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RELATIVE WIDTH MEASURES AND THE PLANK PROBLEM

R. J. GARDNER

A relative width measure in a convex body K in $\mathbb{R}^n$ for a set $\mathcal{G}$ of directions is a Borel probability measure in K such that the measure of the intersection of K with each slab orthogonal to a direction in $\mathcal{G}$ is equal to the relative width of the slab. Such measures are studied in connection with the unsolved plank problem of Th. Bang.

0. Introduction. Tarski's plank problem was solved by Th. Bang [2] when he showed that if a convex body K in $\mathbb{R}^n$ is covered by a finite number of slabs, the sum of their widths is at least the minimum width of K. Bang conjectured that a stronger, and affine invariant, inequality should hold; namely, that the sum of the relative widths of the slabs is at least one (the relative width of a slab is its width divided by the width of K in the same direction). This is still unsolved.

A relative width measure is a Borel probability measure in K such that the measure of the intersection of K with any slab is precisely the relative width of the slab. An example, known to Achimedes, is normalized surface area measure in a ball in $\mathbb{R}^3$; another is the projection of this measure, normalized, in a disc in $\mathbb{R}^2$. If such a measure exists in K, then Bang's conjecture is true for K. This observation has been made several times in the literature, but does not seem to have been thoroughly investigated.

We study these measures, always with Bang's conjecture in mind. For this application, the measures need only have the relative width property for directions corresponding to the covering slabs, and in fact a reduction shows that we need only seek them for coordinate directions. Theorem 1 shows that measures with the latter property always exist in $\mathbb{R}^2$, which generalizes the known special case of Bang's conjecture for two slabs. However, Example 2 shows that even measures with this weaker property do not generally exist for K in $\mathbb{R}^3$.

Section 3 concerns measures with the relative width property for infinite sets of directions. Here, using Fourier transform techniques and particularly a method due to K. Falconer, we show (Theorem 3) that measures with the relative width property for all directions do not exist in the ball in R" for $n > 3$. (After this paper was written, I learned that G. Schwarz also proves this in [23].) In Theorem 4 we refine this result, and show that those in the disc in R^2 and ball in R^3 are essentially the only such measures. Sufficiently 'large' infinite sets of directions also guarantee uniqueness of relative width measures, while in §4 we show that in R^2 and R^3 finite sets of directions do not.

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1. Relative width measures and the plank problem. We shall write int *E* and *dE* for the interior and boundary of a set *E*, respectively.

Suppose *K is a* compact convex set in R", and *6* is a direction (which we identify throughout with the corresponding unit vector in R"). We denote by *W(K, 6)* the *width* of *K* in the direction 0; that is, the distance between two hyperplanes which are orthogonal to *8* and which support *K*. If AT is a fixed compact convex set, and *H* is a convex set, the *relative width* of *H* in the direction *6*, when *W(K, 6) /* 0, is

w(H, 6) = *W(H* n *K, 6)/W(K, 6).*

A *slab* orthogonal to *6* is the closed set between two hyperplanes which are orthogonal to *6*.

A *measure* is a non-negative set function, assumed countably additive unless otherwise stated. Let /z be a Borel probability measure in the compact convex set *K*, and let 0 be a set of directions. We say that *n* is a *relative width measure* in *K* for G if

$$\mu(S \cap K) = w(S, \theta)$$

whenever *S* is a slab orthogonal to some *6* G G.

Suppose now that G = *{6\,..., 6k}* is a finite set of directions, *K* is a compact convex set in R" and 5"; is a slab orthogonal to 0(, 1 < / < *k*. If *K* c \/(*Si*, is it true that *J2iW(Si, 6i)* >—1? This is Bang's plank problem (see [3]). It is an affine invariant form of Tarski's plank problem **[24]**, solved by Bang in [2] when he showed that the weaker inequality

$$\sum_i W(S_i \cap K, \theta_i) \geq \min_\theta\{W(K, \theta)\} \tag{1}$$

holds.

It is not a new observation that relative width measures are relevant to Bang's plank problem. This has essentially been noted by G. Hajós and A. Rényi **[20]**, **D.** Ohmann **[17]** and **J.** W. Green **[12]**, among others. For, if *n* is a relative width measure in *K* for 0 = *{6\,..., θ_k}*, then

$$(2) \qquad £>(S,\cdot DK) \geq \mu(K) = 1 \Rightarrow \sum_i w(S_i, \theta_i) \geq 1,$$
$$i$$

as required.

Since there seems to be some confusion about the status of Bang's problem, we shall briefly survey attempts to solve it. In [17], D. Ohmann shows that it suffices to consider a convex body *K* in R" covered by *n* slabs *S\,...,S_n,* with 5, 'orthogonal to the /th coordinate axis. Further, a *signed* relative width measure is constructed in *K* for these directions, but this is not enough for the implication (2), even when *fi(Sj* n *K)* > 0 for each /. These and other remarks on the problem may be found in [4], where it is also shown that Bang's conjecture is true for covers of *K* by two slabs. Other proofs of this special case are given in **[1]** and **[15].**

Let /; be the length of the longest chord of *K* parallel to *dt,* *i* = \,...,*k.* Then Bang's proof actually shows that *Y.t* *w(St K)lh* ^ 1> which is stronger than (1). According to a translation, the paper **[14]** in Chinese only confirms this result, despite its title. Another attempt in **[18]** breaks down; the error is pinpointed in the review by C. A. Rogers cited under **[18].**

Lastly, we note variations in the proof of Bang's result ([6], **[10],** special cases in [7], [9]) and the interesting paper of R. Alexander **[1]** which relates Bang's problem for *K* a square to an unsolved problem of Davenport.

We begin by considering those properties of *pi* necessary to derive (2). We actually only need *ju* to be finitely additive and to satisfy *fi(S* n *K)* ≤ *w(S,6)* for each slab *S* orthogonal to *6.* Assuming only this, we note that the support of *//* must lie in *K* for a straightforward application to the plank problem. For if *K* c R^2, 0 contains at least two directions, and /z(R^2 - *K)* > 0, it is easy to find slabs *Si* and ^2 orthogonal to any two of these directions such that *K c S\J) S2* and *fi(S* U *S2)* = *c* < 1, from which we can only deduce that *c* ≤ *w(S\, 61)* + *w(S2, 62).* Now the first two lemmas show that we lose no generality in making our other stronger assumptions on *fi.*

LEMMA 1. *Suppose fi is a finitely additive Borel probability measure in K, such that fi(S* n *K)* ≤ *w{S,6) for all slabs S orthogonal to 6. Then n(S DK) = w(S, 6) for all such slabs.*

Proof. Let *S* be any slab orthogonal to *6.* We can find two disjoint slabs £2 and S3, both orthogonal to *6,* with *K* c *\J,Si* and *52iW(Si,0)* = *l.* Then

$$1 \leq £ / * (\$ \text{ n } K) \leq n\{S_x\ C\backslash K) + w(S_2, d) + w(S_3, 6),$$
$$i$$

so *fi(Si nK)* _> *l-w(S*₂, *d)-w(S*₂, *6)* = *w(S!,d),* giving */i(Si f\K)* = *w(Si,6)* as required.

LEMMA 2. *Suppose n is a finitely additive probability measure in K, defined on the algebra si generated by sets SDK, where S is a slab orthogonal to d, and 6 belongs to a fixed set* 0 *ofdirections. Suppose also that fi{S* n *K)* = *w(S, 6) for such sets. Then fi can be extended to a relative width measure in K for* G.

Proof. Let *A e srf.* Then *A* = U/?=i *Aj,* where for each *i* there is a polytope *Pi* such that

$$(\mathrm{int}\, P_i) \cap K \subset A_i \subset P_i \cap K,$$

and moreover for each face of *Pi* there is a *6* e © such that this face is orthogonal to *6.*

For each *f*, let *Hi* be a finite union of open slabs, each orthogonal to some *6* € 0, such that

$$Pi\text{-}intP_t \quad a \quad Hi,$$

and the sum of the relative widths of the slabs in *Hi* is less than *e/m.*

Let *Q* = *(Pi -Hi)DK* for each */,* and *C* = (J£i *Q.* Then *C* is compact, *C c A,* and

$$\mu(C) \geq \mu\left(\bigcup_i P_i \cap K\right) - \mu\left(\bigcup_i H_i \cap K\right) > \mu(A) - \varepsilon,$$

where we have used the fact that P; n *K, H_t* n *K* and C all belong to *si.*

This shows that *fi* is inner regular on J/ with respect to the compact sets. By Henry's extension theorem [22, p. 51, Theorem 16], *fi* can be extended to a countably additive Borel measure.

REMARKS, (i) The extension provided by Henry's theorem is unique if the algebra *si* contains a base for the topology of R". This will be the case if 0 contains *n* linearly independent directions. However, even if *sf* does contain a base, there may be more than one relative width measure in *K* for 0 (see Theorem 5).

(ii) Given any set © of directions, we may of course define *fi* on the sets *S* n *K* of Lemma 2 by *fi(S* n *K)* = *w(S, 6).* However, /i may not extend to the algebra *sf;* Example 1 shows this.

2. Existence of relative width measures. Here and throughout *X* denotes linear Lebesgue measure in R".

Suppose *K* is a convex body in R" and *6* is a direction. Let / be a chord of *K* meeting the two supporting hyperplanes to *K* which are orthogonal to *6.* Define /i in *K* by /i(*B*) = *X{B* n /)//!(/). Clearly *n* is a relative width measure in *K* for {#}.

If *n* is a relative width measure in *K* for ©, and *4>* is a nonsingular affine transformation, there is a corresponding relative width measure in *cj)(K).* Define /z^"1 on #(A") by

$$(\mu\phi^{-1})(B) = \mu(\phi^{-1}(B))$$

for each Borel set *B* in ^(A"). If *6* e ©, and /f is a hyperplane orthogonal to *6,* then 0(//) is a hyperplane orthogonal to some direction *6'.* If 8' = {^':0e ©}, it is easy to see that *n<j)~l* is a relative width measure in *<f)(K)* for ©'.

THEOREM 1. *Let Kbea convex body in* R" *and 6\, 6i two directions. Then there is a relative width measure in Kfor {61,62}.*

Proof. If *n* > *2,* let *P* be the span of the directions *6* and *9₂,* and let *F denote projection onto *P.* Let *E* be any Borel subset of *K* for which *¥ is a bijection from *E* to *F(X) (such a set exists; see **[16,** 4D.13]). If // is a relative width measure in *R(AT) for {0j, 02}» define /i in A by

$$\tilde\mu(B) = \mu(\Psi(B \cap E))$$

for Borel sets *B c K.* Then /i is a relative width measure in *K* for {#1, #2}- So it suffices to consider the case *n* = *2.*

By using an affine transformation, and the remarks preceding this theorem, we may assume that *6* and *62* are parallel to the coordinate axes, and *K* is contained in the unit square / and meets all its sides. Let a1, <2 be the x-coordinates of any two points in the intersections of *dK* with the bottom and top sides of /, respectively, and *b\b₂* the *y*-coordinates of any two points in the intersections of *dK* with the right and left sides of /, respectively. The four points so obtained form the vertices of a quadrilateral $Q \subset K$ (which may degenerate to a triangle).

We consider two cases.

Case (i) $b_1 \le b_2$ and $a_1 \le a_2$, or $b_1 \ge b_2$ and $a_1 \ge a_2$. Let l_i, $1 \le i \le 4$, be the sides of Q (which cannot in this case be degenerate) labelled clockwise with l_1 the segment joining the points $(a_1, 0)$ and $(0, b_2)$. Let m be the line segment joining the points (a_1, b_1) and (a_2, b_2). For Borel sets B in K define

$$\begin{aligned} H(B) = {} & [a_1 b_2 \lambda(B \cap l_1)/\lambda(l_1)] + [a_2(1 - b_2)\lambda(B \cap l_2)/\lambda(l_2)] \\ & + [(1 - a_2)(1 - b_1)\lambda(B \cap l_3)/\lambda(l_3)] \\ & + [(1 - a_1) b_1 \lambda(B \cap l_4)/\lambda(l_4)] \\ & + [(a_2 - a_1)(b_2 - b_1)\lambda(B \cap m)/\lambda(m)]. \end{aligned}$$

Of course, H is the sum of suitable multiples of λ restricted to the line segments l_i and m. To check that H is a relative width measure in K for the coordinate directions is now simply a matter of computation.

Case (ii) $b_1 \ge b_2$ and $a_2 \ge a_1$ (or $b_1 \le b_2$ and $a_2 \le a_1$). Let l_i, $1 \le i \le 4$, be the sides of Q as in Case (i). Since m may not be contained in Q, and even if it is, H as defined above does not work, we use instead the two diagonals d_1 (joining $(a_1, 0)$ and $(a_2, 1)$) and d_2 (joining $(0, b_2)$ and $(1, b_1)$).

We seek non-negative multiples α_i and β_j of λ normalized on l_i and d_j, respectively, such that the sum of these measures is the required relative width measure.

Assuming $b_1 \ge b_2$ and $a_2 \ge a_1$ and considering slabs of the form $\{(x, y): 0 \le x \le c,\ c \le a_1\}$ we see that the equation

$$\frac{\alpha_1}{a_1} + \frac{\alpha_2}{a_2} + \beta_2 = 1$$

must be satisfied. Slabs of the form $\{(x,y): a_1 \le x \le c,\ c \le a_2\}$ and $\{(x,y): a_2 \le x \le c,\ c \le 1\}$ yield similar equations, and we obtain three more by looking at horizontal slabs. In addition, we require

$$\sum \alpha_i + \sum \beta_j = 1$$

for a probability measure.

The equations can be solved by setting $ft_x = 0$ or $p_2 = 0$. If $fi_x = 0$, we obtain:

$$a_x = a_1 b_1 b_2/\Delta_1,$$

$$a_2 = a_2(l-b_x)(l-b_2)/A_2,$$

$$a_3 = (l-a_2)(l-b_x)(\backslash-b_2)/A_2,$$

$$014 = (1 - a_x)b_x b_2/A_x \quad \text{and}$$

$$p_2 = (b_x - b_2)(a_x b_x(\wedge - b_2) - a_2 b_2(\wedge - b_x) - a_x a_2(b_x - b_2))/A_x \cdot A_2$$

where $Ai = b_2 + a_x(b_x - b_2)$ and $A2 = (1 - a_2)(b_x - b_2)$.

These are the solutions if the second factor in the numerator of $fl2$ is non-negative. If not, we set $p_2 = 0$, obtaining solutions for a,- from those above by interchanging a_x and b_2, a_2 and b_x, and a_2 and «4. We also get

$$fix = (a_2-a_x)(a_2 b_2(\wedge - a_x) - a_x b_x(\wedge - a_2) - b_x b_2(a_2 - a_x))/A'_1 \cdot A'_2$$

where A'_i is the expression corresponding to A_i. Denoting the second factor in the numerator of p_x by T_x, we see that $T_x + T_2 = 0$, so that $p_x > 0$ if $r_2 < 0$.

If $a_x = a_2$ and $b_x = b_2$, we solve as in Case (i). Degenerate cases where Q reduces to a triangle may be solved as above, and here the measure is supported by the boundary of the triangle.

It follows that Bang's conjecture is true for two slabs, and so Theorem 1 can be regarded as a generalization of this known result. In fact, only Case (i) of Theorem 1 is needed for this. To see this, suppose $K c S_x U S_2$, where S_x and S_2 are slabs orthogonal to the coordinate axes. Let $R = S_x$ n S_2, and let Q be the quadrilateral obtained by drawing tangent lines to the convex hull of K and R at the vertices of R. Now Q c S_x U S2, and we may assume Q is inscribed in the unit square. Then Q is as in Case (i) of Theorem 1. Since Q is wider than K in the coordinate directions, the existence of a relative width measure in Q gives the result for K.

In fact, the measure from Case (i) of Theorem 1 was found by analyzing the proof of Bang's conjecture for two slabs in [15].

In view of the difficulty of finding a relative width measure in K for two directions, it is surprising that there are convex bodies which have relative width measures for *all* directions; namely, the ellipses in R^2 and ellipsoids in R^3 (see Theorem 3). It would perhaps be unreasonable to expect all convex bodies to support such measures, but the point is that by Ohmann's reduction of Bang's problem, we only require a relative width measure in a convex body K in R" for the n coordinate directions. However, we show in Example 2 that this may not exist, even when $n = 3$.

The proof of the following lemma is a slightly modified version of ([8], Theorem 3), due to K. Falconer, which deals only with absolutely continuous //.

LEMMA 3. *Suppose K is a compact convex set in* R", O *is a set of directions, and JU is a relative width measure in Kfor ©. Let the centroid of K with respect to the density dfi be at the origin, and the support function of K be k(d). Then there is a second-degree homogeneous polynomial p(6) such that* $k^2(6) = p(6)$ *for all 6 € 0.*

Proof. Let *6* € ©, let *L* be the line through the origin in the direction *8*, and suppose that *v* is the projection of *fi* on *L*. Since // is a relative width measure in *K* for *{6},* we have

$$\nu = (k(\theta) + k(-\theta))^{-1}\lambda_L,$$

where *Xi* is A restricted to *L*. Now if / is a 'ridge function'—a function on *K* which depends only on $x \cdot 6 = t$ for all x e *K*—then

$$\int_{K} f(x)d\mu(x) = \int_{-k(-\theta)} f(t)du(t)$$

$$= (k(\theta) + k(-\theta))^{-1} \int_{-k(-6)}^{k(6)} f(t)dt.$$

Since the centroid of *K* with respect to *dfi* is at the origin, we have

$$0 = \left[\int_K x d\mu(x)\right] \cdot 6 = \int_K (x \cdot 0) d\mu(x)$$

$$= (k(6) + k(-6))^{-1} \int_{-k(-6)} t dt,$$

giving $k(6) = k(-6)$.

Taking second moments, we get

$$\int_K (x \cdot 6)^2 d\mu(x) = (k(6) + k(-6))^{-1} \int_{-k(-O)}^{k(6)} t^2\, dt = k^2(6)/3.$$

The left-hand side is a second-degree homogeneous polynomial in $6 = (6_1, \dots, 6_n)$, giving the result.

EXAMPLE 1. A convex body T in $\mathbf{R}^2$ with no relative width measure for a certain set of three directions.

Let T be the triangle with vertices at the origin, (0,1) and (1,0), and 9 the set of directions orthogonal to the edges of T. Suppose *ju* were a relative width measure in T for 0, and let c be the centroid of T relative to *dfi.* From Lemma 3 we require only the fact that if *t(6)* is the support function of T with respect to c, then *t(8) = t(—6)* for 0 € 0. From the coordinate directions, we get *c = {j, ¹j),* which contradicts *t(6) = t(—d)* for *6* orthogonal to the hypoteneuse of T.

EXAMPLE 2. A convex body U in R^3 with no relative width measure for the three coordinate directions.

Let U be the tetrahedron in R^3 with vertices at the origin, (1,0,0), (0, 1,0) and (0, 0,1). If a relative width measure */i* with centroid c existed, then, as in Example 1, using the x- and y-directions, *c —* *(\\a)* for some $a \geq 0$. Since c e U, $a = 0$, contradicting symmetry of the support function at c in the z-direction.

Many examples such as those above could be obtained in the same way. Let us note, however, that there are convex bodies in R^3 without relative width measures for the coordinate directions, to which Lemma 3 cannot be applied. One such is the regular octahedron centered at the origin, with axes in the coordinate directions. This has all the required symmetry, and another proof is needed to show that there is no measure; we omit this here. Despite the non-existence of a relative width measure for this regular octahedron or for the tetrahedron in Example 2, Bang's conjecture is true for 3 slabs covering these polyhedra orthogonal to the coordinate directions. To see this, apply Bang's theorem in its stronger form mentioned in § 1.

3. Existence and uniqueness for infinite 0. In this section we apply Fourier transform techniques to study the existence and uniqueness of relative width measures in convex bodies for infinite sets of directions.

THEOREM 2. *Let Kbea compact convex set in* R", *and* 0 *a (necessarily infinite) set ofdirections with the property that each analyticfunction on* R" *vanishing on each line through the origin parallel to some 6* e 0, *is identically zero. Then ifft is a relative width measure in Kfor ©, fi is unique.*

Proof. Suppose *//;,* / = 1,2, are relative width measures in K for 0. Let *fij* be the Fourier transform of *fij.* Then for </eR" with *d/\d\ = 6* € 0 and *x • 6 = t,* we have

$$\hat{\mu}_i(d) = \int_{JR»} e^{-ix-} dfii(x) = (k(6) + k(-6)y^{l} \; {}^{/} \int_{J-k(-e)}^{rH6} <?\text{-W} dt,$$

as in the proof of Lemma 3, since $e\sim^{lx \cdot d}$ is a ridge function. Here we are assuming that K contains the origin, and $k(6)$ is the support function of K. It follows that *jü\(d) = piiid)* for each d on a line through the origin parallel to some 6 e 0. It is known (see [5, p. 272]) that *p,j* is analytic, so by our assumptions *fi\ - fij* is identically zero. Since a measure is uniquely determined by its Fourier transform [5, Theorem 8.2.4], the result follows.

REMARK. In $R^{2,}$ any infinite set of directions satisfies the conditions of Theorem 2. In R^3 more is required, as easy examples show; however, a set of directions which is infinite in each of an infinite set of planes would do.

The following result was first proved in [23], in the language of probability theory.

THEOREM 3. *There is a relative width measure v_n in the unit ball B^n in* R" *for the unit sphere, S^{n-1}, the set of all directions, if and only if* ***n = 2 or 3.***

Proof. Let d € R", *d\d* = 6 and $x \cdot 6 = t$. Then the Fourier transform $\hat{v}_n$ of v_n must satisfy

$$\hat{v}_n(d) = f\ e^{-ixd}dv_n(x) = \backslash\ \int^1 e^{-i|d|t}$$

$$= \begin{cases} f(\sin|d|)/|d| & (|d| \neq 0), \\ 11 & (|d| = 0). \end{cases}$$

For $n = 2$, this is the Fourier transform of the function

$$g(x) = \begin{cases} (1 - |x|^2)^{-1/2} & (|x| < 1), \\ 0 & (|x| \geq 1), \end{cases}$$

so that *vi* is the measure with this density function (and is therefore absolutely continuous with respect to Lebesgue measure in R^2).

For $n = 3$, it is known (see, for example, [11, p. 199]) that £3 is the Fourier transform of the distribution *d(\x\ - l)/4n,* so that *z*ß is normalized surface area measure in $2?^3$.

If $n > 3$, the Fourier transform of $\hat{v}_n$ is known, since it appears in the solution of the wave equation [11, pp. 197-9], and is a distribution of higher order and not a measure. However, a more straightforward way to see that no measure exists for $n > 3$ is as follows. Suppose v_n to exist, and let // be the projection of v_n onto any 3-dimensional coordinate plane L, so that for Borel sets B in L, we have $fi(B)$ = $u_n(B \times R^{n-3})$. Then fi is a relative width measure in B^3 for S^2 that $jx = VT$, by Theorem 2. By the above, the support of UT, lies in S^2, so the support of v_n is contained in $(S^2 \times R^{n-3})$ in B^n, which lies in $R^3 \times \{0\}$, and hence in a coordinate hyperplane. This is impossible, since v_n must vanish on each hyperplane.

The above theorem uses ideas of K. Falconer [8, Theorem 3] for n = 2 and 3. The existence of v_2 and VT, have long been known. Indeed, u_2 is the projection of (1/3/2), and that 1ß has the right properties was observed by Archimedes!

THEOREM 4. *Let K be a convex body in* R", *and* 0 *a set ofdirections as in Theorem 2. Suppose a relative width measure n in K for* 0 *exists, the centroid ofK with respect to d/x is at the origin, and k(6) is the support function ofK. Then K contains a line segment I, or a 2-dimensional ellipse E or a 3-dimensional ellipsoid E, such that k(6) agrees with the support function of I or E, respectively, for 0 e* 0, *and ix is either*

(i) *normalized linear Lebesgue measure in I, or*

(ii) *the canonical relative width measure in E obtained by a suitable affine transformation of* v_2 *or* $v^\wedge$, *respectively.*

Proof. Lemma 3 shows that $k^2(d) = p(d)$ for 6 e 0, where $p(6)$ is a second-degree homogeneous polynomial in 6. It follows that $k(6)$ agrees with the support function of an m-dimensional ellipsoid for 6 € 0, for some m with $1 \leq m \leq n$. (For the non-degenerate case $m = n$, the proof is given in [19, p. 825, second paragraph]. In the degenerate case, suppose $p(8)$ contains only the variables $9\backslash, \ldots, 0_m$ for $m < n$. Then there is an m-dimensional ellipsoid E in R^m whose support function $h(6)$ satisfies $h^2(6) = p(9)$. Now note that the support function of E regarded as a subset of R" is still $h(6)$.) By taking an affine transformation we may assume that this ellipsoid is B^m. Therefore fi has the relative width property in B^m for 0, and even though the support of n may a priori not lie inside B^m, we may use Theorems 2 and 3 to see that m $\leq$ 3 and $/n$ is an affine image of u_2 in B^2 or ß3 in B^3; except in the degenerate case m = 1, when $k(6)$ agrees with the support function of a line segment for 6 e 0 and fi must be normalized linear Lebesgue measure on this segment (this is the case n = 1 in Theorem 3).

Since the support of // must lie inside K, we see that K contains the line segment, ellipse or ellipsoid, completing the proof.

Theorem 4 goes quite far in characterizing relative width measures for infinite 0. Although constraints are also placed on K, it need not actually be an ellipsoid. To see this, let 0 be an infinite set of directions in R^2 which is sparse enough to allow the existence of a convex body K, containing B^2 but different from B^2, such that for 6 e 0, K and B^2 have common supporting lines parallel to 6.

4. Uniqueness for finite 0. The results of §3 raise the question whether certain finite sets of directions might force uniqueness of the corresponding relative width measures. The next theorem uses a result from the theory of Radon transforms to show that this is not so, at least in dimensions two and three.

THEOREM 5. *Let n = 2or3, and let* 0 *be a finite set ofdirections in* R". *Then there are two different relative width measures in B^n for* 0.

Proof. Let $n \geq 2$. Denote by $H(t, 6)$ the hyperplane in R" orthogonal to 6 at distance \A from the origin. By [13, Proposition 7.6], if K is any compact set there is a function / supported in K, infinitely differentiate on K and not identically zero, such that

$$\int_{>H(t,0)} f(x)dm(x) = O$$

for all / and 8 e 0, where m is $(n - 1)$-dimensional Lebesgue measure in the hyperplane $H(t,8)$.

To deal with the case $n = 2$, take $K = B^2$ and let / be any such function; then / is bounded, $|/(x)| \leq M$ say. Let $g(x)$ be the density function for the measure v_2 (see Theorem 3), and note that $g(x) \geq (l/2n)$ for JC e $intB^2$. It follows that if we define

$$H(E)= \int_E f(g(x)-(f(x)/2nM))dx$$

for Borel sets E in B^2, then \i is a Borel measure in B^2 which is different from v_2.

Let 6 G 0, and let S be the slab orthogonal to 8 bounded by the hyperplanes $H(t_1,d)$ and $H(t_2,8)$. Then

$$\mu(S) = \int_S g(x)\,dx - \frac{1}{2\pi M}\int_S f(x)\,dx$$

$$= u_2(S) - \frac{1}{2nM} \int_{t_1}^{t_2} \int_{H(t,d)} f(x)dm(x)dt = u_2(S),$$

so that n is a relative width measure in B^2 for 0, as required.

Now let $n = 3$. If f is a spherical surface harmonic of order n on S^2, and θ is a direction in R^3 such that $f(\theta) = 0$, then

$$\int_{S^2 \cap H(t,\theta)} f \, d\lambda_t = 0,$$

where λ_t is normalized linear measure in the circle $S^2 \cap H(t, \theta)$. (This follows from [25, p. 100], which uses the expansion of f involving Legendre polynomials.) Suppose that $\Theta = \{\theta_1, \dots, \theta_k\}$ is a finite set of directions. By [21, §18], the vector space of (pure) harmonics on S^2 of order n has dimension $2n + 1$. Therefore if $2n + 1 > k$, by linear algebra there is a spherical harmonic f of order n such that $f(\theta_i) = 0$ for $i = 1, \dots, k$. If $|f| \leq M$ on S^2 and E is a Borel set in R^3, we put

$$\mu(E) = \int_{E \cap S^2} (1 - f(x)/M) \, d\nu_3(x).$$

The proof that μ is a relative width measure in R^3 for Θ which is different from ν_3 now follows that from the case $n = 2$ above.

It remains open whether there is a relative width measure in B^n, $n \geq 4$, for the coordinate directions (or any finite set of directions).

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DEPARTMENT OF MATHEMATICAL SCIENCES
KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
DHAHRAN 31261, SAUDI ARABIA