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ON THE ILIEFF-SENDOV CONJECTURE

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The well-known Ilieff-Sendov conjecture asserts that for any polynomial $p(z) = \prod_{k=1}^{n}(z-z_k)$ with $|z_k| \le 1$, each of the disks $|z-z_k| \le 1$ $(1 \le k \le n)$ must contain a critical point of p. This conjecture is proved for polynomials of arbitrary degree n with at most four distinct zeros. This extends a result of Saff and Twomey.

1. Introduction. The Gauss-Lucas Theorem states that all the critical points of a polynomial $p(z)$ lie in the convex hull of its zeros. This is a result concerning the position of all the zeros of $p'(z)$ relative to all the zeros of $p(z)$. Suppose we focus attention on any arbitrarily fixed zero of $p(z)$ and ask for the location of a zero of $p'(z)$ relative to it. This leads to the well-known conjecture of Ilieff and Sendov [3; Problem 4.5] which asserts that if $p(z)$ has the form

$$(1) \qquad p(z) = \prod_{k=1}^{n}(z-z_k), \qquad |z_k| \le 1 \quad (1 \le k \le n)$$

then each of the disks $|z - z_k| \le 1$ $(1 \le k \le n)$ contains a zero of $p'(z)$. The polynomial $p(z) = z^n - 1$ shows that "1" is sharp. This conjecture is nearly a quarter of a century old and has been verified in some special cases, most notably if $p(z)$ has the form (1) and if

(A) $2 \le n \le 5$ [1, 6, 8],

(B) $p(0) = 0$ [10],

(C) $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, $a_k \le 0$ $(0 \le k \le n-1)$ [11],

(D) $p(z)$ has only real zeros [7],

(E) $p(z) = \sum_{j=0}^{3} a_j z^{n_j}$ $(0 = n_0 < n_1 < n_2 < n_3)$ [12], or

(F) the vertices of the convex hull of the zeros of $p(z)$ all lie on $|z| = 1$ [10],

(G) the convex hull of the zeros of $p(z)$ is a triangular region [12],

(H) $p(z) = (z - z_1)^{n_1}(z - z_2)^{n_2}(z - z_3)^{n_3}$ [9].

The last case (H), due to Saff and Twomey, states that the conjecture is true for any polynomial of the form (1) with at most three distinct zeros. The purpose of this paper is to establish the conjecture for any polynomial with at most four distinct zeros.

Observe that (H) follows immediately from (G). A problem posed by Schmeisser [12] related to (G) is to determine whether the conjecture is true if the convex hull of the zeros of $p(z)$ is a quadrangular region. Our result verifies a special case of his problem.

2. Main results. We can now state our main result.

THEOREM. *If* $p(z) = (z - z_1)^{n_1}(z - z_2)^{n_2}(z - z_3)^{n_3}(z - z_4)^{n_4}$ *with* $|z_k| \leq 1$ $(1 \leq k \leq 4)$, *then each of the disks* $|z - z_k| \leq 1$ $(1 \leq k \leq 4)$ *contains a zero of* $p'(z)$.

Before embarking on the proof we briefly illustrate the idea by giving a simple proof of the Saff-Twomey result. Suppose $p(z) = (z - z_1)^{n_1}(z - z_2)^{n_2}(z - z_3)^{n_3}$ with $|z_k| \leq 1$. Let us distinguish one of the zeros, say z_3, and for convenience let $z_3 = a$. Thus $p(z) = (z - a)^{n_3}(z - z_1)^{n_1}(z - z_2)^{n_2}$. We wish to show $|z - a| \leq 1$ contains a zero of $p'(z)$. If $n_3 > 1$ we are done, so suppose $n_3 = 1$. It is then clear that $p'(z) = n(z-z_1)^{n_1-1}(z-z_2)^{n_2-1}(z-\zeta_1)(z-\zeta_2)$ $(n_1+n_2+1 = n)$ and if we let $q(z) = p(z)/(z - a)$ then also $p'(z) = (z - a)q'(z) + q(z)$. Hence we see that $p'(a) = q(a)$ and so

$$n(a - z_1)^{n_1-1}(a - z_2)^{n_2-1}(a - \zeta_1)(a - \zeta_2) = (a - z_1)^{n_1}(a - z_2)^{n_2}.$$

We may suppose that $|a - \zeta_1| \leq |a - \zeta_2|$ and get

$$n|a - \zeta_1|^2 \leq |a - z_1|\,|a - z_2| \leq 4.$$

Hence $|a - \zeta_1| \leq 2/\sqrt{n}$. If $n \geq 4$, we are done; while if $n = 2$ or 3 we already know the conjecture is true. The proof of the theorem is based on this simple idea, however we need some preliminary results first.

Fix integers n and m with $n \geq m \geq 2$ and let $\mathcal{P}_n(m)$ denote the class of all monic polynomials of degree n with at most m distinct zeros in $|z| \leq 1$:

$$(2) \qquad p(z) = \prod_{k=1}^{m}(z - z_k)^{n_k}, \qquad |z_k| \leq 1 \quad (1 \leq k \leq m),$$

where $n_k \geq 1$ and $\sum_{k=1}^{m} n_k = n$. From (2) we get

$$(3) \qquad p'(z) = n\left[\prod_{k=1}^{m}(z - z_k)^{n_k-1}\right]\left[\prod_{j=1}^{m-1}(z - \zeta_j)\right].$$

Let

$$I(z_k) = \begin{cases} \min\limits_{1 \le j \le m-1} |z_k - \zeta_j|, & \text{if } n_k = 1, \\ 0, & \text{if } n_k > 1, \end{cases}$$

$$I(p) = \max_{1 \le k \le m} I(z_k) \quad \text{and} \quad I(\mathscr{P}_n(m)) = \sup_{p \in \mathscr{P}_n(m)} I(p).$$

The Ilieff-Sendov conjecture says that $I(\mathscr{P}_n(m)) \le 1$.

LEMMA 1. (i) *There exists an extremal polynomial* $p^* \in \mathscr{P}_n(m)$ *such that* $I(p^*) = I(\mathscr{P}_n(m))$.

(ii) p^* *has a zero on each subarc of* $|z| = 1$ *of length* π.

This lemma is essentially proved in [5, 7] but for completeness sake and a slightly easier proof we present it here.

Proof. Since $|p(z)| \le (1 + r)^n$, where $|z| \le r$, for any $p \in \mathscr{P}_n(m)$, it is clear that $\mathscr{P}_n(m)$ is a normal family in C. Each polynomial in $\mathscr{P}_n(m)$ is monic and has at most m distinct zeros and hence $\mathscr{P}_n(m)$ is compact. By definition there exists a sequence $\{p_k\} \subset \mathscr{P}_n(m)$ such that $I(p_k) \to I(\mathscr{P}_n(m))$ as $k \to \infty$. Choose a convergent subsequence (call it $\{p_k\}$ again) so that $p_k \to p^*$ uniformly on compact subsets of C. Let

$$p^*(z) = \prod_{k=1}^{m} (z - z_k^*)^{n_k^*}, \quad |z_k^*| \le 1, \quad \sum_{k=1}^{m} n_k^* = n \tag{4}$$

and so

$$p^{*\prime}(z) = n \left[\prod_{k=1}^{m} (z - z_k^*)^{n_k^* - 1} \right] \left[\prod_{j=1}^{m-1} (z - \zeta_j^*) \right]. \tag{5}$$

Assume $I(p^*) < I(\mathscr{P}_n(m))$. Then $I(p^*) = I(\mathscr{P}_n(m)) - 3\varepsilon$, for some $\varepsilon > 0$. Choose $0 < \delta < \varepsilon$ so that $p^* \neq 0$ in $0 < |z - z_j^*| \le \delta$ $(1 \le j \le m)$. Thus all the zeros of p^* lie in $\Omega = \bigcup_{j=1}^{m} \{z : |z - z_j^*| < \delta\}$ and (by definition of $I(p^*)$) at least one zero of $p^{*\prime}$ lies in each of the disks $|z - z_j^*| \le I(p^*)$ $(1 \le j \le m)$. By Hurwitz' Theorem, for all $k \ge K_0$ sufficiently large, all the zeros of p_k lie in Ω and each disk $|z - z_j^*| \le I(p^*) + \varepsilon$ will contain a zero of p_k'. Hence we see that $I(p_k) \le I(p^*) + \varepsilon + \delta < I(p^*) + 2\varepsilon = I(\mathscr{P}_n(m)) - \varepsilon$. Letting $k \to \infty$ we get $I(\mathscr{P}_n(m)) \le I(\mathscr{P}_n(m)) - \varepsilon$ and a contradiction is reached. This proves (i).

To prove (ii) we first assert that the extremal polynomial p^* of the form (4) must have at least one zero on $|z| = 1$. Suppose not. Then $r = \max_{1\le k\le m} |z_k^*| < 1$. Define $\tilde{p}$ by

$$\tilde{p}(z) = \frac{1}{r^n} p^*(rz) = \prod_{k=1}^{m} \left(z - \frac{z_k^*}{r}\right)^{n_k^*}.$$

Clearly $\tilde{p} \in \mathcal{P}_n(m)$ and since $\tilde{p}'(z) = \frac{1}{r^{n-1}} p^{*\prime}(rz)$, we see that

$$I(\tilde{p}) = \max_{\substack{1\le k\le m\\ n_k=1}} \left[\min_{1\le j\le m-1} \left|\frac{z_k^*}{r} - \frac{\zeta_j^*}{r}\right|\right] = \frac{1}{r} I(p^*) > I(p^*).$$

Contradiction.

Next, we assert that p^* must have at least two distinct roots on $|z| = 1$. Suppose not. Then p^* has the form (4), with say $|z_m^*| = 1$ and $|z_k^*| < 1$ $(1 \le k \le m-1)$. By a rotation, we can assume $z_m^* = 1$. Let $s = (1-r)/2$, where $r = \max_{1\le k\le m-1} |z_k^*|$. If $\tilde{p}(z) = p^*(z+s)$, then again $\tilde{p} \in \mathcal{P}_n(m)$ and $I(\tilde{p}) = I(p^*)$. Hence $\tilde{p}$ is also extremal. However, it is easy to see that its zeros $\tilde{z}_k = z_k^* - s$ satisfy $|\tilde{z}_k| < 1$ $(1 \le k \le m)$. Contradiction.

We have shown that the extremal polynomial p^* has the form (4) with $|z_m^*| = |z_{m-1}^*| = 1$ and $z_m^* \ne z_{m-1}^*$. Assume now that there is some arc on $|z| = 1$ of length $L > \pi$, say $L = 2(\pi - \beta)$ for some $0 < \beta < \pi/2$, on which $p^* \ne 0$. By a rotation, we may suppose that $z_m^* = e^{i\alpha}$ and $z_{m-1}^* = e^{-i\alpha}$ for some $0 < \alpha < \beta < \pi/2$. (Thus $p^*(e^{i\theta}) \ne 0$ for $\alpha < |\theta| < \pi$.) By relabeling the zeros suppose that $|z_1^*| \le |z_2^*| \le \cdots \le |z_{n_0}^*| < 1$ and $|z_{n_0+1}^*| = \cdots = |z_{m-2}^*| = 1$ (put $n_0 = 0$ if all the zeros of p^* lie on $|z| = 1$). Define r and s as follows:

$$r = \begin{cases} 0, & \text{if } n_0 = 0,\\ \max_{1\le k\le n_0} |z_k^*|, & \text{if } n_0 \ne 0,\end{cases}$$

and

$$s = \min\left\{\cos\alpha, \frac{1-r}{2}\right\}.$$

Again consider $\tilde{p}(z) = p^*(z+s)$ and note that $\tilde{p} \in \mathcal{P}_n(m)$ with $I(\tilde{p}) = I(p^*)$. An easy check shows that the zeros $\tilde{z}_k = z_k^* - s$ of $\tilde{p}$ all lie in $|z| < 1$. Contradiction. This completes the proof of the lemma. ∎

LEMMA 2. *The Ilieff-Sendov conjecture is true if*

(i) $p(z) = \prod_{k=1}^{n}(z - z_k)$ *and* $2 \le n \le 4$, *or*

(ii) $p(z) = (z - e^{i\theta_0})^2 \prod_{k=1}^{n-2}(z - z_k)$ *and* $2 \le n \le 7$ $(\theta_0 \in \mathbb{R})$.

The first part of this lemma is known and there are various proofs, but the proof of (ii) gives (i) along the way so we include it here. This now makes the paper completely self-contained.

Proof. Let $p(z) = \prod_{k=1}^{n}(z - z_k)$ with $|z_k| \le 1$. Then $p'(z) = n\prod_{j=1}^{n-1}(z - \zeta_j)$. Distinguish one of the zeros, say z_n (call it a) and by a rotation let

$$p(z) = (z-a)\prod_{k=1}^{n-1}(z-z_k), \qquad 0 \le a \le 1, \qquad |z_k| \le 1.$$

We may also suppose that $0 < a < 1$. If $a = 0$, the conjecture is trivially true for this zero. If $a = 1$, we are also done. Indeed, if we let $q(z) = \prod_{k=1}^{n-1}(z - z_k)$. Then we see that $p''(1)/p'(1) = 2q'(1)/q(1)$ and so

$$\sum_{j=1}^{n-1}\frac{1}{1-\zeta_j} = \sum_{k=1}^{n-1}\frac{2}{1-z_k}. \tag{6}$$

Supposing $\mathrm{Re}\{1/(1-\zeta_1)\} \ge \mathrm{Re}\{1/(1-\zeta_j)\}$ $(1 \le j \le n-1)$ as we may, then from (6) we get

$$(n-1)\,\mathrm{Re}\left\{\frac{1}{1-\zeta_1}\right\} \ge 2\sum_{k=1}^{n-1}\mathrm{Re}\left\{\frac{1}{1-z_k}\right\} \ge (n-1).$$

Hence $\mathrm{Re}\{1/(1-\zeta_1)\} \ge 1$ or $|\zeta_1 - \tfrac{1}{2}| \le \tfrac{1}{2}$ so certainly $|\zeta_1 - 1| \le 1$ (cf. [2]). Hence suppose $0 < a < 1$.

If we put $z = T(w) = (w-a)/(aw-1)$, we have

$$p(T(w)) = \tilde{p}(w)(aw-1)^{-n}, \tag{7}$$

where

$$\tilde{p}(w) = Aw[w^{n-1} + b_{n-1}w^{n-2} + \cdots + b_1]. \tag{8}$$

From (7), the zeros of $\tilde{p}(w)$ are $0, w_1, w_2, \ldots, w_{n-1}$ where $w_k = T^{-1}(z_k)$ $(1 \le k \le n-1)$. Thus we get

$$|b_1| = \prod_{k=1}^{n-1}|w_k| \le 1 \tag{9}$$

and

$$|b_{n-1}| = \left|\sum_{k=1}^{n-1} w_k\right| \le n-1. \tag{10}$$

Differentiating (7) gives

$$\frac{dp(T(w))}{dw} = \frac{dp(T(w))}{dz}\,\frac{dz}{dw} = -a(aw-1)^{-n-1}D_{1/a}\tilde{p}(w),$$

where

(11) $$D_{1/a}\tilde{p}(w) = np(w) + (1/a - w)p'(w)$$

is the polar derivative of p with respect to $1/a$ (see Marden [4]). Hence we arrive at

(12) $$p'(T(w)) = -a(aw-1)^{-n-1}\frac{dw}{dz}D_{1/a}\tilde{p}(w)$$

(where ' denotes differentiation with respect to z). A brief calculation using (8) and (11) gives

(13) $$D_{1/a}\tilde{p}(w) = B\left[w^{n-1} + \cdots + \left(\frac{b_1}{n+ab_{n-1}}\right)\right] = B\prod_{k=1}^{n-1}(w-\gamma_k),$$

where $|\gamma_1| \le |\gamma_2| \le \cdots \le |\gamma_{n-1}|$. It follows from (9), (10) and (13) that

(14) $$\prod_{k=1}^{n-1}|\gamma_k| = \left|\frac{b_1}{n+ab_{n-1}}\right| \le \frac{1}{n-a(n-1)}.$$

Let us now suppose that $|\gamma_1| \le \mu$. Then from (12) we have

$$D_{1/a}\tilde{p}(\gamma_1) = -\frac{1}{a}(a\gamma_1 - 1)^{n+1}\frac{dz}{dw}p'(T(\gamma_1)) = 0.$$

Hence $p'(\zeta_0) = 0$, where $\zeta_0 = T(\gamma_1)$ and so $p'(z)$ has a zero ζ_0 such that

$$|\gamma_1| = |T^{-1}(\zeta_0)| = \left|\frac{\zeta_0 - a}{a\zeta_0 - 1}\right| \le \mu.$$

Thus we get $(\zeta_0 - a)/(a\zeta_0 - 1) = re^{i\theta}$ for some $\theta \in \mathbb{R}$, $0 < r \le \mu$. This gives $\zeta_0 = (a - re^{i\theta})/(1 - are^{i\theta})$ and we conclude that

$$|\zeta_0 - a| = \frac{r(1-a^2)}{|1-are^{i\theta}|} \le \frac{\mu(1-a^2)}{1-a\mu}.$$

Hence if $\mu(1-a^2)/(1-a\mu) \le 1$, or equivalently,

(15) $$\mu \le \frac{1}{1+a-a^2},$$

then $|\zeta_0 - a| \le 1$. It remains to show that (15) holds for the cases stated in the lemma.

Suppose first that $2 \leq n \leq 4$. Then from (14) we have

$$|\gamma_1| \leq \left[\frac{1}{n-a\{n-\backslash)_m}\right]^{1/(n-1)} \equiv \mu.$$

It is easy to check that (15) holds for this *fi* and $2 \leq n \leq 4$ (and any $0 < a < 1$). This proves (i).

Suppose next that $p(z)$ has the form stated in (ii). Then from (7) (for some we conclude that $p(w)$ has a double root at $T\sim^{l(e}$ UQ e R). From (11) we see that $y_{n}\text{-}\backslash = e^{m}$ is a zero of $D\backslash j_a p\{w)$. Using (14) we have the following

nitti-nittK $\dfrac{1}{n - a\{n-1)}$ ($n\text{-}l$, $k=1$; $n\text{-}\backslash$, $k=1$)

and so

$$|\gamma_1| \leq \left[\frac{1}{n-a(n-1)}\right]^{1/(n-2)} \equiv \mu.$$

For this n and $2 \leq n \leq 7$, it is simple to check that (15) holds for any $0<a<1$. •

We should point out that this proof can be slightly modified to give a proof of Laguerre's Theorem (see [4] for another proof).

Proofoftheorem. For fixed $n \geq m \geq 3$, let $p(z)$ be of the form (2). From Lemma 1, an extremal polynomial for $3^{\circ}{}_n(m)$ exists and without loss of generality we may assume p is extremal i.e., $I(p) = I\hat{}nim))$. Distinguish one of the zeros, say z_m and let $z_m = a$. We may suppose too that $0 \leq a \leq 1$. We want to show $\backslash z - a\backslash \leq 1$ contains a critical point of p. If $n_m > 1$, we are done so suppose $n_m = 1$. Thus we have

$$p(z) = (z-a)\prod_{k=\backslash}^{m-1}(z-z_k)^{n_k}, \qquad \backslash z_k\backslash < \backslash, \quad \sum_{k=1}^{m-1} n_k = n-1 \tag{16}$$

and

$$p'(z) = n\left[\prod_{k=1}^{m-1}(z-z_k)^{n_k-1}\right]\left[\prod_{j=1}^{m-1}(z-\zeta_j)\right]. \tag{17}$$

As in the proof of Lemma 2 we may suppose that $0 < a < 1$.

If we let $q(z) = Yl\hat{}ii^2 \sim zk)$ Using we see that $P'(a) = Qi^{a)}$- (16) and (17), we obtain after cancellations

$$n\prod_{j=1}^{m-\backslash}(a-\zeta_j) = \prod_{k=1}^{m-\backslash}(a-z_k).$$

Assuming that $|a - \zeta_1| \le |a - \zeta_j|$ $(1 \le j \le m - 1)$, as we may, we get

$$n|a - \zeta_1|^{m-1} \le \prod_{k=1} |a - z_k| \le |a - z_1||a - z_2| 2^{m-3}.$$

Hence we have the estimate

$$(18) \qquad |a - \zeta_1| \le \left[\frac{2^{m-3} |a - z_1||a - z_2|}{n} \right]^{1/(m-1)},$$

where z_1 and z_2 are any two distinct zeros of $p(z)$ other than $z = a$.

Suppose first that $n \ge 2^{m-1}$. From (18) we see that $|a - \zeta_1| \le 1$ and we are done. Hence suppose $2 \le n < 2^{m-1}$.

Now we restrict ourselves to $m = 4$. Thus we need only consider polynomials of the form (16) of degree $2 \le n < 8$. By Lemma 2(i) we need only consider $n = 5,6$ and 7. Thus, our polynomial has the form

$$p(z) = (z - a)(z - z_1)^{n_1}(z - z_2)^{n_2}(z - z_3)^{n_3}$$

where $0 \le n_1 \le n_2 \le n_3$ and $n_1 + n_2 + n_3 = n - 1$. Since $n = 5,6$ or 7, we must have $n_3 \ge 2$ in each case. Recall that $p(z)$ is an extremal polynomial for $\mathcal{P}_n(m)$ and hence by Lemma 1 it will have a zero on each subarc of $|z| = 1$ of length π. We are thus led to two cases:

Case 1. $p(z)$ has exactly two distinct zeros on $|z| = 1$.

Case 2. $p(z)$ has more than two distinct zeros on $|z| = 1$.

Case 2 is easily disposed of as follows. In this case (recall $0 < a < 1$), we must have $|z_1| = |z_2| = |z_3| = 1$ and since $n_3 \ge 2$, we see that p has a zero of order two on $|z| = 1$. Since $n \le 7$, we may apply Lemma 2(ii) and we are done.

In order for Case 1 to hold, the two zeros must be negatives of each other, say z_0 and $-z_0$. Making use of the estimate (18) we find that

$$|a - \zeta_1| \le \left[\frac{2|a - z_0||a + z_0|}{n} \right]^{1/3} \le \left[\frac{4}{n} \right]^{1/3}$$

and so $|a - \zeta_1| \le 1$ for $n \ge 4$. By Lemma 2(i), the conjecture is true for $n = 2$ and 3. The proof of the theorem is complete. $\square$

3. Remarks. The Ilieff-Sendov conjecture can be interpreted physically in terms of force fields. Indeed, if ζ_0 is a critical point of $p(z) = \prod_{k=1}^{n} (z - z_k)^{n_k}$ that is not a zero of p, then ζ_0 is an equilibrium point in the plane field in which the force exerted on a particle by a point charge at z_k (with charge n_k) is inversely proportional to its distance from z_k. The Ilieff-Sendov conjecture asserts that either the disk of radius one about each zero z_k contains at least one of these equilibrium points or else $n_k > 1$. In general these cases are mutually exclusive. For example if $p(z) = (z-1)^3(z-i)$, then the disk $|z-1| \leq 1$ contains no such equilibrium point. Since the conjecture is trivially true for the zero z_k if $n_k > 1$, the conjecture is only interesting if z_k is a simple zero. In this case we are led to the problem of determining just how close a critical point can then be to a simple zero. We thus pose the following problem:

> Let Q_n denote the set of all polynomials for the form $P(z) = z\prod_{k=1}^{n-1}(z - z_k)$, with $|z_k| \geq 1$ $(1 \leq k \leq n-1)$. Determine the largest constant $c_n > 0$ such that $p'(z) \neq 0$ in $|z| < c_n$ for all $p \in Q_n$.

Clearly $c_n < 1$ and we expect that the critical point nearest the zero $z = 0$ is an equilibrium point in the force field as described above. By concentrating all the charge at one point, together with a single charge at $z = 0$, it is reasonable to conjecture that $c_n = 1/n$ (consider $p(z) = z(z-e^{i\theta})^{n-1}$).

Finally, we must point out that recently a proof of the Ilieff-Sendov conjecture was announced (by title) in the *Abstracts of the American Mathematical Society* (June 1986) by V. I. Istrătescu. He claims his method uses a notion called "bare points" and "a Krein-Milman theorem". The proof of our main result is completely self-contained, relies on classical methods and indicates that a proof of the full conjecture must be delicate in nature.

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