{for a.e. } (w_1, w_0) \text{ in } \big(W_1 \times W_0,\, \mathfrak{C}_1 \otimes \mathfrak{B}_z^3(W_0), Q\big) \\ &\text{i.e., for all } w_1 \in W \text{ for a.e. } w_0 \text{ in } \big(W_0,\, \mathfrak{B}_z^3(W_0),\, m_W\big). \end{aligned}$$

From this and from (2.0), we have the $\mathfrak{B}_z^3(W_0)^*$-measurability of $Q^{(\cdot)}(A_1)$.

Next consider the case where $A_1 \in \mathfrak{B}_z^0(W_1)$. For ψ as defined above we have

$$\psi^{-1}\big(\mathfrak{B}_z^0(W_1) \otimes \mathfrak{B}_z^0(W_0)\big) \subset \mathfrak{F}_z$$

and, denoting $z = (s, t)$,

$$\begin{aligned} &\psi^{-1}\big(\mathfrak{C}_1 \otimes \big(\mathfrak{B}_z^4(W_0) \vee \mathfrak{B}_z^5(W_0) \vee \mathfrak{B}_z^6(W_0)\big)\big) \\ &\qquad = \sigma\big\{B(u', v') - B(u, v') - B(u', v) + B(u, v) \text{ where } s < u \\ &\qquad\qquad\qquad \text{or } t < v \text{ and } u < u' \text{ and } v < v'\big\}. \end{aligned}$$

The two σ-fields on the right sides of the last two expressions are independent with respect to P since B is an $\{\mathfrak{F}_z\}$-Brownian motion on $(\Omega, \mathfrak{F}, P; \mathfrak{F}_z)$. Therefore $\mathfrak{B}_z^0(W_1) \times \mathfrak{B}_z^0(W_0)$ and

$$\mathfrak{C}_1 \otimes \big(\mathfrak{B}_z^4(W_0) \vee \mathfrak{B}_z^5(W_0) \vee \mathfrak{B}_z^6(W_0)\big)$$

are independent with respect to Q. With

$$\mathfrak{A}_1 = \mathfrak{B}_z^0(W_1) \otimes \mathfrak{B}_z^0(W_0),\ \mathfrak{A}_2 = \mathfrak{C}_1 \otimes \mathfrak{B}_z^0(W_0)$$

and $\mathfrak{A}_3 = \mathfrak{C}_1 \otimes (\mathfrak{B}_z^4(W_0) \vee \mathfrak{B}_z^5(W_0) \vee \mathfrak{B}_z^6(W_0))$ and noting

$$\mathfrak{B}_z^0(W_0) \vee \mathfrak{B}_z^4(W_0) \vee \mathfrak{B}_z^5(W_0) \vee \mathfrak{B}_z^6(W_0) = \mathfrak{B}(W_0)$$

we have for $A_1 \in \mathfrak{B}_z^0(W_1)$

$$Q\big(A_1 \times W_0 \,|\, \mathfrak{C}_1 \otimes \mathfrak{B}_z^0(W_0)\big) = Q\big(A_1 \times W_0 \,|\, \mathfrak{C}_1 \otimes \mathfrak{B}(W_0)\big).$$

From this and from (2.0) follows the $\mathfrak{B}_z^0(W_0)^*$-measurability of $Q^{(\cdot)}(A_1)$.

The case where $A_1 \in \mathfrak{B}_z^j(W_1)$ where $j = 1$ or 2 can be treated likewise. □

Let $x \in \partial W$ be fixed. For $i = 1$ and 2, let (X_i, B_i) be a solution of the stochastic differential system (1.1) and (1.2) on an equipped probability space $(\Omega_i, \mathfrak{F}_i, P_i;\ \mathfrak{F}_{i,z})$. Let Q_i be the probability distribution of (X_i, B_i) on $(W_i \times W_0, \mathfrak{B}(W_i \times W_0))$ and let $Q_i^{w_0}(A_i)$, $(A_i, w_0) \in \mathfrak{B}(W_i) \times W_0$, be a regular conditional probability of Q_i under the projection π_i of $W_i \times W_0$ onto W_0.

Let $\Omega = W_1 \times W_2 \times W_0$. On $\mathfrak{B}(\Omega) = \mathfrak{B}(W_1 \times W_2 \times W_0)$ define a probability measure P by setting

$$P(A_1 \times A_2 \times A_0) = \int_{A_0} Q_1^{w_0}(A_1) Q_2^{w_0}(A_2) m_W(dw_0) \tag{2.1}$$

$$\text{for } A_i \in \mathfrak{B}(W_i), \qquad i = 0, 1, 2.$$

Let $\mathfrak{F}$ be the completion of $\mathfrak{B}(\Omega)$ with respect to P and let $\mathfrak{N}$ be the collection of the null sets in $(\Omega, \mathfrak{F}, P)$. Then let

$$\mathfrak{B}_z = \mathfrak{B}_z(W_1) \otimes \mathfrak{B}_z(W_2) \otimes \mathfrak{B}_z(W_0)$$

and

$$\mathfrak{F}_z = \bigcup_{\varepsilon > 0} \sigma(\mathfrak{B}_{s+\varepsilon, t+\varepsilon} \cup \mathfrak{N}) \quad \text{for } z = (s, t) \in \mathbf{R}_+^2. \tag{2.2}$$

We then have an equipped probability space $(\Omega, \mathfrak{F}, P;\ \mathfrak{F}_z)$.

LEMMA 2. *On the equipped probability space $(\Omega, \mathfrak{F}, P;\ \mathfrak{F}_z)$ defined by* (2.1) *and* (2.2), *let a 2-parameter stochastic process B_0 be defined by setting*

$$B_0(z, \omega) = w_0(z) \quad \text{for } \omega = (w_1, w_2, w_0) \in \Omega. \tag{2.3}$$

Then B_0 is an $\{\mathfrak{F}_z\}$-Brownian motion on $(\Omega, \mathfrak{F}, P;\ \mathfrak{F}_z)$ with $\partial B_0 = 0$.

Proof. Clearly B_0 is an $\{\mathfrak{F}_z\}$-adapted stochastic process with continuous sample functions and $\partial B_0 = 0$. Thus, to show that B_0 is an $\{\mathfrak{F}_z\}$-Brownian motion it remains to show that for $z < z'$

$$E\big[\exp\{iuB_0((z, z'])\} \,|\, \mathfrak{F}_z^1 \vee \mathfrak{F}_z^2\big] = \exp\Big\{-\frac{u^2}{2} m_L((z, z'])\Big\} \quad \text{for } \mathbf{u} \in R,$$

where

$$B_0\big((z,z']\big) = B_0(s',t') - B_0(s,t') - B_0(s',t) + B_0(s,t)$$

for $z = (s,t)$ and $z' = (s',t')$ and m_L is the Lebesgue measure on $\mathbf{R}^2$. For this, it is sufficient to show that for every $A_i \in \mathfrak{B}_z^3(W_i)$, $i = 0, 1$ and 2

$$\text{(2.4)} \qquad \mathbf{E}\Big[\exp\big\{iuB_0\big((z,z']\big)\big\}1_{A_1\times A_2\times A_0}\Big] = \exp\Big\{-\frac{u^2}{2}m_L\big((z,z']\big)\Big\}P(A_1\times A_2\times A_0).$$

Now by (2.1) and (2.3), the left side of (2.4) is equal to

$$\int_{W_0} \exp\big\{iuw_0\big((z,z']\big)\big\}Q_1^{w_0}(A_1)Q_2^{w_0}(A_2)1_{A_0}(w_0)m_W(dw_0).$$

Since $Q_i^{w_0}(A_i)$ is a $\mathfrak{B}_z^3(W_0)^*$-measurable function of $w_0 \in W_0$ for our $A_i \in \mathfrak{B}_z^3(W_i)$ for $i = 1$ and 2 by Lemma 1, we have the independence of $w_0((z,z'])$ and $Q_1^{w_0}(A_2)Q_2^{w_0}(A_2)1_{A_0}(W_0)$ as random variables on $(W_0, \mathfrak{B}(W_0)^*, m_W)$ where $\mathfrak{B}(W_0)^*$ is the completion of $\mathfrak{B}(W_0)$ with respect to m_W. The last integral is then equal to

$$\int_{W_0} \exp\big\{iuw_0\big((z,z']\big)\big\}m_W(dw_0)\cdot\int_{A_0} Q_1^{w_0}(A_1)Q_2^{w_0}(A_2)m_W(dw_0) = \exp\Big\{-\frac{u^2}{2}m_L\big((z,z']\big)\Big\}P(A_1\times A_2\times A_0)$$

which is equal to the right side of (2.4). This completes the proof. □

LEMMA 3. *Let μ and ν be two probability measures on $(S, \mathfrak{B}(S))$ where S is a complete separable metric space and $\mathfrak{B}(S)$ is the σ-field of Borel sets in S. Let D be the diagbonal in $S \times S$, i.e.,*

$$D = \{(s_1, s_2) \in S\times S;\ s_1 = s_2\}.$$

If $(\mu\times\nu)(D) = 1$, then there exists a unique $s_0 \in S$ such that $\mu(\{s_0\}) = \nu(\{s_0\}) = 1$.

Proof. Let ρ be the metric on S. Then $\rho(s_1, s_2)$ for $s_1, s_2 \in S$ is a continuous function on $S\times S$ in its product topology and is thus $\mathfrak{B}(S\times S)$ measurable. Then the diagonal D being the subset of $S\times S$ on which ρ is equal to 0 is a member of $\mathfrak{B}(S\times S)$. Thus $(\mu\times\nu)(D)$ is defined.

Suppose $(\mu\times\nu)(D) = 1$. If $\mu \neq \nu$ on $\mathfrak{B}(S)$ then there exists $A \in \mathfrak{B}(S)$ such that $\mu(A) \neq \nu(A)$, say $\mu(A) > \nu(A)$. Then $\nu(A^c) > 0$ so that

$$(\mu\times\nu)(A\times A^c) = \mu(A)\nu(A^c) > 0.$$

But $(A \times A^c) \cap D = \varnothing$ and this implies $(\mu \times \nu)(A \times A^c) = 0$, contradicting the last inequality. Therefore $\mu = \nu$ on $\mathfrak{B}(S)$.

If there exists $A \in \mathfrak{B}(S)$ such that $\mu(A) \in (0,1)$ then $\mu(A^c) \in (0,1)$ also so that

$$(\mu \times \nu)(A \times A^c) = \mu(A)\mu(A^c) \in (0,1).$$

But this contradicts the equality $(\mu \times \nu)(A \times A^c) = 0$ which is implied by $(A \times A^c) \cap D = \varnothing$. Therefore no $A \in \mathfrak{B}(S)$ can have $\mu(A) \in (0,1)$ and consequently $\mu(A) = 0$ or 1 for every $A \in \mathfrak{B}(S)$.

Since a separable metric space is a Lindelöf space, for every positive integer n there exist countably many closed spheres in S, each with diameter n^{-1}, whose union is S. The μ-measure of each of these spheres is either 0 or 1. No two spheres with μ-measure 1 can be disjoint for otherwise we would have $\mu(S) \geq 2$. Let K_n be the closed set which is the intersection of all those spheres with μ-measure 1. Then $\mu(K_n) = 1$ and the diameter $\delta(K_n) \leq 1/n$. Consider the sequence of closed sets K_n, $n = 1, 2, \cdots$. By the same reason as above $K_n \cap K_m \neq \varnothing$ for $n \neq m$. If we let $C_n = \bigcap_{m=1}^{n} K_m$ then we have a decreasing sequence of closed sets C_n, $n = 1, 2, \cdots$ with $\mu(C_n) = 1$ and $\delta(C_n) \leq n^{-1}$ for every n. Since S is a complete metric space and $\delta(C_n) \downarrow 0$ as $n \to \infty$ there exists $s_0 \in S$ such that $\bigcap_{n=1}^{\infty} C_n = \{s_0\}$. Then $\mu(\{s_0\}) = \lim_{n \to \infty} \mu(C_n) = 1$. Since $\mu(S) = 1$ such $s_0 \in S$ is unique. □

3. Proof of the Theorem. With fixed $x \in \partial W$ assume that the stochastic differential system (1.1) and (1.2) has a solution on some equipped probability space and assume that (1.1) satisfies the pathwise uniqueness condition.

For $i = 1$ and 2 let (X_i, B_i) be a solution of (1.1) and (1.2) on an equipped probability space $(\Omega_i, \mathfrak{F}_i, P_i;\ \mathfrak{F}_{i,z})$. Let $Q_i, Q_i^{w_0}(A_i)$ and $(\Omega, \mathfrak{F}, P;\ \mathfrak{F}_z)$ be as in the construction in §2 following the proof of Lemma 1.

Let B_0 be the $\{\mathfrak{B}_z\}$-Brownian motion on the equipped probability space $(\Omega, \mathfrak{F}, P;\ \mathfrak{F}_z)$ defined by (2.3). Introduce two 2-parameter stochastic processes Y_i for $i = 1$ and 2 on $(\Omega, \mathfrak{F}, P;\ \mathfrak{F}_z)$ by setting

$$(3.1) \quad Y_i(z, \omega) = w_i(z) \quad \text{for } z \in \mathbf{R}_+^2 \quad \text{and} \quad \omega = (w_1, w_2, w_0) \in \Omega.$$

Then (Y_i, B_0) and (X_i, B_i) have the same probability distribution Q_i on $(W_i \times W_0, \mathfrak{B}(W_i \times W_0))$ so that (Y_i, B_0) is a solution of (1.1) and (1.2) on $(\Omega, \mathfrak{F}, P;\ \mathfrak{F}_z)$ for $i = 1$ and 2. Thus by the pathwise uniqueness condition we have $Y_1 = Y_2$, i.e.,

$$Y_1(\cdot, \omega) = Y_2(\cdot, \omega) \quad \text{for a.e. } \omega \text{ in } (\Omega, \mathfrak{F}, P),$$

in other words,

$$w_1 = w_2 \quad \text{for } P \text{ a.e. } \omega = (w_1, w_2, w_0) \in \Omega.$$

Since P is defined by (2.1), this implies that there exists a null set N_0 in $(W_0, \mathfrak{B}(W_0), m_W)$ such that

$$(3.2) \quad (Q_1^{w_0} \times Q_2^{w_0})\{(w_1, w_2) \in W_1 \times W_2;\ w_1 = w_2\} = 1 \quad \text{for } w_0 \in N_0^c.$$

Since W_1 and W_2 are copies of W which is a complete separable metric space (3.2) implies according to Lemma 3 that for every $w_0 \in N_0^c$ there exists a unique $w \in W$ such that

$$(3.3) \qquad Q_1^{w_0}(\{w\}) = Q_2^{w_0}(\{w\}) = 1.$$

Let F be a function defined by

$$(3.4) \qquad F(w_0) = w \quad \text{for } w_0 \in N_0^c$$

where w on the right side is the unique element in W satisfying (3.3) for our $w_0 \in N_0^c$. Thus

$$Q_1^{w_0} = Q_2^{w_0} = \delta_{F(w_0)} \quad \text{on } \mathfrak{B}(W) \text{ for } w_0 \in N_0^c.$$

Let us verify that F satisfies the condition 1° in our Theorem. Thus, for $z \in \mathbf{R}_+^2$, let $A \in \mathfrak{B}_z(W)$. Then

$$F^{-1}(A) = \{w_0 \in W_0;\ Q_1^{w_0}(\{w\}) = 1 \text{ for some } w \in A\}$$
$$= \{w_0 \in W_0;\ Q_1^{w_0}(A) = 1\}.$$

According to Lemma 1, $A \in \mathfrak{B}_z(W)$ implies that $\mathfrak{B}_z(W_0)^*$-measurability of $Q_1^{w_0}(A)$ as a function of $w_0 \in W_0$. Thus $F^{-1}(A) \in B_z(W_0)^*$, i.e., F is $\mathfrak{B}_z(W_0)^*/\mathfrak{B}_z(W)$ measurable.

To verify the condition 3° in the Theorem, note that from (3.4), (3.1) and (2.3)

$$F[B_0(\cdot, \omega)] = Y_i(\cdot, \omega) \quad \text{for } B_0(\cdot, \omega) \in N_0^c \text{ for } i = 1 \text{ and } 2.$$

Then since B_0 and Y_i are the images of B_i and X_i in $\Omega = W_1 \times W_2 \times W_0$ the last equality implies

$$F[B_i(\cdot, \omega_i)] = X_i(\cdot, \omega_i) \quad \text{for a.e. } \omega_i \text{ in } (\Omega_i, \mathfrak{F}_i, P_i) \text{ for } i = 1 \text{ and } 2$$

proving 3°. Note also that since F is common to $i = 1$ and 2 and is defined up to a null set in $(W_0, \mathfrak{B}(W_0), m_W)$, we have the uniqueness of F up to a null set in $(W, \mathfrak{B}(W), m_W)$.

Finally if $(\Omega_3, \mathfrak{F}_3, P_3;\ \mathfrak{F}_{3,z})$ is an equipped probability space on which there exists an $\{\mathfrak{F}_{3,z}\}$-Brownian motion B_3 with $\partial B_3 = 0$, then

(X_3, B_3) with X_3 defined by $X_3 = F[B_3]$ has the same probability distribution on $(W \times W, \mathfrak{B}(W \times W))$ as (Y_1, B_0) so that (X_3, B_3) is a solution of (1.1) and (1.2) on $(\Omega_3, \mathfrak{F}_3, P_3; \mathfrak{F}_{3,z})$. Thus condition 2° of the Theorem is satisfied. This completes the proof. □

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Received February 5, 1986.

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