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# FIXED POINTS OF $S^1$-FIBRATIONS

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**Let $M$ be a $S^1$-fibration over a space $B$ and $f\colon M \to M$ a map over $B$. We give some results when $f$ can be deformed over $B$ to a fixed point free map. When the fibration is principal then we compute $\check{H}^{n-1}(\mathrm{Fix}(f), k)$ where $n = \dim M$ and we find $g$ homotopic to $f$ over $B$ which minimize the fixed points.**

**Introduction.** In [1] or [2], A. Dold defined a fixed point index for fibre-preserving maps, i.e. for every map $f\colon U \subset E \to E$ which commutes with the projection $p\colon E \to B$ he defines an index $I(f)$ s.t. $I(f) \neq 0$ implies that every map $g$ homotopic to $f$ through a fibre-preserving homotopy has at least one fixed point. (We call fibre-preserving homotopy a homotopy over $B$). From [1] one can see that this index is not easy to compute even in the case where the fibration is

$$S^1 \times S^1 \xrightarrow{p} S^1,$$

$p$ is the projection in the first coordinate and $f(x, y) = (x, xy)$. The purpose of this paper is to study the fixed point of a fibre-preserving map $f\colon M \to M$ where $M$ is a $S^1$-fibration over a space $B$ and $M, B$ are compact manifolds without boundary.

The paper is divided in 3 parts: In Part I we give a criterion, in terms of the fundamental group, for $f$ to be deformed over $B$ to a fixed point free map. This is Proposition 1.3. Some corollaries of this result are given. In Part II we look at orientable $S^1$-fibrations. We give a lower bound for the number of Nielsen classes over $B$ of $f$ as well as the topological dimension of this class. This is Theorem 2.5.

In Part III we state the question of realizing a homotopy class over $B$ by a map $f$ s.t. $\mathrm{Fix}(f)$ = the set of fixed points of $f$ is minimal in the sense we will describe. We will answer this question in the case where the fibration and the total space are orientable. This is Theorem 3.7.

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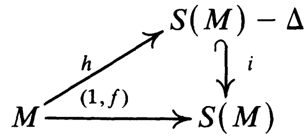
**Part I. Detecting fixed points.** Let

$$\begin{array}{ccc} S^1 & \to & M \\ & & \downarrow \\ & & B \end{array}$$

be a $S^1$-fibration over $B$ where $M, B$ are compact manifolds without boundary and $f\colon M \to M$ be a map over $B$ i.e. $p \circ f = p$. For the study of the category of spaces over $B$ see [**1**] and [**2**]. By a deformation over $B$ we mean a fibre-preserving homotopy. Let $M \times_B M$ be the fibre square which we denote by $S(M)$ and $\Delta$ the diagonal in $S(M)$. Now we will recall Proposition 2.4 of [**3**].

PROPOSITION 1.1. *The map $f$ can be deformed over $B$ to a fixed point free map if and only if there is a map $h\colon M \to S(M) - \Delta$ which makes the diagram below commutative up to homotopy.*



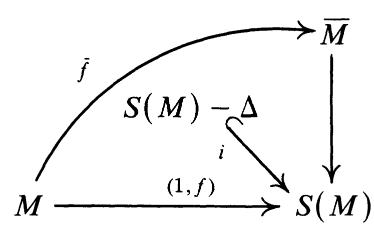

In [**3**], they consider fibrations $F \to M \to B$ where $F$ is a manifold of dimension greater than or equal to three. Under this hypothesis they show that the homotopy fibre of the inclusion $i\colon M \times M - \Delta \to M \times M$ is at least 1-connected. Therefore by general obstruction theory we can always lift the map $(1, f)$ over the 2-skeleton of $M$ and the obstructions to lift over the higher dimensional skeletons are cohomology classes. On the other hand if $F$ is the circle $S^1$ or a 2-dimensional surface, different from $S^2$ or $RP^2$, the obstructions to lift $(1, f)$ are no longer cohomology classes. In these cases the problem of lifting $(1, f)$ can be treated in terms of $\Pi_1$. The case where $F$ is a 2-dimensional surface is much more complicated than the case where $F = S^1$. We return to the case $F = S^1$.

Let $x_0 \in M$ be a base point of $M$ and let us assume that $f(x_0) \neq x_0$. Denote $(x_0, f(x_0))$ the base point of $S(M) - \Delta$ and $S(M)$.

PROPOSITION 1.2. *The map h exists if and only if*

$$(1, f)_\#(\Pi_1(M, x_0)) = i_\#(\Pi_1(S(M) - \Delta; (x_0, f(x_0)))).$$

*Proof*. Let $\overline{M}$ be the covering space of $S(M)$ which corresponds to the subgroup $(1, f)_\#(\Pi_1(M, x_0))$. So we have the commutative diagram:



where $\bar{f}$ is a lifting of $(1, f)$ which exists by elementary properties of covering spaces.

Now let us assume that

$$(1, f)_{\#}(\Pi_1(M, x_0)) = i_{\#}(\Pi_1(S(M) - \Delta; (x_0, f(x_0)))).$$

Then there is a map $j$: $S(M) - \Delta \to \overline{M}$ which is a lifting of $i$. By Proposition 2.1. of [3] we have

$$\Pi_i(S(M), S(M) - \Delta) \approx \Pi_i(S^1, S^1 - y_0) = 0, \qquad i > 1.$$

So $j$ induces isomorphisms in all homotopy groups. Since $S(M) - \Delta$ and $\overline{M}$ are CW-complexes, there exists $l$: $\overline{M} \to S(M) - \Delta$ which is a homotopy inverse of $j$. Take $h = l \circ \bar{f}$.

Now suppose that $h$ exists. Since $i \circ h$ is homotopic to $(1, f)$, it follows that

$$(1, f)_{\#}(\Pi_1(M, x_0)) \subset i_{\#}(\Pi_1(S(M) - \Delta, (x_0, f(x_0)))).$$

Let $p_1$: $S(M) \to M$ be the projection on the first coordinate. We have the fibration

$$\begin{array}{ccc} S^1 - \{x_0\} & \to & S(M) \\ & & \downarrow \\ & & M \end{array}$$

Therefore $p_1$: $\Pi_1(S(M)) \to \Pi_1(M)$ is an isomorphism. Since $p_1 \circ h \approx \mathrm{id}$ we have that $h_{\#}$ is an isomorphism and the equality $(1, f)_{\#}(\Pi_1(M, x_0)) = i_{\#}(\Pi_1(S(M) - \Delta, (x_0, f(x_0))))$ follows.

PROPOSITION 1.3. *A map $f$ can be deformed over $B$ to a fixed point free map if and only if*

$$(1, f)_{\#}(\Pi_1(M, x_0)) = i_{\#}(\Pi_1(S(M) - \Delta, (x_0, f(x_0)))).$$

*Proof.* This follows directly from Proposition 1.1. and 1.2.

Let us consider a fibre preserving map $A$: $M \to M$ whose restriction to each fibre is the antipodal map. Such a map exists because $M \to B$ is a locally trivial $S^1$-fibration. Without loss of generality let us assume that $f(x_0) = A(x_0)$.

PROPOSITION 1.4. *If* $\mathrm{im}(i_{\#}) = \mathrm{im}(1, f)_{\#}$ *then $A_{\#} = f_{\#}$. Conversely if $\Pi_1(S^1) \to \Pi_1(M)$ is injective or surjective then $A_{\#} = f_{\#}$ implies* $\mathrm{im}(i_{\#}) = \mathrm{im}(1, f)_{\#}$.

*Proof*. The map $(1, A)$: $M \to S(M)$ is a left inverse of $p_1$. Since $\Pi_1(S(M) - \Delta) \to \Pi_1(M)$ is an isomorphism, (see the proof of Proposition 2.2) it follows that

$$(1, A)_{\#}\Pi_1(M) \to \Pi_1(S(M) - \Delta)$$

is an isomorphism. Therefore $\operatorname{im}(i_{\#}) = \operatorname{im}(i \circ (1, A))_{\#}$. But

$$\operatorname{im}(i \circ (1, A))_{\#} = \operatorname{im}(1, f)_{\#}$$

is equivalent to $(i \circ (1, A))_{\#}(\alpha) = (1, f)_{\#}(\alpha)$ for every $\alpha \in \Pi_1(M)$. This implies $A_{\#}(\alpha) = f_{\#}(\alpha)$.

For the second part let us assume first that

$$\Pi_1(S^1) \to \Pi_1(M)$$

is injective. From the diagram below

$$\begin{array}{ccc}
\Pi_1(S^1) & & \Pi_1(S^1) \\
\downarrow j_2 & & \downarrow j \\
\Pi_1(S(M)) & \xrightarrow{p_2} & \Pi_1(M) \\
\downarrow p_1 & & \downarrow p_1 \\
\Pi_1(M) & \xrightarrow{p} & \Pi_1(B)
\end{array}$$

we have

$$p_{1\#} \circ (1, A)_{\#}(\alpha) = p_{1\#}(1, f)_{\#}(\alpha) \Rightarrow [(1, A)_{\#}(\alpha) - (1, f_{\#}(\alpha)] = j_{2\#}(\beta)$$

for some $\beta \in \Pi_1(S^1)$. So

$$p_{2\#}[(1, A)_{\#}(\alpha) - (1, f)_{\#}(\alpha)] = A_{\#}(\alpha) - f_{\#}(\alpha) = j_{\#}(\beta) = 0.$$

Since $j_{\#}$ is injective we have $\beta = 0$. Therefore $(1, A)_{\#}(\alpha) = (1, f)_{\#}(\alpha)$ and the result follows. Finally let us assume that $j_{\#}$: $\Pi_1(S^1) \to \Pi_1(M)$ is surjective. We have the diagram:

$$\begin{array}{ccc}
S^1 & \underset{(1, f|_{S^1})}{\overset{(1, A|_{S^1})}{\rightrightarrows}} & S^1 \times S^1 \\
j \downarrow & & \downarrow j_1 \times j_2 \\
M & \underset{(1, f)}{\overset{(1, A)}{\rightrightarrows}} & S(M)
\end{array}$$

From the fact that $f_{\#} = A_{\#}$ and using the long exact sequence in homotopy of the fibration $S^1 \to M \to B$ we have that $(A|_{S^1})_{\#} = (f|_{S^1})_{\#}$. Given $\alpha \in \Pi_1(M)$, there exists $\beta \in \Pi_1(S^1)$ s.t. $j_{\#}(\beta) = \alpha$. So we have

$$\begin{aligned}
(1, A)_{\#}(\alpha) + (1, f)_{\#}(\alpha) &= (1, A)_{\#} j_{\#}(\alpha) - (1, f)_{\#} j_{\#}(\alpha) \\
&= (j_1 \times j_2)_{\#}(1, A|_{S^1})_{\#}(\beta) - (1, f|_{S^1})(\alpha) = 0
\end{aligned}$$

and the result follows.

COROLLARY 1.5. *Let $S^1 \to K \to S^1$ be the $S^1$-fibration where $K$ is the Klein bottle. Then the $1_K$: $K \to K$ $1_K$ = identity map cannot be deformed over $B$ to a fixed point free map.*

COROLLARY 1.6. *Let*

$$\begin{array}{ccc} f\colon B \times S^1 & \longrightarrow & B \times S^1 \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle p_1} & \\ & B & \end{array}$$

*be a fibre-preserving map. Then $f$ can be deformed over $B$ to a fixed point free map if and only if $f = (1, g)$ where $g$: $B \times S^1 \to S^1$ is homotopic to $p_2$: $B \times S^1 \to S^1$ defined by $p_2(x, y) = y$.*

*Proof*. The "if" part is clear. So let us assume that $f$ can be deformed over $B$ to a fixed point free map. By Proposition 1.4. we have $f_\# = A_\#$ and therefore $p_{2\#}f_\# = p_{2\#}A_\#$ or $(p_2 \circ f)_\# = (p_2 \circ A)_\#$. But this means that

$$(p_2 \circ f)^* = (p_2 \circ A)^*\colon H^1(S^1) \to H^1(B \times S^1)$$

and consequently $p_2 \circ f$ is homotopic to $p_2 \circ A$ which is homotopic to $p_2$. So the result follows.

REMARK. In general $f_\# = A_\#$ does not imply $\operatorname{im}(1, f)_\# = \operatorname{im}(i)_\#$. We can construct a counter-example with $B = S^1 \times S^1 \times S^2$ and the fibration is the induced fibration from the universal $S^1$-fibration by the map $g$: $S^1 \times S^1 \times S^2 \to K(Z, 2)$ which is represented by $\alpha_2 \otimes 1 + 2 \otimes \beta_2 \in H^2(S^{\times}S^1 \times S^2)$, $\alpha_2$, $\beta_2$ being generators of $H^2(S^1 \times S^1)$, $H^2(S^2)$ respectively.

**Part II. The homology of the fixed points set.** We will start by recalling some results of [**4**].

Let $x, y \in \operatorname{Fix}(f)$, where $f$ is a fibre-preserving map and $S^1 \to M \to B$ is a $S^1$-fibration. We say that $x$ is equivalent to $y$ over $B$ if there is a path $\lambda$: $[0, 1] \to M$ s.t. $\lambda(0) = x$, $\lambda(1) = y$ and $\lambda$ is homotopic to $f(\lambda)$ rel$\{x, y\}$ over $B$.

DEFINITION 2.1. The equivalence classes are called the Nielsen classes of $f$ over $B$.

PROPOSITION 2.2. *If $M$ is compact then the number of Nielsen classes over $B$ is finite.*

*Proof*. This is Lemma 2.1 of **[4]**.

In **[4]** he also defines essential Nielsen classes and the Nielsen number of $f$. Now we will define a lower bound for the number of non-empty Nielsen classes of $f$ over $B$ for the case of a principal $S^1$-fibration. I believe it would be interesting to compare this number with the Nielsen number as defined in **[4]**.

Let $S^1 \to M \to B$ be an orientable $S^1$-fibration and $\Theta$: $S^1 \times M \to M$ the $S^1$-action. Given $f$: $M \to M$ a fibre-preserving map, there is a map $\theta_f$: $M \to S^1$, which satisfies the equation $f(x) = \theta_f(x) \cdot x$, where $\theta_f(x) \cdot x$ means $\Theta(\theta_f(x), x)$.

PROPOSITION 2.3. *Given an orientable fibration and a map $f$ then* $\mathrm{Fix}(f) = \theta_f^{-1}(1)$ *where* $1 \in S^1$.

*Proof*. Obvious.

Let $i(f)$ denote the number of elements of the group

$$\Pi_1(S^1)/\theta_{f\#}(\Pi_1(M)).$$

PROPOSITION 2.4. *If $i(f) = \infty$ then $f$ can be deformed over $B$ to a fixed point free map.*

*Proof*. If $i(f) = \infty$ then $\theta_{f\#}$ is the constant map. Therefore $\theta_f$ is homotopic to the constant map equal to $-1 \in S^1$. Therefore $f$ is homotopic over $B$ to the antipodal map $A$.

THEOREM 2.5. *Let $i(f) = r < \infty$. If $g$ is homotopic to $f$ over $B$ then there exist at least $r$ Nielsen classes $F_1, \dots, F_r$ such that $\check{H}^{n-1}(F_i, K) \neq 0$ (Čech cohomology) where $K$ is $Z$ or $\mathbf{Z}_2$ depending on whether $M$ is an orientable or a non-orientable $n$-dimensional compact manifold.*

*Proof*. Let us first assume that $r = 1$. Then we have the following commutative diagram

$$\begin{array}{ccccccc}
H_1(M) & \to & H_1(M, M - \mathrm{Fix}(f)) & \to & H_0(M - \mathrm{Fix}(f)) & & \\
\downarrow & & \downarrow & & \downarrow & & \\
H_1(S^1) & \to & H_1(S^1, S^1 - \{1\}) & \to & H_0(S^1 - \{1\}) & \to & \tilde{H}_0(S^1)
\end{array}$$

where the coefficients are in $\mathbf{Z}$ or $\mathbf{Z}_2$. Since $H_1(M) \to H_1(S^1)$ is surjective then $H_1(M, M - \mathrm{Fix}(f)) \neq 0$. So by Poincaré Duality it follows that $\check{H}^{n-1}(\mathrm{Fix}(f)) \neq 0$ and the result follows.

Now let $i(f) = r$ and $S^1 \xrightarrow{p_r} S^1$ be the $r$-fold covering map. There is a lifting $\overline{\Theta}_f$: $M \to S^1$ where

$$\operatorname{Fix}(f) = \bigcup_{k=0}^{r-1} \overline{\Theta}_f^{-1}(e^{2\Pi i K/r})$$

and we have that $\overline{\Theta}_{f*}$: $\Pi_1(M) \to \Pi_1(S^1)$ is surjective. By the case $r = 1$ we have that

$$\check{H}^{n-1}\left(\overline{\Theta}_f^{-1}(e^{2\Pi i K'/r})\right) \neq 0.$$

It is easy to see that

$$x \in \overline{\Theta}_f^{-1}(e^{2\Pi i K/r}), \qquad y \in \Theta_f^{-1}(e^{2\Pi i K'/r})$$

and $K \neq K'$ then $x$ and $y$ do not belong to the same Nielsen class. Therefore the result follows.

REMARK. (1) The Proposition 2.2. and Theorem 2.5. suggest what should be a function $g \in [f]$ over $B$ s.t. Fix($g$) is minimal. The definition will be given in Part III.

(2) I have not been able to extend the definition of this lower bound for non-orientable fibrations.

**Part III. The realization problem.** From now on let us assume that $S^1 \to M \to B$ is a principal $S^1$-fibration and $M$ is a compact orientable $n$-manifold. Let $f$: $M \to M$ be a fibre-preserving map.

DEFINITION 3.1. We say that Fix($g$) is minimal, where $g$ is homotopic to $f$ over $B$, if Fix($g$) is an $n - 1$-submanifold with $i(f)$ connected components.

PROPOSITION 3.2. *Given $f$: $M \to M$ we can find $g$ homotopic to $f$ over $B$ such that* Fix($g$) *is an $n - 1$-submanifold.*

*Proof*. Let $\theta_f$: $M \to S^1$ be as defined in Part II. Now we can deform $\theta_f$ to a map $\overline{\Theta}$: $M \to S^1$ such that $1 \in S^1$ is a regular value. Therefore $g(x) = \overline{\Theta}(x)$. $x$ is homotopic to $f$ over $B$ and Fix($g$) $= \overline{\Theta}^{-1}(1)$ is an $n - 1$-submanifold.

PROPOSITION 3.3. *The homology class* $[\overline{\Theta}^{-1}(1)]$ *represented by the submanifolds* $\overline{\Theta}^{-1}(1)$ *is the Poincaré dual of the 1-dimensional cohomology class* $\overline{\Theta}^*(i_1)$ *where* $\overline{\Theta}^*$: $H^1(S^1, Z) \to H^1(M, \mathbf{Z})$ *and $i_1$ is the generator of* $H^1(S^1, Z)$.

*Proof*. See [**6**].

Recall that $H^1(M, \mathbf{Z})$ is a free abelian group and $H_{n-1}(M, \mathbf{Z})$ is also a free abelian group by Poincaré Duality.

PROPOSITION 3.4. *If* $\overline{\Theta}_{\#}$: $\pi_1(M) \to \pi_1(S^1)$ *is surjective then* $\overline{\Theta}^*(i_1)$ *is indivisible.*

*Proof*. Let $\overline{\Theta}^*(i_1) = \lambda\alpha$, $\lambda \in \mathbf{R}$, $\alpha \in H^1(M, \mathbf{Z})$.
So

$$\left\langle i_1, h\overline{\Theta}_{\#}(x)\right\rangle = \left\langle\overline{\Theta}^*(i_1), \quad h(x)\right\rangle = \langle\lambda\alpha, h(x)\rangle = \lambda\langle\alpha, h(x)\rangle$$

where $h$ is the Hurewicz homomorphism, $x \in \Pi_1(M)$ and $\langle\ ,\ \rangle$ is the evaluation. Therefore $\operatorname{im}\overline{\Theta}_{\#} \subset \lambda \cdot \mathbf{Z}$. Since $\overline{\Theta}_{\#}$ is surjective we have $\lambda = 1$ and the result follows.

PROPOSITION 3.5 (*D. Sullivan*). *Given an indivisible homology class of* $H_{n-1}(M, \mathbf{Z})$ *then it can be represented by a connected* $n - 1$*-submanifold.*

*Proof*. See [**5**] or the appendix.

THEOREM 3.6. *If* $\theta_f$: $\Pi_1(M) \to \Pi_1(S^1)$ *is surjective then* $f$ *can be deformed over* $B$ *to a map* $g$ *s.t.* Fix($g$) *is a connected* $n - 1$*-submanifold.*

*Proof*. Since $\theta_{f\#}$: $\pi_1(M) \to \pi_1(S^1)$ is surjective, by Proposition 3.4. $\theta_f$ defines an $n - 1$-homology class of $M$ which is indivisible. By Proposition 3.5 there is a connected $n - 1$-submanifold $N$ which represents this class. Now let us take a tubular neighborhood of this submanifold. This neighborhood is homeomorphic to $N \times (-\varepsilon, \varepsilon)$. Then we define $\overline{\Theta}$: $M \to S^1$ such that $\overline{\Theta}^{-1}(1) = N$ and 1 is a regular value of $\overline{\Theta}$. By Proposition 3.3 $\overline{\Theta}$ is homotopic to $\theta_f$ and $g(x) = \overline{\Theta}(x) \cdot x$ is a function such as we are looking for.

Finally the main result.

THEOREM 3.7. *Given* $f$: $M \to M$ *there is a map* $g$ *homotopic to* $f$ *over* $B$ *such that* Fix($f$) *is minimal.*

*Proof*. Let $\tilde{\Theta}_f$: $M \to S^1$ be a lifting of $\theta_f$ i.e. $p_r \circ \tilde{\Theta}_f = \theta_f$ where $p_r$ is the $r$-fold cover of $S^1$. By Theorem 3.6 $\tilde{\Theta}_f$ is homotopic to a map $\tilde{\theta}$: $M \to S^1$ such that $\tilde{\Theta}^{-1}(1)$ is a connected $n - 1$-submanifold. Let $\phi$: $S^1 \to S^1$ be a diffeomorphism homotopic to the identity which sends the set $\{e^{2\Pi iK/r} \mid K = 0, 1, \ldots, r-1\}$ into a small neighbourhood of 1 whose points are regular values of $\tilde{\Theta}$. Let $\overline{\Theta} = p_r \circ \phi^{-1} \circ \tilde{\Theta}$. Then $g(x) = \overline{\Theta}(x) \cdot x$ is a function such as we are looking for.

**Appendix.** Now let us sketch the proof of Proposition 3.5. (This is due to Prof. D. Sullivan.)

*Proof.* Let $M$ be a compact orientable manifold of dimension $n$ and $N \subset M$ an $n-1$-compact embedded submanifold. Suppose $n \geq 3$ and $N$ has more than 1 connected component. Call $N_1$, $N_2$ two components. Given $p \in N_1$, $q \in N_2$ there is a path $\lambda$ in $M$ such that $\lambda(0) = p$, $\lambda(1) = q$ since $M$ is connected. We can assume that $\lambda[0,1] \cap N$ is a finite set $\{a_1, \ldots, a_t\}$ and $a_1 = p$, $a_t = q$. Let $\lambda$ have the natural orientation. At each point $a_i$ we have the intersection number of $\lambda$ and $N$ which is $+1$, $-1$ or 0. We can assume that the intersection number of $a_i$ is either $+1$ or $-1$, otherwise we deform $\lambda$ in such a way that $a_i$ is not in the intersection. Now let us suppose that the total intersection number of $\lambda$ and $N$ is equal to zero. Then we can find 2 consecutive points, $a_i$, $a_{i+1}$ such that one has intersection number $+1$ and the other has intersection number $-1$. Now we apply surgery, replacing two small discs around $a_i$, $a_{i+1}$ by a tube around the arc from $a_i$ to $a_{i+1}$. The new manifold represents the same homology class. Since $m \geq 3$ the following fact is true: if $a_i$, $a_{i+1}$ belong to the same component of $N$ then the new submanifold has the same number of components as $N$, otherwise the number of components decreases by one. Because the total intersection number is zero we can continue this process and end up with a submanifold $N'$ with less components than $N$. If $N'$ is not connected we apply the above procedure again until we get a connected submanifold.

Now let me show that it is always possible to connect one point of $N_1$ to a point of $N_2$ by a curve which has total intersection number zero. Let $p \in N_1$, $q \in N_2$ and $\lambda$ a curve from $p$ to $q$ such that $\lambda[0,1] \cap N$ is finite. Call $r$ the intersection number of $\lambda$ and the submanifold $N$. Since $[N] \in H_{m-1}(M, \mathbf{Z})$ is indivisible by Poincaré Duality we can pass by $g$ an embedded circle $\phi$: $[0,1] \to M$ $\phi(0) = \phi(1) = g$ which has total intersection number $+1$ with $N$. Given a number $s$ let $s \cdot \phi = \underbrace{\phi * \cdots * \phi}_{s}$ where $*$ is the composition of paths. Let $T$ be a tubular neighborhood of $\phi[0,1]$. Since $M$ is orientable then $T \approx D^{n-1} \times S^1$ where $D^{n-1}$ is the $n-1$-disc. Now we can deform $s\phi$ to $\bar{\phi}_s$ in such a way that $\bar{\phi}_s([0,1])$ is an embedded circle. Finally let $\phi'_s$ be a small deformation of $\bar{\phi}_s$ such that $\phi'_s(1) = g' \neq g$ and $g' \in N_2$ and near $g$. Now consider the following curve $\lambda * \phi'_s$. Call $I_\lambda(g)$ and $I_{\phi'_s}(g)$ the intersection numbers of $g$ as points of $\lambda$ and $\phi'_s$

respectively. If $I_\lambda(g) = -I_{\phi'_s}(g)$ then let $s = r - I_\lambda(g)$. If $I_\lambda(g) = I_{\phi'_s}(g)$ let $s = r$. Then we have that the total intersection number of $\lambda * \phi'_s$ is zero.

Now let $m = 2$. The fact that, in this case, an indivisible homology class can be represented by an embedded circle is classical and was known by Poincaré.

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