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BOTT MAPS AND THE COMPLEX PROJECTIVE PLANE: A CONSTRUCTION OF R. WOOD'S EQUIVALENCES

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To the memory of Dr. Shichirô Oka

Let $\mathrm{U}(\infty)$, $\mathrm{O}(\infty)$ and $\mathrm{Sp}(\infty)$ be the direct limits of the finite-dimensional unitary, orthogonal and symplectic groups under inclusion, and let $\mathbf{P}_2\mathbf{C}$ be the complex projective plane. Then, by a result of R. Wood in K-theory, there exist homotopy equivalences from $\mathrm{U}(\infty)$ to the space of based maps $\mathbf{P}_2\mathbf{C} \to \mathrm{O}(\infty)$, and to the space of based maps $\mathbf{P}_2\mathbf{C} \to \mathrm{Sp}(\infty)$. In this paper we give an explicit construction of such homotopy equivalences, and prove Wood's theorem by using classical results of R. Bott and elementary homotopy theory.

Introduction. It is well-known that, in topological K-theory, there are natural isomorphisms

$$\widetilde{K\mathrm{U}}^*(X) \to \widetilde{K\mathrm{O}}^*(X \wedge \mathbf{P}_2\mathbf{C}) \quad \text{and} \quad \widetilde{K\mathrm{U}}^*(X) \to \widetilde{K\mathrm{Sp}}^*(X \wedge \mathbf{P}_2\mathbf{C}),$$

where $\mathbf{P}_2\mathbf{C}$ is the complex projective plane. This result is originally due to R. M. W. Wood, and his method for giving such isomorphisms can be found in [**9**] (see also [**1**; §2] or [**6**; §1]).

Now let $\mathrm{U}(\infty)$, $\mathrm{O}(\infty)$ and $\mathrm{Sp}(\infty)$ be the infinite-dimensional unitary, orthogonal and symplectic groups respectively, and let $\tilde{\mathscr{C}}(X; Y)$ denote the space of basepoint-preserving continuous maps from X to Y (equipped with the compact-open topology). Then the result of Wood mentioned above implies:

THEOREM (0.1) (*R. Wood*). *There are homotopy equivalences from* $\mathrm{U}(\infty)$ *to the space* $\tilde{\mathscr{C}}(\mathbf{P}_2\mathbf{C}; \mathrm{O}(\infty))$, *and to the space* $\tilde{\mathscr{C}}(\mathbf{P}_2\mathbf{C}; \mathrm{Sp}(\infty))$.

The main purpose of this paper is to construct such homotopy equivalences explicitly. In §4 we shall define certain maps

$$\chi_n^{\mathrm{O}}\colon \mathrm{U}(2n) \to \tilde{\mathscr{C}}(\mathbf{P}_2\mathbf{C}; \mathrm{O}(8n)) \quad \text{and} \quad \chi_n^{\mathrm{Sp}}\colon \mathrm{U}(n) \to \tilde{\mathscr{C}}(\mathbf{P}_2\mathbf{C}; \mathrm{Sp}(2n)),$$

and in §5 we shall show (Theorem (5.4)) that these give rise to homotopy equivalences

$$\chi_\infty^{\mathrm{O}}\colon \mathrm{U}(\infty) \to \tilde{\mathscr{C}}(\mathbf{P}_2\mathbf{C}; \mathrm{O}(\infty)) \quad \text{and} \quad \chi_\infty^{\mathrm{Sp}}\colon \mathrm{U}(\infty) \to \tilde{\mathscr{C}}(\mathbf{P}_2\mathbf{C}; \mathrm{Sp}(\infty))$$

in direct limits. Thus we shall give another proof of (0.1) which does not use vector bundle theory. This work may be regarded as a continuation of [**10**] and [**11**], and indeed our proof of (0.1) is accomplished by the techniques used there. A by-product of our work is the result that, even for $n < \infty$, the maps χ_n^{O} and χ_n^{Sp} induce isomorphisms of homotopy groups in sufficiently low dimensions.

Throughout this paper we shall keep the notation of [**10**] and [**11**]. In particular, we denote by comm(A, B) the commutator $ABA^{-1}B^{-1}$.

1. Preliminaries. We begin by fixing our notation. Let I_n be the $n \times n$ identity matrix. We put

$$J_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \in \mathrm{SO}(2n),$$

$$T_n = \mathrm{diag}(I_n, -I_n) = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} \in \mathrm{O}(2n),$$

$$K_n = \mathrm{diag}(J_n, -J_n) = \begin{pmatrix} 0 & -I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n \\ 0 & 0 & -I_n & 0 \end{pmatrix} \in \mathrm{SO}(4n),$$

$$S_n = \mathrm{diag}(I_n, J_nT_n, I_n) = \begin{pmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & I_n \end{pmatrix} \in \mathrm{O}(4n).$$

Here $\mathrm{diag}(A_1, A_2, \dots, A_r)$ denotes the square matrix with blocks A_1, $A_2, \dots, A_r$ down the main diagonal and zeroes elsewhere. Also we let $P_n \in \mathrm{O}(2n)$ be the $2n \times 2n$ permutation matrix defined in [**10**; §1]. This matrix represents the transformation

$$(x_1, \dots, x_n, y_1, \dots, y_n) \mapsto (x_1, y_1, \dots, x_n, y_n)\colon \mathbf{R}^{2n} \to \mathbf{R}^{2n}$$

(so that $\det(P_n) = (-1)^{n(n-1)/2}$), and we put

$$Q_n = P_{2n}\mathrm{diag}(P_n, P_n) \in \mathrm{O}(4n), \qquad R_n = P_{4n}\mathrm{diag}(Q_n, Q_n) \in \mathrm{SO}(8n).$$

Further, as in [**10**; §1], we put

$$\mathrm{dec}(X + iY) = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}, \quad \mathrm{deq}(Z + jW) = \begin{pmatrix} Z & -\overline{W} \\ W & \overline{Z} \end{pmatrix}$$

where X, Y are arbitrary $n \times n$ real matrices and Z, W are arbitrary $n \times n$ complex matrices, and where i $(\in \mathbf{C})$ and j are the standard generators of the algebra $\mathbf{H}$ of quaternions.

For brevity, we write $\mathrm{O}(2n)/\mathrm{U} = \mathrm{O}(2n)/\mathrm{U}(n)$, $\mathrm{U}(2n)/\mathrm{Sp} = \mathrm{U}(2n)/\mathrm{Sp}(n)$, $\mathrm{U}(n)/\mathrm{O} = \mathrm{U}(n)/\mathrm{O}(n)$, and $\mathrm{Sp}(n)/\mathrm{U} = \mathrm{Sp}(n)/\mathrm{U}(n)$. Here the spaces $\mathrm{O}(2n)/\mathrm{U}(n)$, $\mathrm{U}(2n)/\mathrm{Sp}(n)$ are defined by using the embeddings

$$A \mapsto P_n \mathrm{dec}(A) P_n^{-1}\colon \mathrm{U}(n) \to \mathrm{O}(2n),$$
$$A \mapsto P_n \mathrm{deq}(A) P_n^{-1}\colon \mathrm{Sp}(n) \to \mathrm{U}(2n)$$

induced by the canonical isomorphisms

$$(x_1 + iy_1, \ldots, x_n + iy_n) \mapsto (x_1, y_1, \ldots, x_n, y_n)\colon \mathbf{C}^n \to \mathbf{R}^{2n},$$
$$(z_1 + jw_1, \ldots, z_n + jw_n) \mapsto (z_1, w_1, \ldots, z_n, w_n)\colon \mathbf{H}^n \to \mathbf{C}^{2n}.$$

We denote by κ_n^{U} the latter embedding $\mathrm{Sp}(n) \to \mathrm{U}(2n)$, by ι_n^{U} the inclusion map $\mathrm{O}(n) \to \mathrm{U}(n)$, and by $\xi_n^{\mathrm{U}/\mathrm{Sp}}$ (resp. by $\xi_n^{\mathrm{U}/\mathrm{O}}$) the obvious projection map from $\mathrm{U}(2n)$ onto $\mathrm{U}(2n)/\mathrm{Sp}$ (resp. from $\mathrm{U}(n)$ onto $\mathrm{U}(n)/\mathrm{O}$).

Let G denote either O or Sp. We further put

$$G(2n)/(G \times G) = G(2n)/P_n \mathrm{diag}(G(n) \times G(n)) P_n^{-1}$$

with $\mathrm{diag}(G(n) \times G(n)) = \{\mathrm{diag}(A, B) \mid A \in G(n), B \in G(n)\} \subset G(2n)$, and write $\xi_n^{G/(G\times G)}$ for the projection map from $G(2n)$ onto $G(2n)/(G \times G)$(cf. [**11**; §1]).

2. Bott maps for the orthogonal and symplectic groups. Here we recall classical results of Bott, which will be used in §5. Let $\Omega(X)$ denote the space of loops on X, and let $\Omega_0(X)$ denote the arcwise-connected component of the trivial loop. Consider the following maps:

$$\omega_n^{\mathrm{O}}\colon \mathrm{O}(2n)/\mathrm{U} \to \Omega(\mathrm{O}(2n)), \quad \omega_n^{\mathrm{O}/\mathrm{U}}\colon \mathrm{U}(2n)/\mathrm{Sp} \to \Omega(\mathrm{O}(4n)/\mathrm{U}),$$
$$\omega_n^{\mathrm{U}/\mathrm{Sp}}\colon \mathrm{Sp}(2n)/(\mathrm{Sp} \times \mathrm{Sp}) \to \Omega_0(\mathrm{U}(4n)/\mathrm{Sp}),$$
$$\omega_n^{\mathrm{Sp}/(\mathrm{Sp}\times\mathrm{Sp})}\colon \mathrm{Sp}(n) \to \Omega(\mathrm{Sp}(2n)/(\mathrm{Sp} \times \mathrm{Sp})),$$
$$\omega_n^{\mathrm{Sp}}\colon \mathrm{Sp}(n)/\mathrm{U} \to \Omega(\mathrm{Sp}(n)), \quad \omega_n^{\mathrm{Sp}/\mathrm{U}}\colon \mathrm{U}(n)/\mathrm{O} \to \Omega(\mathrm{Sp}(n)/\mathrm{U}),$$
$$\omega_n^{\mathrm{U}/\mathrm{O}}\colon \mathrm{O}(2n)/(\mathrm{O} \times \mathrm{O}) \to \Omega_0(\mathrm{U}(2n)/\mathrm{O}),$$
$$\omega_n^{\mathrm{O}/(\mathrm{O}\times\mathrm{O})}\colon \mathrm{O}(n) \to \Omega(\mathrm{O}(2n)/(\mathrm{O} \times \mathrm{O}))$$

where ω_n^{O}, $\omega_n^{\mathrm{O}/\mathrm{U}}$, ω_n^{Sp} and $\omega_n^{\mathrm{Sp}/\mathrm{U}}$ are the maps defined in [**10**; §2], and where the maps $\omega_n^{\mathrm{U}/\mathrm{Sp}}$, $\omega_n^{\mathrm{U}/\mathrm{O}}$, $\omega_n^{\mathrm{O}/(\mathrm{O}\times\mathrm{O})}$ and $\omega_n^{\mathrm{Sp}/(\mathrm{Sp}\times\mathrm{Sp})}$ are defined as follows:

$$\begin{aligned}&\omega_n^{\mathrm{U}/\mathrm{Sp}}\left(\xi_n^{\mathrm{Sp}/(\mathrm{Sp}\times\mathrm{Sp})}\left(P_n A P_n^{-1}\right)\right)(t)\\ &\quad = \xi_{2n}^{\mathrm{U}/\mathrm{Sp}}\left(Q_n S_n \exp\left(\frac{\pi}{2}\, tiT_{2n}\right) S_n \mathrm{deq}(A) S_n \exp\left(-\frac{\pi}{2} tiT_{2n}\right) S_n Q_n^{-1}\right)\end{aligned}$$

where $A \in \mathrm{Sp}(2n)$, $t \in [0,1]$;

$$\omega_n^{\mathrm{U/O}}\left(\xi_n^{\mathrm{O/(O\times O)}}\left(P_n A P_n^{-1}\right)\right)(t) = \xi_{2n}^{\mathrm{U/O}}\left(P_n \exp\left(\frac{\pi}{2} t i T_n\right) A \exp\left(-\frac{\pi}{2} t i T_n\right) P_n^{-1}\right)$$

where $A \in \mathrm{O}(2n)$, $t \in [0,1]$;

$$\omega_n^{G/(G\times G)}(A)(t) = \xi_n^{G/(G\times G)}\left(P_n \exp\left(\frac{\pi}{2} t J_n\right) \operatorname{diag}(A, I_n) \exp\left(-\frac{\pi}{2} t J_n\right) P_n^{-1}\right)$$

where $A \in G(n)$, $t \in [0,1]$, and $G = \mathrm{O}$ or Sp as in §1. Then the direct limit maps

$$\omega_\infty^{\mathrm{O}}\colon \mathrm{O}(\infty)/\mathrm{U} \to \Omega(\mathrm{O}(\infty)),\ \omega_\infty^{\mathrm{O/U}}\colon \mathrm{U}(\infty)/\mathrm{Sp} \to \Omega(\mathrm{O}(\infty)/\mathrm{U}), \text{etc.},$$

where we have put $\omega_\infty^{\mathrm{O}} = \varinjlim \omega_n^{\mathrm{O}}$, $\mathrm{O}(\infty)/\mathrm{U} = \varinjlim \mathrm{O}(2n)/\mathrm{U}$, etc., are defined in the usual way,[1] and the Bott periodicity theorems for the orthogonal and symplectic groups are immediate consequences of the following:

THEOREM (2.1) (*see* [**2**], [**3**], [**4**], [**5**], *and also* [**8**; §24]). *The maps* $\omega_\infty^{\mathrm{O}}$, $\omega_\infty^{\mathrm{O/U}}$, $\omega_\infty^{\mathrm{U/Sp}}$, $\omega_\infty^{\mathrm{Sp/(Sp\times Sp)}}$, $\omega_\infty^{\mathrm{Sp}}$, $\omega_\infty^{\mathrm{Sp/U}}$, $\omega_\infty^{\mathrm{U/O}}$ *and* $\omega_\infty^{\mathrm{O/(O\times O)}}$ *are homotopy equivalences.*

3. The maps $\nu_n^{\mathrm{U/Sp}}$ and $\nu_n^{\mathrm{U/O}}$. For later use, we define here the maps $\nu_n^{\mathrm{U/Sp}}\colon \mathrm{U}(2n)/\mathrm{Sp} \to \mathrm{U}(4n)/\mathrm{Sp}$ and $\nu_n^{\mathrm{U/O}}\colon \mathrm{U}(n)/\mathrm{O} \to \mathrm{U}(2n)/\mathrm{O}$ as follows:

$$\nu_n^{\mathrm{U/Sp}}\left(\xi_n^{\mathrm{U/Sp}}\left(P_n A P_n^{-1}\right)\right) = \xi_{2n}^{\mathrm{U/Sp}}\left(Q_n S_n \operatorname{diag}(A, I_{2n}) S_n Q_n^{-1}\right) \quad \text{for } A \in \mathrm{U}(2n);$$

$$\nu_n^{\mathrm{U/O}}\left(\xi_n^{\mathrm{U/O}}(A)\right) = \xi_{2n}^{\mathrm{U/O}}\left(P_n \operatorname{diag}(A, I_n) P_n^{-1}\right) \quad \text{for } A \in \mathrm{U}(n).$$

Consider now the direct limits $\nu_\infty^{\mathrm{U/Sp}} = \varinjlim \nu_n^{\mathrm{U/Sp}}$ and $\nu_\infty^{\mathrm{U/O}} = \varinjlim \nu_n^{\mathrm{U/O}}$. Then by an elementary argument used in [**5**; §1], we can see:

LEMMA (3.1). *The map* $\nu_\infty^{\mathrm{U/Sp}}$ (*resp.* $\nu_\infty^{\mathrm{U/O}}$) *is homotopic to the identity map of* $\mathrm{U}(\infty)/\mathrm{Sp}$ (*resp. of* $\mathrm{U}(\infty)/\mathrm{O}$).

For a proof, see Appendix 1. An immediate consequence of this lemma is that $\nu_\infty^{\mathrm{U/Sp}}$ and $\nu_\infty^{\mathrm{U/O}}$ are homotopy (self-) equivalences. We shall use this fact in §5.

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[1]Strictly speaking, for example $\omega_\infty^{\mathrm{O}}$ is defined as the composition of the dierct limit map $\varinjlim \omega_n^{\mathrm{O}}\colon \varinjlim \mathrm{O}(2n)/\mathrm{U} \to \varinjlim \Omega(\mathrm{O}(2n))$ and the canonical bijection $\varinjlim \Omega(\mathrm{O}(2n)) \to \Omega(\varinjlim \mathrm{O}(2n))$. But here and throughout we simply write $\omega_\infty^{\mathrm{O}} = \varinjlim \omega_n^{\mathrm{O}}$, etc., by abuse of notation.

4. Definition of the maps χ_n^{O} and χ_n^{Sp}. We continue to use the notation of §1. For each $(z_0, z_1, z_2) \in \mathbf{C}^3$, let us now put

$$L_n(z_1, z_2) = \operatorname{diag}(z_1 I_n, \bar{z}_1 I_n) + z_2 i T_n J_n = \begin{pmatrix} z_1 I_n & -z_2 i I_n \\ -z_2 i I_n & \bar{z}_1 I_n \end{pmatrix},$$

$$M_n(z_0, z_1, z_2) = \operatorname{dec}(S_n z_0 I_{4n} S_n) + K_{2n} \operatorname{dec}(S_n L_{2n}(z_1, z_2) S_n)$$

$$= \left(\begin{array}{cc|cc|cc|cc}
x_0 I_n & 0 & -x_1 I_n & -y_2 I_n & -y_0 I_n & 0 & y_1 I_n & -x_2 I_n \\
0 & x_0 I_n & -y_2 I_n & -x_1 I_n & 0 & -y_0 I_n & -x_2 I_n & -y_1 I_n \\
\hline
x_1 I_n & y_2 I_n & x_0 I_n & 0 & -y_1 I_n & x_2 I_n & -y_0 I_n & 0 \\
y_2 I_n & x_1 I_n & 0 & x_0 I_n & x_2 I_n & y_1 I_n & 0 & -y_0 I_n \\
\hline
y_0 I_n & 0 & y_1 I_n & -x_2 I_n & x_0 I_n & 0 & x_1 I_n & y_2 I_n \\
0 & y_0 I_n & -x_2 I_n & -y_1 I_n & 0 & x_0 I_n & y_2 I_n & x_1 I_n \\
\hline
-y_1 I_n & x_2 I_n & y_0 I_n & 0 & -x_1 I_n & -y_2 I_n & x_0 I_n & 0 \\
x_2 I_n & y_1 I_n & 0 & y_0 I_n & -y_2 I_n & -x_1 I_n & 0 & x_0 I_n
\end{array}\right),$$

$$N_n(z_0, z_1, z_2) = z_0 I_{2n} + j L_n(z_1, z_2) = \begin{pmatrix} (z_0 + j z_1) I_n & i j z_2 I_n \\ i j z_2 I_n & (z_0 + j \bar{z}_1) I_n \end{pmatrix}$$

with $z_r = x_r + i y_r$, $x_r \in \mathbf{R}$, $y_r \in \mathbf{R}$ $(r = 0, 1, 2)$, and consider the unit 4-sphere

$$\mathbf{S}(\mathbf{C}^2 \times \mathbf{R}) = \{(w_0, w_1, w_2) \in \mathbf{S}(\mathbf{C}^3) \mid w_2 \in \mathbf{R}\},$$

where

$$\mathbf{S}(\mathbf{C}^3) = \{(w_0, w_1, w_2) \in \mathbf{C}^3 \mid |w_0|^2 + |w_1|^2 + |w_2|^2 = 1\}.$$

Then we can see by elementary calculations that

$$M_n(w_0, w_1, w_2) \in \mathrm{O}(8n) \quad \text{and} \quad N_n(w_0, w_1, w_2) \in \mathrm{Sp}(2n)$$

for all $(w_0, w_1, w_2) \in \mathbf{S}(\mathbf{C}^2 \times \mathbf{R})$. Bearing this in mind, we define the maps χ_n^{O} and χ_n^{Sp} mentioned in the introduction, as follows:

If $(w_0, w_1, w_2) \in \mathbf{S}(\mathbf{C}^2 \times \mathbf{R})$, then we put

$$\chi_n^{\mathrm{O}}(P_n A P_n^{-1})([w_0 : w_1 : w_2]) = R_n \operatorname{comm}(M_n(w_0, w_1, w_2), \operatorname{dec}(S_n \operatorname{diag}(A, I_{2n}) S_n)) R_n^{-1}$$

for $A \in \mathrm{U}(2n)$, and

$$\chi_n^{\mathrm{Sp}}(A)([w_0 : w_1 : w_2]) = P_n \operatorname{comm}(N_n(w_0, w_1, w_2), \operatorname{diag}(A, I_n)) P_n^{-1}$$

for $A \in \mathrm{U}(n)$. If $(w_0, w_1, w_2) \in \mathbf{S}(\mathbf{C}^3)$ and $w_2 \neq 0$, then we put

$$\chi_n^{\mathrm{O}}(P_n A P_n^{-1})([w_0 : w_1 : w_2]) = \chi_n^{\mathrm{O}}(P_n A P_n^{-1})([w_0 \bar{w}_2/|w_2| : w_1 \bar{w}_2/|w_2| : |w_2|])$$

for $A \in \mathrm{U}(2n)$, and

$$\chi_n^{\mathrm{Sp}}(A)([w_0 : w_1 : w_2]) = \chi_n^{\mathrm{Sp}}(A)([w_0\bar{w}_2/|w_2| : w_1\bar{w}_2/|w_2| : |w_2|])$$

for $A \in \mathrm{U}(n)$. Here $[w_0 : w_1 : w_2]$ denotes the point of $\mathbf{P}_2\mathbf{C}$ corresponding to $(w_0, w_1, w_2) \in \mathbf{S}(\mathbf{C}^3)$.

We leave it to the reader to check that χ_n^{O} and χ_n^{Sp} are well-defined.

5. The main theorem. As before let $\tilde{\mathscr{C}}(X; Y)$ denote the space of based maps $X \to Y$. Henceforth we use the following conventions (see also Appendix 2):

(1) Let $\mathbf{P}_1\mathbf{C} = \{[z_0 : z_1] \mid (z_0, z_1) \in \mathbf{C}^2,\ (z_0, z_1) \neq (0,0)\}$ be the complex projective line. Then each element f of $\tilde{\mathscr{C}}(\mathbf{P}_1\mathbf{C}; Y)$ is regarded as an element of $\Omega^2(Y) = \Omega(\Omega(Y))$ by putting

$$f(u)(v) = f([\cos(\pi v) + i\sin(\pi v)\cos(\pi u) : \sin(\pi v)\sin(\pi u)])$$

for $u, v \in [0,1]$. In this way we identify $\tilde{\mathscr{C}}(\mathbf{P}_1\mathbf{C}; Y)$ with the double loop space of Y.

(2) Also we identify $\tilde{\mathscr{C}}(\mathbf{P}_2\mathbf{C}/\mathbf{P}_1\mathbf{C}; Y)$ with the 4th iterated loop space of Y in the following way: Let q: $\mathbf{P}_2\mathbf{C} \to \mathbf{P}_2\mathbf{C}/\mathbf{P}_1\mathbf{C}$ be the canonical map, and let

$$(*)\quad \begin{cases} w_0(u, v) = \cos(\pi v) + i\sin(\pi v)\cos(\pi u), \\ w_1(s, t, u, v) = \sin(\pi v)\sin(\pi u)(\cos(\pi t) + i\sin(\pi t)\cos(\pi s)), \\ w_2(s, t, u, v) = \sin(\pi v)\sin(\pi u)\sin(\pi t)\sin(\pi s). \end{cases}$$

Then each $g \in \tilde{\mathscr{C}}(\mathbf{P}_2\mathbf{C}/\mathbf{P}_1\mathbf{C}; Y)$ is regarded as an element of $\Omega^4(Y)$ by

$$g(s)(t)(u)(v) = g(q([w_0(u, v) : w_1(s, t, u, v) : w_2(s, t, u, v)])).$$

With these understood, consider now the diagrams

$$\begin{tikzcd}
 & \mathrm{Sp}(n) \arrow[rrr, "\kappa_n^{\mathrm{U}}"] \arrow[d, "\omega_n^{\mathrm{Sp}/(\mathrm{Sp}\times\mathrm{Sp})}"] & & {} \arrow[ddddd, "n\chi_n^{\mathrm{O}}"] & \mathrm{U}(2n) \arrow[r, "\xi_n^{\mathrm{U}/\mathrm{Sp}}"] & \mathrm{U}(2n)/\mathrm{Sp} \arrow[d, "\nu_n^{\mathrm{U}/\mathrm{Sp}}"] \\
 & \Omega(\mathrm{Sp}(2n)/(\mathrm{Sp}\times\mathrm{Sp})) \arrow[d, "\Omega(\omega_n^{\mathrm{U}/\mathrm{Sp}})"] & & & & \mathrm{U}(4n)/\mathrm{Sp} \arrow[dd, "\omega_{2n}^{\mathrm{O}/\mathrm{U}}"] \\
 & \Omega^2(\mathrm{U}(4n)/\mathrm{Sp}) \arrow[d, "\Omega^2(\omega_{2n}^{\mathrm{O}/\mathrm{U}})"] & & & & \\
(5.1)_n & \Omega^3(\mathrm{O}(8n)/\mathrm{U}) \arrow[d, "\Omega^3(\omega_{4n}^{\mathrm{O}})"] & (5.1\mathrm{a}) & & (5.1\mathrm{b})_n & \Omega(\mathrm{O}(8n)/\mathrm{U}) \arrow[d, "\Omega(\omega_{4n}^{\mathrm{O}})"] \\
 & \Omega^4(\mathrm{O}(8n)) \arrow[d, equals] & & & & \Omega^2(\mathrm{O}(8n)) \arrow[d, equals] \\
 & \tilde{\mathscr{C}}(\mathbf{P}_2\mathbf{C}/\mathbf{P}_1\mathbf{C}; \mathrm{O}(8n)) \arrow[rr] & & \tilde{\mathscr{C}}(\mathbf{P}_2\mathbf{C}; \mathrm{O}(8n)) \arrow[rr] & & \tilde{\mathscr{C}}(\mathbf{P}_1\mathbf{C}; \mathrm{O}(8n))
\end{tikzcd}$$

and

$$(5.2)_n \qquad \begin{tikzcd}
O(n) \arrow[r, "\iota_n^{U}"] \arrow[d, "\omega_n^{O/(O\times O)}"] & U(n) \arrow[ddddd, "{}_n\chi_n^{Sp}"] \arrow[r, "\xi_n^{U/O}"] & U(n)/O \arrow[d, "\nu_n^{U/O}"] \\
\Omega(O(2n)/(O\times O)) \arrow[d, "\Omega(\omega_n^{U/O})"] & & U(2n)/O \arrow[dd, "\omega_{2n}^{Sp/U}"] \\
\Omega^2(U(2n)/O) \arrow[d, "\Omega^2(\omega_{2n}^{Sp/U})"] \arrow[r, phantom, "(5.2a)"] & \arrow[r, phantom, "(5.2b)_n"] & \\
\Omega^3(Sp(2n)/U) \arrow[d, "\Omega^3(\omega_{2n}^{Sp})"] & & \Omega(Sp(2n)/U) \arrow[d, "\Omega(\omega_{2n}^{Sp})"] \\
\Omega^4(Sp(2n)) \arrow[d, equal] & & \Omega^2(Sp(2n)) \arrow[d, equal] \\
\widetilde{\mathscr{C}}(\mathbf{P}_2\mathbf{C}/\mathbf{P}_1\mathbf{C}; Sp(2n)) \arrow[r] & \widetilde{\mathscr{C}}(\mathbf{P}_2\mathbf{C}; Sp(2n)) \arrow[r] & \widetilde{\mathscr{C}}(\mathbf{P}_1\mathbf{C}; Sp(2n))
\end{tikzcd}$$

where the labelled maps are as defined before and the bottom rows are induced by the obvious cofibration $\mathbf{P}_2\mathbf{C}/\mathbf{P}_1\mathbf{C} \leftarrow \mathbf{P}_2\mathbf{C} \leftarrow \mathbf{P}_1\mathbf{C}$. Taking the direct limits and writing $\chi_\infty^{O} = \varinjlim \chi_n^{O}$, $\chi_\infty^{Sp} = \varinjlim \chi_n^{Sp}$, etc., we then get the diagrams $(5.1)_n$ and $(5.2)_n$ for $n = \infty$, in which all rows are (Hurewicz) fibration sequences.

PROPOSITION (5.3). *The diagrams* $(5.1)_n$ *and* $(5.2)_n$ *for* $n \leq \infty$ *are homotopy-commutative.*

This will be proved in §6, the next section. Our main theorem is the following, which is a refinement of Theorem (0.1):

THEOREM (5.4). *The maps* χ_∞^{O} *and* χ_∞^{Sp} *are homotopy equivalences, and*:

(i) *the homomorphism* $(\chi_n^{O})_*\colon \pi_r(U(2n)) \to \pi_r(\widetilde{\mathscr{C}}(\mathbf{P}_2\mathbf{C}; O(8n)))$ *induced by* χ_n^{O} *is isomorphic for* $r \leq 4n-1$ *with* $(r,n) \neq (3,1)$;

(ii) *the homomorphism* $(\chi_n^{Sp})_*\colon \pi_r(U(n)) \to \pi_r(\widetilde{\mathscr{C}}(\mathbf{P}_2\mathbf{C}; Sp(2n)))$ *induced by* χ_n^{Sp} *is isomorphic for* $r \leq 2n-1$.

Proof. The part for $n = \infty$ is obtained by an easy five-lemma argument: Combining Theorem (2.1), Lemma (3.1) and Proposition (5.3), and noting J. H. C. Whitehead's theorem (and Theorem 3 of [7]), we see that χ_∞^{O} and χ_∞^{Sp} are homotopy equivalences.

The remaining part is proved as follows.[2] Consider the commutative diagram

$$\begin{array}{ccc} \pi_r(\mathrm{U}(\infty)) & \xrightarrow{(\chi^{\mathrm{O}}_\infty)_*} & \pi_r(\tilde{\mathscr{C}}(\mathbf{P}_2\mathbf{C};\mathrm{O}(\infty))) \\ \uparrow & & \uparrow \\ \pi_r(\mathrm{U}(2n)) & \xrightarrow{(\chi^{\mathrm{O}}_n)_*} & \pi_r(\tilde{\mathscr{C}}(\mathbf{P}_2\mathbf{C};\mathrm{O}(8n))) \end{array}$$

where the verticals are the canonical homomorphisms. Then the left-hand vertical is an isomorphism for $r \le 4n-1$, while the right-hand vertical is an isomorphism for $r \le 8n-6$. (Note that $(\mathrm{O}(\infty), \mathrm{O}(8n))$ is $(8n-1)$-connected.) Hence (i) follows. The assertion (ii) can be verified analogously.

REMARK. One can easily check that for $(r, n) = (3, 1)$ the homomorphism $(\chi^{\mathrm{O}}_1)_*\colon \pi_3(\mathrm{U}(2)) \to \pi_3(\tilde{\mathscr{C}}(\mathbf{P}_2\mathbf{C};\mathrm{O}(8)))$ is monomorphic but not epimorphic.

6. Proof of Proposition (5.3). First we shall show that the subdiagrams $(5.1\mathrm{b})_n$ and $(5.2\mathrm{b})_n$ are homotopy-commutative. For this, consider the maps

$\Theta^{\mathrm{O}}_{2n}(r)\colon \mathrm{U}(4n)/\mathrm{Sp} \to \Omega^2(\mathrm{O}(8n))$ and $\Theta^{\mathrm{Sp}}_{2n}(r)\colon \mathrm{U}(2n)/\mathrm{O} \to \Omega^2(\mathrm{Sp}(2n))$

defined in [**10**; §4], where $r \in [0, 1]$. If in $(5.1\mathrm{b})_n$ and $(5.2\mathrm{b})_n$ we replace the map

$$\Omega\left(\omega^{\mathrm{O}}_{4n}\right)\circ\omega^{\mathrm{O/U}}_{2n}\colon \mathrm{U}(4n)/\mathrm{Sp} \to \Omega(\mathrm{O}(8n)/\mathrm{U}) \to \Omega^2(\mathrm{O}(8n))$$

by $\Theta^{\mathrm{O}}_{2n}(0)$ and the map

$$\Omega\left(\omega^{\mathrm{Sp}}_{2n}\right)\circ\omega^{\mathrm{Sp/U}}_{2n}\colon \mathrm{U}(2n)/\mathrm{O} \to \Omega(\mathrm{Sp}(2n)/\mathrm{U}) \to \Omega^2(\mathrm{Sp}(2n))$$

by $\Theta^{\mathrm{Sp}}_{2n}(0)$ respectively, then the resulting diagrams are strictly commutative, as seen by direct calculations. On the other hand, as mentioned in [**10**; §4], we have

$$\Theta^{\mathrm{O}}_{2n}(1) = \Omega\left(\omega^{\mathrm{O}}_{4n}\right)\circ\omega^{\mathrm{O/U}}_{2n} \quad \text{and} \quad \Theta^{\mathrm{Sp}}_{2n}(1) = \Omega\left(\omega^{\mathrm{Sp}}_{2n}\right)\circ\omega^{\mathrm{Sp/U}}_{2n}.$$

Hence the homotopy-commutativity of $(5.1\mathrm{b})_n$ and $(5.2\mathrm{b})_n$ for $n < \infty$ follows, and considering the direct limits $\Theta^{\mathrm{O}}_\infty(r)$ and $\Theta^{\mathrm{Sp}}_\infty(r)$, we see that $(5.1\mathrm{b})_\infty$ and $(5.2\mathrm{b})_\infty$ are also homotopy-commutative.

[2] This proof was communicated to the author by S. Oka.

Next we shall prove the homotopy-commutativity of $(5.1\mathrm{a})_n$ and $(5.2\mathrm{a})_n$. For $r, s, t, u, v \in [0,1]$, let

$$F_{2n}(r,u,v) \in \mathrm{O}(8n) \quad \text{and} \quad G_{2n}(r,u,v) \in \mathrm{Sp}(2n)$$

be as defined in [**10**; §4], and put

$$V_n(s,t,u) = \exp\left(\frac{\pi}{2}uK_{2n}\right)\mathrm{dec}\left(S_n\exp\left(\frac{\pi}{2}tiT_{2n}\right)\exp\left(\frac{\pi}{2}sJ_{2n}\right)S_n\right) \in \mathrm{O}(8n),$$
$$W_n(s,t,u) = \exp\left(\frac{\pi}{2}ujI_{2n}\right)\exp\left(\frac{\pi}{2}tiT_n\right)\exp\left(\frac{\pi}{2}sJ_n\right) \in \mathrm{Sp}(2n).$$

Further, put $V_n(s,t) = V_n(s,t,0)$, $W_n(s,t) = W_n(s,t,0)$, and define the maps

$$\Pi_n^{\mathrm{O}}(r)\colon \mathrm{Sp}(n) \to \Omega^4(\mathrm{O}(8n)) \quad \text{and} \quad \Pi_n^{\mathrm{Sp}}(r)\colon \mathrm{O}(n) \to \Omega^4(\mathrm{Sp}(2n))$$

for each $r \in [0,1]$, as follows:

$$\begin{aligned}&\Pi_n^{\mathrm{O}}(r)(A)(s)(t)(u)(v)\\&\quad = R_nV_n(rs,rt,ru)C_n(A;r,s,t,u,v)(V_n(rs,rt,ru))^{-1}R_n^{-1}\end{aligned}$$

where $A \in \mathrm{Sp}(n)$ and

$$\begin{aligned}&C_n(A;r,s,t,u,v)\\&\quad = \mathrm{comm}\left((V_n(s,t))^{-1}F_{2n}(r,u,v)V_n(s,t), \mathrm{dec}(S_n\mathrm{diag}(\mathrm{deq}(A), I_{2n})S_n)\right);\end{aligned}$$

$$\begin{aligned}&\Pi_n^{\mathrm{Sp}}(r)(A)(s)(t)(u)(v)\\&\quad = P_nW_n(rs,rt,ru)D_n(A;r,s,t,u,v)(W_n(rs,rt,ru))^{-1}P_n^{-1}\end{aligned}$$

where $A \in \mathrm{O}(n)$ and

$$\begin{aligned}&D_n(A;r,s,t,u,v)\\&\quad = \mathrm{comm}\left((W_n(s,t))^{-1}G_{2n}(r,u,v)W_n(s,t), \mathrm{diag}(A, I_n)\right).\end{aligned}$$

Then for $r = 0$, we have

$$F_{2n}(0,u,v) = I_{8n}\cos(\pi v) + J_{4n}\sin(\pi v)\cos(\pi u) + K_{2n}\sin(\pi v)\sin(\pi u),$$
$$G_{2n}(0,u,v) = I_{2n}\cos(\pi v) + iI_{2n}\sin(\pi v)\cos(\pi u) + jI_{2n}\sin(\pi v)\sin(\pi u),$$

and calculations show that

$$\begin{aligned}&(V_n(s,t))^{-1}F_{2n}(0,u,v)V_n(s,t)\\&\quad = M_n(w_0(u,v), w_1(s,t,u,v), w_2(s,t,u,v)),\end{aligned}$$

$$\begin{aligned}&(W_n(s,t))^{-1}G_{2n}(0,u,v)W_n(s,t)\\&\quad = N_n(w_0(u,v), w_1(s,t,u,v), w_2(s,t,u,v))\end{aligned}$$

where $w_0(u,v)$, $w_1(s,t,u,v)$ and $w_2(s,t,u,v)$ are given by the formulae $(*)$ at the beginning of §5 and where $M_n(z_0,z_1,z_2)$ and $N_n(z_0,z_1,z_2)$ are as defined in §4. Hence we see that the map $\chi_n^{\mathrm{O}} \circ \kappa_n^{\mathrm{U}}$ is just the composite map

$$\mathrm{Sp}(n) \overset{\Pi_n^{\mathrm{O}}(0)}{\to} \Omega^4(\mathrm{O}(8n)) = \tilde{\mathscr{C}}(\mathbf{P}_2\mathbf{C}/\mathbf{P}_1\mathbf{C}; \mathrm{O}(8n)) \to \tilde{\mathscr{C}}(\mathbf{P}_2\mathbf{C}; \mathrm{O}(8n))$$

and the map $\chi_n^{\mathrm{Sp}} \circ \iota_n^{\mathrm{U}}$ is equal to the composition

$$\mathrm{O}(n) \overset{\Pi_n^{\mathrm{Sp}}(0)}{\to} \Omega^4(\mathrm{Sp}(2n)) = \tilde{\mathscr{C}}(\mathbf{P}_2\mathbf{C}/\mathbf{P}_1\mathbf{C}; \mathrm{Sp}(2n)) \to \tilde{\mathscr{C}}(\mathbf{P}_2\mathbf{C}; \mathrm{Sp}(2n))$$

(where the unlabelled arrows are the maps induced by the canonical surjection $\mathbf{P}_2\mathbf{C} \to \mathbf{P}_2\mathbf{C}/\mathbf{P}_1\mathbf{C}$). Also, noting the equalities

$$\begin{aligned}
&(V_n(s,t))^{-1} F_{2n}(1,u,v) V_n(s,t) \\
&\qquad = (V_n(s,t,u))^{-1} \exp(\pi v J_{4n}) V_n(s,t,u), \\
&(W_n(s,t))^{-1} G_{2n}(1,u,v) W_n(s,t) \\
&\qquad = (W_n(s,t,u))^{-1} \exp(\pi v i I_{2n}) W_n(s,t,u),
\end{aligned}$$

we see by calculations that

$$\Pi_n^{\mathrm{O}}(1) = \Omega^3(\omega_{4n}^{\mathrm{O}}) \circ \Omega^2(\omega_{2n}^{\mathrm{O/U}}) \circ \Omega(\omega_n^{\mathrm{U/Sp}}) \circ \omega_n^{\mathrm{Sp/(Sp\times Sp)}},$$

$$\Pi_n^{\mathrm{Sp}}(1) = \Omega^3(\omega_{2n}^{\mathrm{Sp}}) \circ \Omega^2(\omega_{2n}^{\mathrm{Sp/U}}) \circ \Omega(\omega_n^{\mathrm{U/O}}) \circ \omega_n^{\mathrm{O/(O\times O)}}.$$

Hence the homotopy-commutativity of $(5.1\mathrm{a})_n$ and $(5.2\mathrm{a})_n$ for $n < \infty$ is clear, and considering $\Pi_\infty^{\mathrm{O}}(r)$ and $\Pi_\infty^{\mathrm{Sp}}(r)$, we conclude that $(5.1\mathrm{a})_\infty$ and $(5.2\mathrm{a})_\infty$ are also homotopy-commutative.

Appendix 1. Proof of Lemma (3.1). For completeness we record a proof of (3.1) here.[3] First, choose a path Λ_n: $[0,1] \to \mathrm{SO}(n+2)$ for each n so that $\Lambda_n(0) = I_{n+2}$ and $\Lambda_n(1)$ is the permutation matrix associated to the 3-cycle: $1 \mapsto n+1$, $n+1 \mapsto n+2$, $n+2 \mapsto 1$. Further, define $\Gamma_n(t) \in \mathrm{SO}(2n)$ inductively by

$$\Gamma_1(t) = I_2 \quad \text{and} \quad \Gamma_{n+1}(t) = \mathrm{diag}(\Gamma_n(t), I_2)\,\mathrm{diag}(I_n, \Lambda_n(t)),$$

where $t \in [0,1]$. Note that $\Gamma_n(1)$ is a $2n \times 2n$ permutation matrix and the corresponding permutation takes r to $2r-1$ for $1 \le r \le n$.

[3] The author learned the techniques of this proof from Chapter 4, §3 of the following book: H. Toda and M. Mimura, The topology of Lie groups (Japanese), Vol. 1, Kinokuniya Sûgaku Sôsho 14-A, Kinokuniya Book-Store, Tokyo, 1978.

It is now easy to see that $\nu_\infty^{U/O}$ is homotopic to the identity map: Consider the family of maps

$$A \mapsto \Gamma_n(t)\,\mathrm{diag}(A, I_n)(\Gamma_n(t))^{-1}\colon \mathrm{U}(n) \to \mathrm{U}(2n) \qquad (t \in [0,1]).$$

By passage to the quotients, these induce maps $\mathrm{U}(n)/\mathrm{O} \to \mathrm{U}(2n)/\mathrm{O}$, and then, since $\Gamma_n(1)\,\mathrm{diag}(A, I_n)(\Gamma_n(1))^{-1} = P_n \mathrm{diag}(A, I_n)P_n^{-1}$ and $\Gamma_n(0) = I_{2n}$, we get a homotopy between $\nu_n^{U/O}$ and the canonical injection $\mathrm{U}(n)/\mathrm{O} \to \mathrm{U}(2n)/\mathrm{O}$ for each n. Taking the direct limit, we get the required homotopy.

Replacing $\mathrm{U}(n)/\mathrm{O}$ by $\mathrm{U}(2n)/\mathrm{Sp}$, and $\Gamma_n(t)$ by the Kronecker product of $\Gamma_n(t)$ and I_2, we can see by the same type of argument that $\nu_\infty^{U/Sp}$ is homotopic to the identity. We leave further details to the reader.

Appendix 2. Note on the conventions mentioned in §5. For brevity we let $I = [0,1]$ here. Let $\mathbf{P}_n\mathbf{C}$ be the n-dimensional complex projective space, and let Y be an arbitrary based space. In §5, we have identified the space $\tilde{\mathscr{C}}(\mathbf{P}_1\mathbf{C}; Y)$ with $\Omega^2(Y)$ and the space $\tilde{\mathscr{C}}(\mathbf{P}_2\mathbf{C}/\mathbf{P}_1\mathbf{C}; Y)$ with $\Omega^4(Y)$. These identifications are based on the following observations:

(1) Let $\mathbf{P}_m\mathbf{R}$ be the m-dimensional real projective space, and put

$$u_0 = \cos(\pi t_1), \quad u_m = \sin(\pi t_1)\sin(\pi t_2)\cdots\sin(\pi t_{m-1})\sin(\pi t_m),$$

$$u_r = \sin(\pi t_1)\sin(\pi t_2)\cdots\sin(\pi t_r)\cos(\pi t_{r+1}) \qquad (1 \le r \le m-1).$$

Then the map $(t_1, t_2, \ldots, t_m) \mapsto [u_0 : u_1 : \cdots : u_m]$ from I^m to $\mathbf{P}_m\mathbf{R}$ defines, by passage to the quotient, a homeomorphism from $I^m/\partial I^m$ to $\mathbf{P}_m\mathbf{R}/\mathbf{P}_{m-1}\mathbf{R}$ (where ∂I^m is the boundary of I^m).

(2) Put $z_r = x_r + iy_r$ $(0 \le r \le n)$. Then the map

$$[x_0 : y_0 : x_1 : y_1 : \cdots : x_n : y_n] \mapsto [z_0 : z_1 : \cdots : z_n]$$

from $\mathbf{P}_{2n+1}\mathbf{R}$ to $\mathbf{P}_n\mathbf{C}$ defines, by restriction and by passage to the quotient, a homeomorphism from $\mathbf{P}_{2n}\mathbf{R}/\mathbf{P}_{2n-1}\mathbf{R}$ to $\mathbf{P}_n\mathbf{C}/\mathbf{P}_{n-1}\mathbf{C}$.

Combining (1) and (2) and taking $m = 2n$, we thus get a homeomorphism from $I^{2n}/\partial I^{2n}$ to $\mathbf{P}_n\mathbf{C}/\mathbf{P}_{n-1}\mathbf{C}$, and hence a homeomorphism from $\tilde{\mathscr{C}}(\mathbf{P}_n\mathbf{C}/\mathbf{P}_{n-1}\mathbf{C}; Y)$ to $\Omega^{2n}(Y)$.

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