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NON-TANGENTIAL LIMIT THEOREMS FOR NORMAL MAPPINGS

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Let X be a relatively compact complex subspace of a hermitian manifold N with hermitian distance d_N. Let Ω be a bounded domain with C^1-boundary in $\mathbf{C}^m$. A holomorphic mapping $f: \Omega \to N$, $f(\Omega) \subset X$, is called a normal mapping if the family $\{f \circ \psi: \psi: \Delta \to \Omega$ is holomorphic$\}$, $\Delta := \{z \in \mathbf{C}: |z| < 1\}$, is a normal family in the sense of H. Wu. Let $\{p_n\}$ be a sequence of points in Ω which tends to a boundary point $\zeta \in \partial\Omega$ such that $\lim_{n\to\infty} d_N(f(p_n), l) = 0$ for some $l \in \overline{X}$. Two sets of sufficient conditions on $\{p_n\}$ are given for a normal mapping $f: \Omega \to X$ to have the non-tangential limit value l, thus extending the results obtained by Bagemihl and Seidel.

1. Introduction. In [2], F. Bagemihl and W. Seidel posed the following question: Given a sequence $\{z_n\}$ in the open unit disc Δ converging to some $\zeta \in \partial\Delta$ and a meromorphic function $f: \Delta \to P_1(\mathbf{C})$ such that $\lim_{n\to\infty} f(z_n) = c$ for some $c \in P_1(\mathbf{C})$, under what conditions on f and $\{z_n\}$ can f have the limit c along some continuum in Δ which is asymptotic at ζ? They answer this question with two interesting sufficient conditions on f and $\{z_n\}$.

In this paper we extend their results to the higher dimensional case. First we shall introduce a few terminologies.

Let Ω be a bounded domain with C^1-boundary in $\mathbf{C}^m$. Then at each $\zeta \in \partial\Omega$, the tangent space $T_\zeta(\partial\Omega)$ and the unit outward normal vector ν_ζ are well-defined. We denote by $\mathbf{C}T_\zeta(\partial\Omega)$ and $\mathbf{C}\nu_\zeta$ the complex tangent space and the complex normal space, respectively. The complex tangent space at ζ is defined as the $(m-1)$ dimensional complex subspace of $T_\zeta(\partial\Omega)$ and given by $\mathbf{C}T_\zeta(\partial\Omega) := \{z \in \mathbf{C}^m: (z, w) = 0, \forall w \in \mathbf{C}\nu_\zeta\}$, $(z, w) = \sum_{j=1}^m z_j \bar{w}_j$.

We say that a subset $S \subset \Omega$ is *asymptotic* at $\zeta \in \partial\Omega$ if $\overline{S} \cap \partial\Omega = \{\zeta\}$ and *non-tangentially asymptotic* at ζ if $S \subset \Gamma_\alpha(\zeta)$ for some $\alpha > 1$, where

$$\Gamma_\alpha(\zeta) := \{z \in \Omega : |z - \zeta| < \alpha\delta_\zeta(z)\}, \tag{1a}$$

$$\delta_\zeta(z) = \min\{p(z, \partial\Omega), p(z, T_\zeta(\partial\Omega))\}, \tag{1b}$$

and p denotes the euclidean distance in $\mathbf{C}^m$. In particular, a curve $\gamma\colon (0,1) \to \Omega$ is non-tangentially asymptotic at ζ if $\gamma(t) \in \Gamma_\alpha(\zeta)$ for some $\alpha > 1$ and all $t \in (0,1)$, and $\lim_{t\to 1-} \gamma(t) = \zeta$.

Let N be a connected paracompact hermitian manifold with hermitian metric h_N which induces the standard topology of N. By d_N we denote the distance function associated with h_N.

By $\operatorname{Hol}(\Omega, N)$ we denote the space of all holomorphic maps $f\colon \Omega \to N$. We say that a mapping $f \in \operatorname{Hol}(\Omega, N)$ has an *asymptotic limit* l at $\zeta \in \partial\Omega$ along the curve γ in Ω, write $\lim_{\gamma \ni z \to \zeta} f(z) = l$, if γ is asymptotic at ζ and $\lim_{t\to 1-} d_N(f(\gamma(t)), l) = 0$, a *radial limit* l at ζ if $\lim_{\varepsilon\to 0+} d_N(f(\zeta - \varepsilon v_\zeta), l) = 0$, a *non-tangential limit* l at ζ if $\lim_{\Gamma_\alpha(\zeta)\ni z\to\zeta} d_N(f(z), l) = 0$ for every $\alpha > 1$ and an *admissible limit* l at ζ if $\lim_{A_\alpha(\zeta)\ni z\to\zeta} d_N(f(z), l) = 0$ for every $\alpha > 0$, where

$$(2)\quad A_\alpha(\zeta) := \{z \in \Omega\colon |(z-\zeta, \nu_\zeta)| < (1+\alpha)\delta_\zeta(z),\ |z-\zeta|^2 < \alpha\delta_\zeta(z)\}.$$

Let M be a connected complex manifold of dimension m. We assume that M is hyperbolic, i.e., the Kobayashi pseudometric k_M is a metric. Denote the infinitesimal Kobayashi metric by K_M. According to H. Royden [10], the Kobayashi metric k_M is the integrated form of K_M. M is hyperbolic if and only if for each $p \in M$, there exists a neighborhood U_p and a constant $a_U > 0$ such that

$$K_M(q,\xi) \geq a_U|\xi| \quad \text{for } (q,\xi) \in U \times \mathbf{C}^m.$$

DEFINITION. A mapping $f \in \operatorname{Hol}(M,N)$ is called *normal* if the family $\{f \circ \psi\colon \psi \in \operatorname{Hol}(\Delta, M)\}$, Δ is the unit disc in $\mathbf{C}$, forms a normal family in the sense of H. Wu [11].

We remark that the definition of normality adopted here does not require M to be homogeneous and coincides with that of [7] when M is homogeneous and N is compact [1], [6]. Therefore, it is a slightly more general notion than that of [7].

2. Preliminary properties of normal mappings. Let X be a relatively compact complex subspace of a hermitian manifold N. We shall denote by $\operatorname{Hol}(M,X)$ the space of all holomorphic maps $f\colon M \to N$ with $f(M) \subset X$.

LEMMA 1. *Let M be a hyperbolic manifold and let X be a relatively compact complex subspace of a hermitian manifold N with hermitian metric h_N. The family $F \subset \operatorname{Hol}(M,X)$ is normal in the sense of H. Wu* *if for each compact subset $E \subset M$ there exists a constant $C(E) > 0$ such that*

$$Qf(p) := \sup_{|\xi|=1} \frac{h_N(f(p), df(p)\xi)}{K_M(p,\xi)} \leq C(E) \tag{3}$$

for all $p \in E$ and all $f \in F$.

Due to the compactness of $\overline{X}$, the proof of Lemma 1 can be carried out in the same way as that of Lemma 2.7 of [7]. Therefore, we omit the proof.

THEOREM 1. *Let M be a hyperbolic manifold (not necessarily homogeneous) and let X be a relatively compact complex subspace of a hermitian manifold N. The following statements are equivalent for $f \in \mathrm{Hol}(M, X)$.*

(a) *f is normal.*

(b) *There exists a constant $Q > 0$ such that*

$$Qf := \sup\{Qf(p) : p \in M\} \leq Q.$$

(c) *There is no P-sequence $\{p_n\}$ in M possessed by f, i.e., there is no sequence $\{q_n\}$ in M such that $\lim_{n\to\infty} k_M(p_n, q_n) = 0$ but $\overline{\lim}_{n\to\infty} d_N(f(p_n), f(q_n)) \geq \varepsilon$ for some $\varepsilon > 0$.*

Proof. (a) $\Rightarrow$ (*b*): Assume that $\{f \circ \psi : \psi \in \mathrm{Hol}(\Delta, M)\}$ is a normal family. By Lemma 1, for each compact $E \subset \Delta$, there exists a constant $Q = Q(E) > 0$ such that

$$h_N(f \circ \psi(0), (f \circ \psi)'(0)) \leq Q \tag{4}$$

for all $\psi \in \mathrm{Hol}(\Delta, M)$. By the definition of K_M at $(p, \xi) \in M \times \mathbf{C}^m$, there exists $\psi \in \mathrm{Hol}(\Delta, M)$ such that $\psi(0) = p$, $\psi'(0)a = \xi$ for $a > 0$ and $a/2 < K_M(p, \xi) \leq a$. Therefore, from (4),

$$h_N(f(p), df(p)\xi) \leq 2QK_M(p, \xi)$$

for all $(p, \xi) \in M \times \mathbf{C}^m$. Namely, $Qf \leq 2Q$.

(b) $\Rightarrow$ (c): If (c) fails to hold, then there exists a sequence $\{p_n\}$ and $\{q_n\}$ in M with $\lim_{n\to\infty} k_M(p_n, q_n) = 0$ but $\overline{\lim}\, d_N(f(p_n), f(q_n)) \geq \varepsilon$ for some $\varepsilon > 0$. It contradicts (b), because (b) implies that $d_N(f(p_n), f(q_n)) \leq Qk_M(p_n, q_n)$.

(c) $\Rightarrow$ (a): If (c) holds, then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $z, w \in \Delta$, $k_\Delta(z, w) < \delta$ implies $d_N(f \circ \psi(z), f \circ \psi(w)) < \varepsilon$

for all $\psi \in \operatorname{Hol}(\Delta, M)$, since otherwise there exists an $\varepsilon > 0$ such that for all $n \in \mathbf{N}$ there exist sequences $\{z_n\}$ and $\{w_n\}$ in Δ with $k_\Delta(z_n, w_n) < 1/n$ but $d_N(f \circ \psi(z_n), f \circ \psi(w_n)) \geq \varepsilon$ for some $\psi \in \operatorname{Hol}(\Delta, M)$. This means that $\{z_n\}$ is a P-sequence for $f \circ \psi$. Since

$$k_M(\psi(z_n), \psi(w_n)) \leq k_\Delta(z_n, w_n) \leq 1/n \to 0,$$

$\{\psi(z_n)\}$ is also a P-sequence for f in M which contradicts (c). Therefore, $\{f \circ \psi : \psi \in \operatorname{Hol}(\Delta, M)\}$ is an equicontinuous family and hence normal since $\overline{X}$ is compact. This proves (a).

Theorem 1 is also proved in [6] for compact N and in [3] for $N =$ the Riemann sphere.

3. Boundary behavior of normal mappings.

THEOREM 2. *Let X and N be given as in Theorem* 1, *and let Ω be a bounded domain with C^1-boundary in $\mathbf{C}^m$. Suppose that S is an arbitrary asymptotic continuum at $\zeta \in \partial\Omega$ such that*

$$\lim_{S \ni z \to \zeta} \frac{p(z, \mathbf{C}\nu_\zeta)}{r(\nu(z))} = 0, \tag{6a}$$

where $r(\nu(z))$ denotes the radius of the largest ball in $\Omega \cap \mathbf{C}T_{\nu(z)}$, centered at $\nu(z)$, the orthogonal projection of z to $\mathbf{C}\nu_\zeta$ and $\mathbf{C}T_{\nu(z)}$ is the hyperplane through $\nu(z)$ that is parallel to $\mathbf{C}T_\zeta(\partial\Omega)$. If $f \in \operatorname{Hol}(\Omega, X)$ is a normal map such that $\lim_{S \ni z \to \zeta} d_N(f(z), l) = 0$ for some $l \in \overline{X}$, then $\lim_{\Gamma_\alpha(\zeta) \ni z \to \zeta} d_N(f(z), l) = 0$ for all $\alpha > 1$.

Proof. By the definition of $r(\nu(z))$, $\Omega \cap \mathbf{C}T_{\nu(z)}$ contains the euclidean ball $B(\nu(z), r(\nu(z)))|_{\mathbf{C}T_{\nu(z)}}$, the restriction to $\mathbf{C}T_{\nu(z)}$.

The distance-decreasing property of the Kobayashi metric implies

$$k_\Omega(z, \nu(z)) \leq \tanh^{-1} \frac{|z - \nu(z)|}{r(\nu(z))}, \tag{7}$$

and hence, as $S \ni z \to \zeta$, $\eta := \nu(z) \to \zeta$ along $\nu(S) := \{\nu(z) : z \in S\}$ from (7). Since f is normal, by Theorem 1, there exists a number $Q > 0$ such that

$$d_N(f(z), f(\nu(z)) \leq Q k_\Omega(z, \nu(z)). \tag{8}$$

Therefore, $\lim_{\nu(S) \ni \eta \to \zeta} d_N(f(\eta), l) = 0$. Let Ω_ζ be the connected component of $\Omega \cap \mathbf{C}\nu_\zeta$ with $\zeta \in \partial\Omega_\zeta$. Then the restriction $f|_{\Omega_\zeta}$ is a normal map from the plane domain Ω_ζ into X. Therefore, it follows from Theorem 4 of [5] with a slight modification that

$$\lim_{\tilde{\Gamma}_\alpha(\zeta)\ni\eta\to\zeta} d_N(f(\eta), l) = 0 \quad \text{for all } \alpha > 1,$$

where $\tilde{\Gamma}_\alpha(\zeta) := \Gamma_\alpha(\zeta)\cap \mathbf{C}\nu_\zeta$. The rest of the proof can easily be carried over from the proof of Proposition 8.2 of [7] to this case with X replaced by d_N.

COROLLARY 1. *Let X and N be given as in Theorem 1 and let Ω be a bounded domain with C^2-boundary in $\mathbf{C}^m$. Let S be an arbitrary asymptotic continuum at $\zeta \in \partial\Omega$ such that*

$$\lim_{S\ni z\to\zeta} \frac{p^2(z, \mathbf{C}\nu_\zeta)}{p(z, \mathbf{C}T_\zeta)} = 0. \tag{6b}$$

If $f \in \mathrm{Hol}(\Omega, X)$ is a normal map such that $\lim_{S\ni z\to\zeta} d_N(f(z), l) = 0$ for some $l \in \overline{X}$, then

$$\lim_{\Gamma_\alpha(\zeta)\ni z\to\zeta} d_N(f(z), l) = 0 \quad \textit{for all } \alpha > 1.$$

Proof. Since Ω is a bounded domain with C^2-boundary in $\mathbf{C}^m$, there exists an $\varepsilon = \varepsilon(\zeta) > 0$ such that the euclidean ball $B_\varepsilon := B(\zeta - \varepsilon\nu_\zeta, \varepsilon)$ is contained in Ω and tangent to $\partial\Omega$ at ζ from inside. The order of tangency in this case is not worse than along the admissible region A_α given in (2). In fact, there exists a constant $C > 0$ such that

$$r(\nu(z)) \geq C|\zeta - \nu(z)|^{1/2}$$

for $z \in S$. See Example 1 of [4]. Therefore,

$$\left[\frac{Cp(z, \mathbf{C}\nu_\zeta)}{r(\nu(z))}\right]^2 \leq \frac{|z - \nu(z)|^2}{|\zeta - \nu(z)|} \leq \frac{p^2(z, \mathbf{C}\nu_\zeta)}{p(z, \mathbf{C}T_\zeta)}. \tag{9}$$

Corollary 1 now follows from Theorem 2 or directly from the Proof of Proposition 8.2 of [7] with minor adjustments.

We now prove the following extensions of the results given in [2].

THEOREM 3. *Let X and N be given as in Theorem 1. Let Ω be a bounded homogeneous domain in $\mathbf{C}^m$ and let $\{p_n\}$ be a sequence of points in Ω which tends to a boundary point $\zeta \in \partial\Omega$ where the outward normal ν_ζ exists, such that*

(a) *there exists a constant $M > 0$ with $k_\Omega(p_n, p_{n+1}) \leq M$ for all n,*

$$\lim_{n\to\infty} \frac{p(p_n, \mathbf{C}\nu_\zeta)}{r(\nu(p_n))} = 0. \tag{b}$$

If $f \in \mathrm{Hol}(\Omega, N)$ *is a normal map which omits* $l \in \overline{X}$ *in* Ω *but* $\lim_{n\to\infty} d_N(f(p_n), l) = 0$ *then*

$$\lim_{\Gamma_\alpha(\zeta)\ni z\to\zeta} d_N(f(z), l) = 0 \quad \text{for all } \alpha > 1.$$

Proof. Let $\varphi_n \in \mathrm{Aut}(\Omega)$ be such that $\varphi_n(p_0) = p_n$ for some fixed point $p_0 \in \Omega$. Then the family $\{g_n\}$, $g_n = f \circ \varphi_n$, omits l for all n and forms a normal family, since f is normal.

For $R > M$, let $B_k(p_0, R) := \{p \in \Omega : k_\Omega(p_0, p) < R\}$. Since Ω is homogeneous, k_Ω is complete and, hence $\overline{B}_k(p_0, R)$ is a compact subset of Ω. So, $\{g_n\}$ has a subsequence $\{g_m\}$ which converges uniformly on $\overline{B}_k$ to $g \in \mathrm{Hol}(\Omega, N)$. Since each g_m omits l on B_k, by the Hurwitz theorem **[8]**, either $g(z) \neq l$ or $g(z) \equiv l$ on $B_k(p_0, R)$. But since $d_N(g_m(p_0), l) = d_N(f(p_m), l) \to 0$, $g(z) \equiv l$ for all $z \in B_k(p_0, R)$. This implies that $f(z) = l$ for all $z \in B_k(p_m, R)$ and all m, i.e., $f(z) \equiv l$ on $\bigcup_{m=1}^{\infty} B_k(p_m, R)$. Since

$$k_\Omega(p_m, \nu(p_m)) \le \tanh^{-1} \frac{|p_m - \nu(p_m)|}{r(\nu(p_m))} \to 0$$

as $n \to \infty$, there exists m_0 such that for all $m \ge m_0$ $k_\Omega(p_m, \nu(p_m)) < R$ which implies $\nu(p_m) \in B_k(p_m, R)$ for all $m \ge m_0$. Let $S := \mathbf{C}\nu_\zeta \cap \bigcup_{m \ge m_0} B_k(p_m, R)$.

Then condition (6a) in Theorem 2 is trivially satisfied and also $\lim_{S\ni z\to\zeta} d_N(f(z), l) = 0$. Therefore, we have

$$\lim_{\Gamma_\alpha(\zeta)\ni z\to\zeta} d_N(f(z), l) = 0$$

for all $\alpha > 1$ by Theorem 2.

THEOREM 4. *Let* X *and* N *be given as in Theorem* 1. *Let* $\{p_n\}$ *be a sequence of points in a bounded domain* $\Omega \subset \mathbf{C}^m$ *which tends to a boundary point* $\zeta \in \partial\Omega$ *where the unit outward normal* ν_ζ *exists such that*

(a) $$\lim_{n\to\infty} k_\Omega(p_n, p_{n+1}) = 0,$$

(b) $$\lim_{n\to\infty} \frac{p(p_n, \mathbf{C}\nu_\zeta)}{r(\nu(p_n))} = 0.$$

If $f \in \mathrm{Hol}(\Omega, X)$ *is a normal map such that* $\lim_{n\to\infty} d_N(f(p_n), l) = 0$ *for some* $l \in \overline{X}$, *then* $\lim_{\Gamma_\alpha(\zeta)\ni z\to\zeta} d_N(f(z), l) = 0$ *for all* $\alpha > 1$.

Proof. Let $\{q_n\}$, $q_n = \nu(p_n)$, be the orthogonal projection of $\{p_n\}$ to $\mathbf{C}\nu_\zeta$. Then

$$k_\Omega(q_n, q_{n+1}) \leq k_\Omega(p_n, p_{n+1}) \tag{10}$$

so that $k_\Omega(q_n, q_{n+1}) \to 0$ as $n \to \infty$. Let γ be a curve in $\Omega \cap \mathbf{C}\nu_\zeta$ joining q_n and q_{n+1} by shortest curves. Since k_Ω is an inner metric, such curves exist for sufficiently large n. Since f is normal, by Theorem 1, there exists $Q > 0$ such that

$$d_N(f(p_n), f(q_n)) \leq Q k_\Omega(p_n, q_n). \tag{11}$$

Therefore, condition (b) together with (7) implies

$$\lim_{n\to\infty} d_N(f(p_n), f(q_n)) = 0,$$

and hence,

$$\lim_{n\to\infty} d_N(f(q_n), l) = 0 \tag{12}$$

by the triangle inequality. We wish to show:

$$\lim_{\gamma \ni z \to \zeta} d_N(f(z), l) = 0. \tag{13}$$

Suppose there is a sequence $\{q'_n\}$ on γ converging to ζ for which f fails to have the limit l. By the compactness of $\overline{X}$ there must be a subsequence $\{q'_m\}$ such that

$$\lim_{m\to\infty} d_N(f(q'_m), l') = 0 \tag{14}$$

for some $l' \in \overline{X}$, $l' \neq l$. We may assume that q'_m are all distinct from the points q_m. For each m, there exists an index n_m such that q'_m lies on the geodesic segment of γ that joins q_{n_m} and q_{n_m+1}. By (10),

$$k_\Omega(q_{n_m}, q'_m) \leq k_\Omega(q_{n_m} q_{n_m+1} \to 0$$

as $m \to \infty$. Since f is normal, for some $Q > 0$ we have

$$d_N(f(q_{n_m}), f(q'_m)) \leq Q k_\Omega(q_{n_m}, q'_m) \to 0$$

as $m \to \infty$. From this and (12) we conclude $\lim_{m\to\infty} d_N(f(q'_m), l) = 0$, contradicting (14). Therefore we have (13). Since condition (6a) of Theorem 2 holds trivially in this case, Theorem 4 follows from Theorem 2.

We remark that if the domain Ω in Theorems 3 and 4 is assumed to have C^2-boundary, then both theorems hold when condition (b) is replaced by

$$\text{(b}'\text{)} \qquad \lim_{n\to\infty} \frac{p^2(p_n, \mathbf{C}\nu_\zeta)}{p(p_n, \mathbf{C}T_\zeta)} = 0$$

in both cases.

Introducing the notion of hypoadmissible limit, J. Cima and S. Krantz have proved the Lindelöf Principle for normal meromorphic functions on domains in $\mathbf{C}^n$ with C^2-boundary in [3]. The author wishes to thank the referee for pointing this out to him.

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