PACIFIC JOURNAL OF MATHEMATICS
Vol. 136, No. 1, 1989

A NONLINEAR ELLIPTIC OPERATOR AND ITS SINGULAR VALUES

P. T. CHURCH AND J. G. TIMOURIAN

The boundary value problem $\Delta u + \lambda u - u^3 = g$ on Ω, $u|\partial\Omega = 0$, where $\Omega \subset \mathbf{R}^n$ $(n \leq 4)$ is a bounded domain, defines a real analytic map A_λ of the Sobolev space $H = W_0^{1,2}(\Omega)$ onto itself. A point $u \in H$ is a fold point if A_λ at u is C^∞ equivalent to $f \times \mathrm{id}\colon \mathbf{R} \times E \to \mathbf{R} \times E$, where $f(t) = t^2$. (1) There is a closed subset $\Gamma_\lambda \subset H$ such that (a) at each point of $A_\lambda^{-1}(H - \Gamma_\lambda)$ the map A_λ is either locally a diffeomorphism or a fold, and (b) for each nonempty connected open subset $V \subset H$, $V - \Gamma_\lambda$ is nonempty and connected; thus Γ_λ is nowhere dense in H and does not locally separate H. Suppose that $n \leq 3$ and the second eigenvalue λ_2 of $-\Delta u$ on Ω with $u|\partial\Omega = 0$ is simple. Define $A\colon H \times \mathbf{R} \to H \times \mathbf{R}$ by $A(u, \lambda) = (A_\lambda(u), \lambda)$. (2) There is a connected open neighborhood V of $(0, \lambda_2)$ in $H \times \mathbf{R}$ such that $A^{-1}(V)$ has three components U_0, U_1, U_2 with $A\colon U_i \to V$ a diffeomorphism for $i = 1, 2$ and $A|U_0\colon U_0 \to V$ C^∞ equivalent to $w \times \mathrm{id}\colon \mathbf{R}^2 \times E \to \mathbf{R}^2 \times E$ defined by $(w \times \mathrm{id})(t, \lambda, \nu) = (t^3 - \lambda t, \lambda, \nu)$.

We continue the study **[BCT-2]** of the equation

$$\Delta u + \lambda u - u^3 = g \quad \text{on } \Omega, \qquad u|\partial\Omega = 0,$$

where $\Omega \subset \mathbf{R}^n$ $(n \leq 4)$ is a bounded domain. If H is the Sobolev space $W_0^{1,2}(\Omega)$, define

$$\langle A_\lambda(u), \varphi \rangle_H = \int_\Omega [\nabla u \nabla \varphi - \lambda u \varphi + u^3 \varphi]$$

for all $\varphi \in C_0^\infty(\Omega)$, and define $A\colon H \times \mathbf{R} \to H \times \mathbf{R}$ by $A(u, \lambda) = (A_\lambda(u), \lambda)$.

Let SA_λ be the singular set (0.1) of the real analytic map A_λ. By Theorem (1.8) and Remark (1.9) there is a closed subset $\Gamma_\lambda \subset A_\lambda(SA_\lambda)$ such that (a) $A_\lambda^{-1}(H - \Gamma_\lambda)$ consists entirely of regular points $(u \notin SA_\lambda)$ and fold points (0.1) and (b) for every nonempty connected open subset V of H, $V - \Gamma_\lambda$ is nonempty and arcwise connected (so that H is not locally separated by Γ_λ at any point). Roughly, this states: most solutions g of $A_\lambda(u) = g$ come from only regular points **[Sm**, p. 862, (1.3)], and of the rest most come from only fold points. The relation between (1.8) and **[Mi]** is discussed in (1.10). A comparable result holds in the domain [**CDT**]: $\operatorname{int} SA = \emptyset$, and if $\Lambda \subset SA$ is the set of nonfold points and $V \subset H \times \mathbf{R}$ is a nonempty connected open subset, then $V - \Lambda$ is nonempty and connected.

There are [**BCT-2**, (3.9)] a connected open neighborhood V of $(0, \lambda_1) \in H \times \mathbf{R}$ and C^∞ diffeomorphisms φ and ψ such that $A|A^{-1}(V)$: $A^{-1}(V) \to V$ (with $n \leq 3$) is $\psi \circ (w \times \mathrm{id}) \circ \varphi$, where $w \times \mathrm{id}$: $\mathbf{R}^2 \times E \to \mathbf{R}^2 \times E$ is given by $(w \times \mathrm{id})(t, \lambda, v) = (t^3 - \lambda t, \lambda, v)$. Now suppose that λ_2 is a simple eigenvalue of $-\Delta$ on Ω (with null boundary conditions). Then there is (2.4) a connected open neighborhood V of $(0, \lambda_2)$ in $H \times \mathbf{R}$ such that $A^{-1}(V)$ has three components U_0, U_1, U_2 with A: $U_i \approx V$ a diffeomorphism for $i = 1, 2$ and $A|U_0$: $U_0 \to V$ being $\psi \circ (w \times \mathrm{id}) \circ \varphi$ above. That $A_\lambda(u) = 0$ has exactly five solutions u for $\lambda_2 < \lambda < \lambda_2 + \varepsilon$ and $\varepsilon > 0$ sufficiently small was previously noted in [**AM**, p. 642, Theorem (3.4)].

The set of (weak) solutions of the boundary value problem for a given g and λ is the point inverse set $A^{-1}(g, \lambda)$, and we are naturally led to a study of the singularities and structure of A, as in this paper. For a more detailed discussion see [**CT-2**, Introduction].

0.1. DEFINITIONS. Let E_1 and E_2 be Banach spaces, let U be open in E_1, let $u \in U$, and let A: $U \to E_2$ be a C^k ($k = 1, 2, \ldots$ or ∞) map. If $DA(u)$ is surjective, we say that u is a *regular point* of A. The *singular set* SA is the set of nonregular points. We say that the map A is *Fredholm at* u *with index* ν if $DA(u)$ is a Fredholm linear map with index ν, i.e., $a = \dim\ker DA(u)$ is finite, $\operatorname{Range} DA(u)$ is closed, and its codimension b in E_2 is finite, with $\nu = a - b$; if A is Fredholm at each point of U, we say that A is a *Fredholm* map.

If $k \geq 2$ with (0) A Fredholm at u with index 0, (1) $\dim\ker DA(u) = 1$ (and therefore range $DA(u)$ has codimension one), and (2) for some (and hence for any) nonzero element $e \in \ker DA(u)$

$$D^2A(u)(e, e) \notin \operatorname{Range} DA(u),$$

then we say that u is a *fold point* of A.

If (2) is replaced by its negation, and we add (3) for some $\omega \in T_uE_1$,

$$D^2A(u)(e, \omega) \notin \operatorname{Range} DA(u),$$

then we say that u is a *precusp point* of A (see [**BCT-1**, p. 3, (1.6)] and [**BCT-2**, (3.1), (3.2)]).

These notions are invariant under coordinate change [**BCT-1**, p. 9, (3.2)].

0.2. THEOREM ([**BC**, *p.* 950], [**BCT-1**, (1.5)] *and* (1.7)). *If A has a fold at $\bar{u}$, then A at u is locally C^{k-2} equivalent* [**BCT-1**, (1.2)] *to*

$$F: \mathbf{R} \times E \to \mathbf{R} \times E, \quad (t, v) \to (t^2, v) \text{ at } (0, 0).$$

If $k \geq 4$, the converse is true.

0.3. NOTATION. An ordered pair in $X \times Y$ is denoted by (x, y), while the inner product of x and y in a Hilbert space H is denoted by $\langle x, y \rangle_H$. Real analytic [**Z**, p. 362, (8.8)] is denoted by C^ω. Assume throughout that Ω is a bounded connected open subset of $\mathbf{R}^n$ ($n \leq 4$). In general, notation follows that in [**BCT-2**] and [**CT-2**].

Church thanks Syracuse University for research leave during 1986–87, and the University of Alberta for its hospitality and support during that period.

1. The exceptional set $\Gamma_\lambda \subset A_\lambda(SA_\lambda)$. Our goal in this section is the proof of Theorem 1.8. "Dimension" is defined in [**HW**, p. 10 and p. 24].

1.1. LEMMA [**B**, *p.* 14, *Proposition*]. *Let M^n be an n-manifold without boundary, and let X be a closed subset. Then:*

(a) $\dim X \leq n-1$ *if and only if X contains no nonempty open subset of M^n; and*

(b) $\dim X \leq n-2$ *if and only if X contains no nonempty open subset of M^n, and for every connected open subset V of M^n, $V - X$ is connected.*

In [**Bo**, p. 14, Proposition] use $L = Z$, the group of integers under addition. See also [**HW**, p. 24; p. 26, Theorem III 1; p. 41, Theorem IV 1; p. 48, Theorem IV 4; and pp. 151–152].

1.2. LEMMA. *Let M^n be an n-manifold without boundary, let E be a connected locally connected topological space, and let X be a closed subspace of $M^n \times E$.*

(a) *If* $\dim(X \cap (M^n \times v)) \leq n-1$ *for every $v \in E$, then X contains no nonempty open subset of $M^n \times E$.*

(b) *If* $\dim(X \cap (M^n \times v)) \leq n-2$ *for every $v \in E$, then for every nonempty connected open subset V of $M^n \times E$, $V - X$ is nonempty and connected.*

(c) *Let $\pi_1 \colon M^n \times E \to M^n$ and $\pi_2 \colon M^n \times E \to E$ be projections. If* $\dim \pi_1(X) \leq n-1$ *and $\pi_2(X)$ contains no nonempty open subset of E,* *then for every nonempty connected open subset V of $M^n \times E$, $V - X$ is nonempty and connected.*

Proof. Conclusion (a) is immediate from (1.1)(a). Let S_1, S_2 and S_3 be the following statements:

(S_1) $\dim(X \cap (M^n \times v)) \leq n - 2$ for every $v \in E$.

(S_2) If B and D are nonempty connected open subsets of M^n and E, respectively, then $(B \times D) - X$ is connected and nonempty.

(S_3) If V is a nonempty connected open subset of $M^n \times E$, then $V - X$ is connected and nonempty.

We first prove that S_1 implies S_2. Let U be a component of $(B \times D) - X$, so U is open in $B \times D$. By (1.1)(b) $(B \times v) - X$ is connected for every $v \in D$, so if $(B \times v) - X$ meets U, then $(B \times v) - X \subset U$. The set $S(U)$ of $v \in D$ such that $(B \times v) - X \subset U$ is nonempty, open, and closed (since $S(U')$ is open for the other components U' of $(B \times D) - X$). Since D is connected, $S(U) = D$, i.e., $U = (B \times D) - X$ so that $(B \times D) - X$ is connected.

Next we prove that S_2 implies S_3. Let V be any connected open subset of $M^n \times E$, let W be a component of $V - X$, and suppose $W \neq V - X$; since $V \cap \overline{W} \subset W \cup X$, $V \cap \overline{W} \neq V$. Let $y \in V \cap \operatorname{bdy} \overline{W}$. There are connected open subsets B and D of M^n and E, respectively, such that $y \in B \times D \subset V$, and thus $(B \times D) \cap W \neq \varnothing$. Since $(B \times D) - X$ is connected open in $V - X$, $(B \times D) - X \subset W$, so $B \times D \subset V \cap \overline{W}$. As a result, $y \in V \cap \operatorname{int}(\overline{W})$, contradicting its choice. Thus $V - X$ is connected, as desired.

Conclusion (b) results from the two previous paragraphs.

We next prove that the hypotheses of (c) imply S_2. By **[HW**, p. 41, Theorem IV 1] there exists $\bar{x} \in B - \pi_1(X)$, and thus

$$\bigcup\{x \times D : x \in B - \pi_1(X)\} \cup \bigcup\{B \times v : v \in D - \pi_2(X)\},$$

call it Y, is a connected subset of $(B \times D) - X$. Let U be the component of $(B \times D) - X$ containing Y, and let $(x, v) \in (B \times D) - X$. There are connected open B' and D' in B and D, respectively, such that $(x, v) \in B' \times D' \subset (B \times D) - X$, and since $(B' \times D') \cap Y \neq \varnothing$, $(x, v) \in U$. Now (x, v) is arbitrary, so $(B \times D) - X$ is connected.

Since S_2 implies S_3, conclusion (c) follows from the previous paragraph.

1.3. Remark. Lemma 1.2 can be generalized with the same proofs. Replace M^n by any connected, locally connected topological space M, replace "$\dim(X \cap (M^n \times v)) \leq n - 1$ [resp., $\dim \pi_1(X) \leq n - 1$]" by (i) "$X \cap (M \times v)$ contains no nonempty open subset of $M \times v$ [resp., M]", and replace "$\dim(X \cap (M^n \times v)) \leq n - 2$" by (i) and (ii) "for every connected open subset V' of $M \times v$, $V' - X$ is connected".

1.4. DEFINITIONS [**Mi**, p. 288]. Let Y be a locally arcwise connected metric space. A subset S of Y does not *disconnect locally* if for every $x \in S$ there exists a fundamental system $\mathscr{B}$ of open spheres with center at x, arcwise connected, and such that, for every $B \in \mathscr{B}$, $B - S$ is still arcwise connected. A subset S of Y is said to be *supermeager* if S is meager (i.e., of first category) and does not disconnect locally.

1.5. LEMMA. *Let Y be a Banach manifold, and let $S \subset Y$ be a countable union of closed subsets of Y. Then S is supermeager if and only if, for every nonempty connected open subset $V \subset Y$, $V - S$ is nonempty and arcwise connected.*

Thus, if E in (1.2) is a Banach manifold, then the conclusion in (1.2)(b) and (c) may be restated: X is supermeager. Lemma 1.5 is true for any locally arcwise connected metric space Y, if $\operatorname{int}_Y S = \varnothing$.

Proof. Assume S is supermeager and write $S = \bigcup_{j=1}^{\infty} S_j$, where each S_j is closed and (1) we may suppose that $S_1 = \varnothing$.

We first prove that (2) each S_j is supermeager. Let $x \in S_j$, let $\mathscr{B}$ be given by (1.4) for S and x, let $B \in \mathscr{B}$, and let $x_1, x_2 \in B - S_j$. Choose arcwise connected open subsets $U_i \subset B - S_j$ with $x_i \in U_i$, and use the Baire Theorem to choose $z_i \in U_i - S$ ($i = 1, 2$). There is an arc in $B - S$ joining z_1 and z_2, and thus a path in $B - S_j$ joining x_1 and x_2; (2) results.

Let $V \subset Y$ be any nonempty connected open subset, and let $y_0, y_1 \in V - S$; we prove that there is a path $\gamma \subset V - S$ joining y_0 to y_1, and thus obtain the desired conclusion. The proof is given in [**Mi**, Proposition 1, beginning at the top of p. 289], except that B is replaced by V, we use (1), and $2b_1 = \min\{1, d(\Phi_1([0; 1]), S_1)\} = 1$. [The word "radius" is omitted in "whose radius is $r \leq \min\{b_1, 1/4\}$".]

1.6. LEMMA. *Let X and Y be C^2 separable manifolds over* (*real*) *Banach spaces, and let $A: X \to Y$ be a C^2 Fredholm map of index* 0. *Let S^*A be the set of $u \in X$ such that either*

(a) $\dim \ker DA(u) > 1$, *or*

(b) *u is a precusp point* (0.1).

Then, for every nonempty connected open set $V \subset Y$, $V - A(S^*A)$ is nonempty and arcwise connected.

The conclusion is equivalent (1.5) to: $A(S^*A)$ is supermeager in Y. (See the following proof.)

Proof. Let RA and CA be the set of $u \in X$ satisfying hypotheses (a) and (b), respectively. For each $u \in CA$ there is [**BCT-1**, p. 9, (3.3)] an open neighborhood W of u and a C^2 diffeomorphism β^{-1} of W onto an open set in $E_1 = \mathbf{R} \times E \times \mathbf{R}$ such that $\beta^{-1}(u) = (0, 0, 0)$, E is a Banach space,

$$A\beta\colon \beta^{-1}(W) \to E_2 = \mathbf{R} \times E \times \mathbf{R}, \quad (t, v, \lambda) \to (h(t, v, \lambda), v, \lambda)$$

with

$$(\partial h/\partial t)(0, 0, 0) = 0, \quad (\partial^2 h/\partial t^2)(0, 0, 0) = 0, \quad \text{and} \quad (\partial^2 h/\partial t\, \partial\lambda)(0, 0, 0) \neq 0.$$

There is [**Sm**, pp. 862–863, (1.6)] an open neighborhood V of $(0, 0, 0)$ such that $\bar{V} \subset W$ and $A|\bar{V}\colon \bar{V} \to Y$ is proper and thus closed. By the Implicit Function Theorem [**Z**, p. 150, 4.B] there are an open neighborhood P of $(0, 0)$ in $\mathbf{R} \times E$, an open interval I about 0 in $\mathbf{R}$, and a C^1 map $\lambda\colon \bar{P} \to \mathbf{R}$ such that

$$S(A\beta) \cap (\bar{P} \times \bar{I}) = \operatorname{graph} \lambda \subset \bar{P} \times \bar{I} \subset V. \tag{1}$$

Define $\mu\colon \bar{P} \to \bar{P} \times \mathbf{R}$ by $\mu(t, v) = (h(t, v, \lambda(t, v)), v, \lambda(t, v))$; since $\partial h/\partial t \equiv 0$ on graph λ and $\partial^2 h/\partial t^2 = 0$ if and only if $\partial\lambda/\partial t = 0$, (2) $C(A\beta) \cap (\bar{P} \times I)$ is the set T of $(t, v, \lambda(t, v))$ for which $\partial\lambda/\partial t = 0$. For each fixed v, define $\mu_v(t) = (h(t, v, \lambda(t, v)), \lambda(t, v))$. According to [**C**, p. 1037, Proposition 4] (3) if $f\colon M^n \to N^p$ is a $C^{\max(n-k,1)}$ map and $R_k(f)$ is the set of points $x \in M^n$ at which $Df(x)$ has rank at most k, then $\dim(f(R_k(f))) \leq k$. It follows that (4) $\mu(T \cap (\mathbf{R} \times v))$ has dimension at most 0. Alternatively, define $\pi_i\colon \mathbf{R}^2 \to \mathbf{R}$ $(i = 1, 2)$ by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. From Sard's Theorem [**Sa**, p. 883] $\pi_i(\mu(T \cap (\mathbf{R} \times v)))$ has dimension 0 $(i = 1, 2)$, and (4) results from (1.2)(c) and (1.1)(b). That $A\beta(T)$ is supermeager follows from (4) and (1.2)(b). Now $\beta C(A\beta) = W \cap CA$ [**BCT-1**, p. 9, (3.2)], and it follows from (1) and (2) that (5) for each $u \in A$, there is an open neighborhood Q of u in X such that $A(\bar{Q} \cap CA)$ is a closed supermeager set in Y.

According to [**Mi**, p. 291, Theorem A] (or [**CT-1**, Theorem 1] and (1.5)) $A(RA)$ is supermeager in Y; since A is locally proper [**Sm**, pp.

862–863, (1.6)], for each $u \in RA$, there is an open neighborhood Q of u in X such that $A(\bar{Q} \cap RA)$ is a closed supermeager set in Y. Since X is separable, there is a countable collection of open sets Q_i of X such that $RA \cup CA \subset \bigcup_i Q_i$ and $A(\bar{Q}_i \cap (RA \cup CA))$ is a closed supermeager subset of Y. The conclusion follows from [**Mi**, p. 288, Proposition 1]: if Y is a Banach space and S is the countable union of closed supermeager subsets of Y, then S is supermeager.

1.7. Hypotheses. In (1.8) assume the following hypotheses on $f : \mathbf{R} \to \mathbf{R}$: (1) f is C^2, (2) $f(0) = 0 = f'(0)$, and (3) for every $s \neq 0$ in $\mathbf{R}$, (a) $f'(s) \geq 0$ and (b) $f''(s) \neq 0$. It follows from the Mean Value Theorem that (4) $f''(0) = 0$ and (5) for every $s \neq 0$ in $\mathbf{R}$, (a) $f'(s) > 0$ and (b) $sf''(s) > 0$.

Let Ω be a bounded domain in $\mathbf{R}^n$ ($n \leq 4$), let $H = W_0^{1,2}(\Omega)$, and formally define $A_\lambda : H \to H$ by

$$\langle A_\lambda(u), \phi\rangle_H = \int_\Omega [\nabla u \nabla \phi - \lambda u \phi + f(u)\phi]$$

for every $\phi \in C_0^\infty(\Omega)$, and $A : H \times \mathbf{R} \to H \times \mathbf{R}$ by $A(u, \lambda) = (A_\lambda(u), \lambda)$. Assume sufficient hypotheses of f and n so that A_λ is C^2 (e.g., f is C^3 and $f^{(3)} \in L^\infty(\Omega)$).

An example is $f(s) = s^3$.

1.8. Theorem. *Let A_λ be as given in* (1.7), *and let CA_λ be the set of singular points not fold points* (0.1). *Then, for every nonempty connected open $V \subset H$, $V - A_\lambda(CA_\lambda)$ is nonempty and* (*arcwise*) *connected. An analogous result holds for A and $H \times \mathbf{R}$.*

Thus $A_\lambda(CA_\lambda)$ is supermeager in H ((1.5) and [**Sm**; pp. 862–863, (1.6)]). The theorem states roughly: most solutions g of $A_\lambda(u) = g$ come from only regular points u [**Sm**], and of the remainder most come from only fold points. For $\lambda < \lambda_1$, A_λ is a diffeomorphism [**BCT-2**, (2.3)], and 0 is the only singular point of A_{λ_1} [**BCT-2**, (2.7)i)].

Proof. Since A_λ is C^1 Fredholm of index 0 [**BCT-2**, (2.5)], $A_\lambda(SA_\lambda)$ is meager in H by the Smale-Sard Theorem [**Sm**, p. 862, (1.3)]. That $A_\lambda(CA_\lambda)$ is supermeager in H will follow from (1.6), once we prove: (1) If $u \in SA_\lambda$, $(u, \lambda) \neq (0, \lambda_i)$ ($i = 1, 2, \ldots$), and $\dim(\ker DA_\lambda(u)) = 1$ with generator e, then there exists $\omega \in H$ such that

$$0 \neq \langle D^2 A_\lambda(u)(e, \omega), e\rangle_H = \int_\Omega f''(u) e^2 \omega.$$

Suppose that (1) fails for $\omega = u$. By (1.7) $sf''(s) > 0$ for $s \neq 0$, so that (2) $ue = 0$ a.e. By (1.7) $f'(0) = 0$ and thus $\int_\Omega f'(u)e\psi = 0$ for every $\psi \in H$; since $\langle DA_\lambda(u) \cdot e, \psi \rangle_H = 0$, $\lambda = \lambda_i$ and $e = \phi_i$, the ith eigenvalue and eigenvector of $-\Delta$ with null boundary conditions on Ω ($i = 1, 2, \dots$). Since ϕ_i is real analytic [**BJS**, p. 136 and pp. 207–210], $\phi_i(x) \neq 0$ a.e., so that $u(x) = 0$ a.e. Thus (1) is satisfied, and the conclusion for A_λ results.

For A note that (1) becomes

$$(1') \qquad \langle D^2A(u,\lambda)((e,0),(0,1)),(e,a)\rangle_{H\times\mathbf{R}} \neq 0,$$

where (e,a) is orthogonal to the codimension 1 subspace Range $DA(u,\lambda)$ and $a = \langle u, Le\rangle_H = \int_\Omega ue$ [**BCT-2**, proof of (3.5)]; (1′) is $-\langle Le, e\rangle = -1 \neq 0$.

1.9. Remark. In case $f(u) = u^3$, A and A_λ are proper [**BCT-2**, (2.8)] so that $\Gamma_\lambda = A_\lambda(CA_\lambda)$ is a closed subset of H satisfying the conditions stated in the introduction. More generally, sufficient conditions for $f(u)$ in (1.7) to be proper are given in [**BCT-2**, (2.9)].

1.10. Remark. In [**Mi**] the author discusses smooth Fredholm maps of index 0, and calls a singular value $y \in A(SA)$ an ordinary value if every $u \in A^{-1}(y)$ is either a fold point or a regular point (0.1). In the introduction [**Mi**, p. 288] she states (1) "Finally we ha[v]e that for a smooth proper Fredholm map of index 0, the critical values y are ordinary value[s] (i.e., y is image of a finite number of singular point[s] in each of which the operator behaves locally making a fold) ex[c]ept [for] a supermeager set". Statement (1) is false in the generality claimed: define $A\colon \mathbf{R} \to \mathbf{R}$ by $A(t) = t^3$.

One may put together [**Mi**, Proposition 1, p. 288; Theorem A, p. 291; and Theorem D, p. 296] to obtain (1) under an additional hypothesis: this result is Lemma 1.6 (see (1.5)), except that she assumes C^4, rather than our C^2 hypothesis in (1.6).

2. The structure of A at $(0, \lambda_2)$. The main result of §2 is (2.4), which gives the structure of $A|A^{-1}(V)\colon A^{-1}(V) \to V$, where A is the map of the introduction, V is an open neighborhood of $(0,\lambda)$, and $\lambda < \lambda_2 + \varepsilon$ for some $\varepsilon > 0$. Theorem 2.4, as well as the other results of §2, applies to a more general map (2.1), used in [**BCT-2**] and [**CT-2**], so that map is now defined.

2.1. Definition [**BCT-2**, (1.2)]. *The abstract map A.* Consider any Hilbert space H over the real numbers and a map $A_\lambda\colon H \to H$ defined by

$$A_\lambda(u) = u - \lambda Lu + N(u),$$

where L and N have the following properties:

(1) L is a compact, self-adjoint, positive linear operator ($\langle Lu, u\rangle_H \geq 0$ and $= 0$ only if $u = 0$). It follows [**D**, pp. 349–350] that H is separable and the eigenvalues λ_m ($m = 1, 2, \dots$) of $u = \lambda Lu$ are positive, $\lambda_m \leq \lambda_{m+1}$, and (if H is infinite dimensional) $\lambda_m \to \infty$ as $m \to \infty$. Let $\{u_m\}$ be an orthonormal basis of H of eigenvectors.

(2) The first eigenvalue λ_1 is simple.

(3) (a) The map N is C^k ($k = 1, 2, \dots$ or ∞ or ω) such that $DN(u)$ is nonnegative self-adjoint ($\langle DN(u)\cdot v, v\rangle_H \geq 0$ for every $v \in H$).

(b) If $\langle DN(u)\cdot u_m, u_m\rangle_H = 0$ for some m ($m = 1, 2, \dots$), then $u = 0$. [Statement (b_1) is: $\langle DN(u)\cdot u_1, u_1\rangle_H = 0$ implies $u = 0$.]

(c) $k \geq 2$ and $D^j N(0) = 0$ for $j = 0, 1, 2$. [Statement c_j) for $j = 0, 1, 2$ is: N is C^j and $D^j N(0) = 0$.]

(d) $k \geq 3$ and $\langle D^3N(u)(v, v, v), v\rangle_H > 0$ for $0 \neq v \in H$.

(e) $D^4N(u) \equiv 0$. From Taylor's Theorem [**Z**, p. 148, Theorem 4.A] it follows that N is real analytic, and assuming (3)(c), $(3!)N(u) = D^3N(0)(u, u, u)$, so that $2DN(u)\cdot v = D^3N(0)(u, u, v)$.

We refer to a map A_λ satisfying (1) and (3)(a) above, and to A defined by $A(u, \lambda) = (A_\lambda(u), \lambda)$, as *abstract* A_λ and A. If a result requires an additional hypothesis from the list above, that fact is explicitly indicated.

2.2. EXAMPLE [**BCT-2**, (1.3)]. *The standard map A.* Our main example of abstract A is the map A of the first paragraph of this paper; it satisfies all the properties of (2.1) and we call it standard A. Here H is the Sobolev space $W_0^{1,2}(\Omega)$ [**B-1**, p. 28], where Ω is a bounded connected open subset of $\mathbf{R}^n$ with $n \leq 4$, and the operators L and N are defined by

$$\langle Lu, \varphi\rangle_H = \int_\Omega u\varphi \quad \text{and} \quad \langle N(u), \varphi\rangle_H = \int_\Omega u^3\varphi$$

for all $\varphi \in C_0^\infty(\Omega)$, the space of C^∞ real valued functions with compact support in Ω. Standard A is proper for $n \leq 3$ [**BCT-2**, (2.8)]. For more information about standard A, see [**BCT-2**, (1.3)], and for a generalization with certain functions $f(u)$ in place of u^3, see [**BCT-2**, (1.4)].

Other examples of (2.1) are given in [**BCT-2**, (1.7) and (1.8)]. The von Kármán equations for the buckling of a thin planar elastic plate yield an operator A satisfying most of the properties of (2.1) (see **[BCT-2**, §4, especially (4.6)]).

If $\lambda_j(u)$ ($j = 1, 2, \ldots$) is the jth eigenvalue of $v - \lambda Lv + DN(u) \cdot v = 0$, then SA (0.1) is the union of the graphs of $\lambda_j \colon H \to \mathbf{R}$ **[CT-2**, (1.5)]. We first consider the action of the group $\mathbf{Z}/2\mathbf{Z}$ on H ($A_\lambda(-u) = -A_\lambda(u)$), and now observe that graph λ_j ($j = 1, 2, \ldots$), the singular set SA, the set of fold points, and the set of cusp points are all invariant under this action.

2.3. Remark. Consider abstract A_λ with (2.1) (3)(c) and (e), $u \in H$ and $\lambda \in \mathbf{R}$. Then:

(i) The eigenvalues $\lambda_j(-u) = \lambda_j(u)$ and their eigenspaces are the same ($j = 1, 2, \ldots$).

(ii) If u is a singular point [resp., fold point, cusp point] (0.1), then so is $-u$ and $\ker DA_\lambda(u) = \ker DA_\lambda(-u)$.

(a) For a fold point u, $\langle D^2 A_\lambda(u)(e, e), e\rangle_H$ ($\int_\Omega ue^3$ in the standard case (2.2)) reverses sign if u is replaced by $-u$.

(b) For a cusp point u,

$$\langle D^3 A_\lambda(u)(e, e, e), e\rangle_H - 3\langle D^2 A_\lambda(u)(e, y), e\rangle_H,$$

which for standard A_λ is

$$\int_\Omega e^4 - 3\int_\Omega ue^2 y$$

(see the proof of **[BCT-2**, (3.6)]), preserves sign if u is replaced by $-u$, where

$$y \in [DA_\lambda(u)]^{-1}(D^2 A_\lambda(u)(e, e))$$

and $y(-u) = y(u)$ (modulo $\ker DA_\lambda(u)$).

(iii) If A_λ is proper and every component of $A_\lambda^{-1}(0)$ is a point, then $A_\lambda^{-1}(0)$ has an odd number m ($m = 1, 3, 5, \ldots$) of points (solutions).

A degree argument does not yield (iii), since 0 may be in $A_\lambda(SA_\lambda)$. If we assume (2.1) (2) (3) (b_1) (c) and (d), by **[BCT-2**, (3.8)] there is an open neighborhood V of $(0, \lambda_1)$ in $H \times \mathbf{R}$ such that $A|A^{-1}(V)\colon A^{-1}(V) \to V$ is C^∞ equivalent to $w \times \mathrm{id}$ given by $(w \times \mathrm{id})(t, \lambda, v) = (t^3 - \lambda t, \lambda, v)$; thus, if u is any fold point of A_λ and $A_\lambda(u) = g$ where $(g, \lambda) \in V$, then $A_\lambda^{-1}(g)$ has precisely two points. As a result, 0 in (iii) cannot be replaced by arbitrary $g \in A_\lambda(SA_\lambda)$. From (2.2), for $n \le 3$ standard A satisfies the hypotheses of (2.3).

Proof. By (2.1) (3)(c) and (e) $DA_\lambda(u) = I - \lambda L + DN(u)$, $3!N(u) = D^3N(0)(u,u,u)$, $2DN(u)\cdot v = D^3N(0)(u,u,v)$, $D^2N(u)(v,w) = D^3N(0)(u,v,w)$, $D^3N(u)(v,w,x) = D^3N(0)(v,w,x)$, and $D^jN(u) \equiv 0$ for $j \geq 4$; thus $D^jN(-u) = (-1)^{j+1}D^j(u)$ $(j = 0, 1, \dots)$. Conclusion (ii) is immediate, and since $\lambda_j(u)$ is the jth eigenvalue $(j = 1, 2, \dots)$ of $v - \lambda Lv + DN(u)\cdot v = 0$ [**CT-2**, (1.1)], conclusion (i) results.

For (iii), from the properness of A_λ, $A_\lambda^{-1}(0)$ is a compact 0-dimensional set; since A_λ is real analytic, $A_\lambda^{-1}(0)$ is finite. Now $A_\lambda(0) = 0$, and if $u \neq 0$ and $A_\lambda(u) = 0$, then $A_\lambda(-u) = 0$, yielding conclusion (iii). Conclusion (iii) is related to Borsuk's Theorem [**D**, p. 21, Theorem 4.1].

2.4. THEOREM. *Consider a C^k $(k = 3$ [resp., ∞]$)$ proper map abstract A satisfying in addition* (2.1) (2) (3)(b)(c)(d) *and* (e), *e.g. standard A with $n \leq 3$* [**BCT-2**, (1.3) *and* (2.8)]; *the symbol $\approx$ below means homeomorphism* [*resp., C^∞ diffeomorphism*]. *Let $\lambda < \lambda_2 + \varepsilon$ for $\varepsilon > 0$ sufficiently small, and if $\lambda_2 \leq \lambda < \lambda_2 + \varepsilon$, assume that λ_2 is a simple eigenvalue of $v = \lambda Lv$, e.g. of $-\Delta$. Then there is a connected open neighborhood V of $(0, \lambda)$ in $H \times \mathbf{R}$ such that $A^{-1}(V)$ has $2m+1$ components U_i with $A(U_i) = V$ $(i = 0, \pm 1, \dots, \pm m)$ and $(0, \lambda) \in U_0$.*

(a) *For $\lambda < \lambda_1$, $m = 0$; for $\lambda_1 < \lambda < \lambda_2$, $m = 1$; for $\lambda_2 < \lambda < \lambda_2 + \varepsilon$, $m = 2$; and $A\colon U_i \approx V$ $(i = 0, \pm 1, \dots, \pm m)$.*

(b) *For $\lambda = \lambda_1$, $m = 0$ and there are φ and ψ such that the diagram*

$$\begin{tikzcd} U_0 \arrow[r, "\approx", "\varphi"'] \arrow[d, "A"'] & \mathbf{R}^2 \times E \arrow[d, "w\times\mathrm{id}"] \\ V \arrow[r, "\approx", "\psi"'] & \mathbf{R}^2 \times E \end{tikzcd}$$

commutes, where $\varphi(0, \lambda_1) = (0,0,0) = \psi(0, \lambda_1)$, E is closed subspace of H and $w(t, \lambda) = (t^3 - \lambda t, \lambda)$ (cf. [**BCT-2**, *figure* 1] *and* [**GG**, *p.* 147]).

(c) *If $\lambda = \lambda_2$, then $m = 1$, $A\colon U_1 \approx V$ $(i = \pm 1)$, and $A|U_0\colon U_0 \to V$ is $\psi(w \times \mathrm{id})\varphi$ as in* (b).

Proof. Conclusion (a) for $\lambda < \lambda_1$ is [**BCT-2**, (2.3)] and (b) is [**BCT-2**, (3.8) (and (3.9))].

The singular set image $(w \times \mathrm{id})S(w \times \mathrm{id})$ separates $\mathbf{R}^2 \times E$ into two components C_1 and C_3 such that if $p \in C_i$, then $(w \times \mathrm{id})^{-1}(p)$ has i points $(i = 1, 3)$; and $S(w \times \mathrm{id})$ separates $\mathbf{R}^2 \times E$ into two components B_1 and B_3, where $w \times \mathrm{id}\colon B_3 \approx C_3$. Because of the equivalence in (b),

$A|U_0\colon U_0 \to V$ has the same property, giving components B_1', B_3', C_1', C_3' with $A\colon B_3' \approx C_3'$. Since λ_1 is simple (2.1), if $(g, \lambda) \in V$ (and V is sufficiently small) then $\lambda < \lambda_2$; thus $S(A|U_0)$ is part of the graph of $\lambda_1\colon H \to \mathbf{R}$ **[CT-2,** (1.5) and (2.2)]. Now (u, λ) is in one component or the other of $U_0 - S(A|U_0)$ depending on whether $\lambda < \lambda_1(u)$ or $\lambda > \lambda_1(u)$. If $T = \{(u, \lambda)\colon \lambda < \lambda_1\}$, then $A|T\colon T \approx T$ **[BCT-2,** (2.3)]. Thus B_3' must be $\{(u, \lambda)\colon \lambda > \lambda_1(u)\}$, $(0, \lambda) \in B_3'$ for $\lambda_1 < \lambda < \lambda_1 + \delta$ for some $\delta > 0$, and (1) $(A|U_0)^{-1}(0, \lambda) = A^{-1}(0, \lambda)$ had three points for such λ.

By **[CT-2,** (3.1) (ii)], (2) if $\lambda_1 < \lambda \leq \lambda_2$, then $(0, \lambda) \notin A(SA)$ except that $A(0, \lambda_2) = (0, \lambda_2) \in A(SA)$ **[BCT-2,** (2.6)]. Since A is proper, the image $A(\operatorname{graph} \lambda_1)$ is closed in $H \times \mathbf{R}$ and $(0, \lambda_2) \notin A(\operatorname{graph} \lambda_1)$. Thus (3) there is an $\varepsilon > 0$ sufficiently small that $(0, \lambda) \notin A(\operatorname{graph} \lambda_1)$ for $\lambda_2 < \lambda < \lambda_2 + \varepsilon$. (4) If, in addition λ_2 is simple, then $(0, \lambda) \notin A(SA)$ by **[CT-2,** (3.1)(i)].

For $\Gamma = \{(0, \lambda)\colon \lambda_1 < \lambda < \lambda_2\}$, $A^{-1}(\Gamma) \to \Gamma$ is a proper local homeomorphism by (2), and thus is a finite-to-one covering map **[P,** p. 128]. Since Γ is simply connected, A maps each component of $A^{-1}(\Gamma)$ homeomorphically onto Γ **[Ma,** p. 159, Theorem 6, or p. 160, Exercise 6.1], and (by (1)) (5) $A^{-1}(0, \lambda)$ has three points for each λ with $\lambda_1 < \lambda < \lambda_2$. Conclusion (a) for $\lambda_1 < \lambda < \lambda_2$ results from **[BCT-2,** (3.7)].

Conclusion (c) for *some* number of components results from **[BCT-2,** (3.6) and (3.7)] and $m = 1$ follows from (5) and (2).

Let $\Lambda = \{(0, \lambda)\colon \lambda_2 < \lambda < \lambda_2 + \varepsilon\}$ where ε is given in (3) and (4). As for Γ above, by (4) each component of $A^{-1}(\Lambda)$ is mapped homeomorphically on Λ. By (c) and the argument of the second paragraph applied to $A|U_0\colon U_0 \to V$ about $(0, \lambda_2)$, there are three components of $A^{-1}(\Lambda)$ inside U_0 for ε sufficiently small; and since by (c) $A\colon U_1 \approx V$ and $A\colon U_{-1} \approx V$, there are five components altogether. Conclusion (a) for $\lambda_2 < \lambda < \lambda_2 + \varepsilon$ results from **[BCT-2,** (3.7)].

That $A_\lambda(u) = 0$ has exactly five solutions u for λ_2 simple and $\lambda_2 < \lambda < \lambda_2 + \varepsilon$ with ε sufficiently small was noted in **[AM,** p. 642, Theorem 3.4]. That it has three solutions for $\lambda_1 < \lambda < \lambda_2$ was noted in **[B-2]**, in each case for a class of maps A including standard A.

2.5. REMARK. For standard A (2.2) and each A_λ with $n \leq 3$, degree A = degree $A_\lambda = 1$ and for U_i given by (2.4) (c) (at λ_2), degree $A|U_i = 1$ for $i = -1, 1$, and degree $A|U_0 = -1$.

Proof. For $(u, \lambda) \in (H \times \mathbf{R}) - SA$ and U a bounded open neighbourhood of (u, λ) such that A maps U diffeomorphically onto its image, let the local degree of A at (u, λ), $\deg A|U = \deg(A, U, A(u, \lambda))$ [**D**, p. 56]. From [**D**, p. 56, (D3)] it is constant on each component of $(H \times \mathbf{R}) - SA$. By [**D**, p. 64, Theorem 8.10] for $(u, \lambda) = (0, \lambda) = A(u, \lambda)$ it is $+1$ if $0 < \lambda < \lambda_1$ and -1 if $\lambda_1 < \lambda < \lambda_2$. From the argument of (2.4), especially the second paragraph, the U_1 and U_{-1} of (2.4)(c) are in the same component as $(0, \lambda)$ for $0 < \lambda < \lambda_1$, and U_0 is in the same component as $(0, \lambda)$ for $\lambda_1 < \lambda < \lambda_2$, and the local conclusions result.

Now degree A means $\deg(A, H \times \mathbf{R}, y)$ [**D**, p. 56 and p. 87] for any $y \in H \times \mathbf{R}$; we may take $y = (0, \lambda)$ for $0 < \lambda < \lambda_1$, so degree $A = 1$. (Since $\sum_{i=0}^{2} \operatorname{degree} A|U_i = 1$, this conclusion is confirmed [**D**, p. 56, (D2)].)

REFERENCES

[AM] A. Ambrosetti and G. Mancini, *Sharp nonuniqueness results for some nonlinear problems*, Nonlinear Anal., (1979), 635–645.

[B-1] M. S. Berger, *Nonlinearity and Functional Analysis*, Academic Press, New York, 1977.

[B-2] ———, *Nonlinear problems with exactly three solutions*, Indiana Univ. Math. J., **28** (1979), 689–698.

[BC] M. S. Berger and P. T. Church, *Complete integrability and perturbation of a nonlinear Dirichlet problem.* I, Indiana Univ. Math. J., **28** (1979), 935–952. Erratum, ibid., **30** (1981), 799.

[BCT-1] ———, *Folds and cusps in Banach spaces, with applications to nonlinear partial differential equations.* I, Indiana Univ. Math. J., **34** (1985), 1–19.

[BCT-2] ———, *Folds and cusps in Banach spaces, with applications to nonlinear partial differential equations.* II, Trans. Amer. Math. Soc., **307** (1988), 225–244.

[BJS] L. Bers, F. John, and M. Schecter, *Partial Differential Equations*, John Wiley and Sons, New York, 1964.

[B] A. Borel, *Seminar on Transformation Groups*, Annals of Mathematical Studies, Number 46, Princeton Univ. Press, Princeton, NJ, 1960.

[C] P. T. Church, *On points of Jacobian Rank k.* II, Proc. Amer. Math. Soc., **16** (1965), 1035–1038.

[CDT] P. T. Church, E. N. Dancer, and J. G. Timourian, preprint.

[CT-1] P. T. Church and J. G. Timourian, *Critical values of Fredholm maps*, Proc. Amer. Math. Soc., (to appear).

[CT-2] ———, *The singular set of a nonlinear elliptic operator*, Michigan Math. J., (to appear).

[D] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.

[GG] M. Golubitsky and V. Guillemin, *Stable Mappings and their Singularities*, Springer-Verlag, New York, 1973.

[HW] W. Hurewitz and H. Wallman, *Dimension Theory*, Princeton Univ. Press, Princeton, NJ, 1948.

[Ma] W. S. Massey, *Algebraic Topology: An Introduction*, Springer-Verlag, New York, 1967 (fourth corrected printing, 1977).

[Mi] A. M. Micheletti, *About differentiable mappings with singularities between Banach spaces*, Analisi Funzionale e Applicazioni, Boll. Un. Mat. Ital. Supp., **1** (1980), 287–301.

[P] R. Palais, *Natural operations on differential forms*, Trans. Amer. Math. Soc., **92** (1959), 125–141.

[Sa] A. Sard, *The measure of the critical values of differentiable maps*, Bull. Amer. Math. Soc., **48** (1942), 883–890.

[Sm] S. Smale, *An infinite dimensional version of Sard's Theorem*, Amer. J. Math., **87** (1965), 861–867.

[Z] E. Zeidler, *Nonlinear Functional Analysis I. Fixed-Point Theorems*, Springer-Verlag, New York, 1986.

Received August 19, 1987 and in revised form March 1, 1988. Both authors partially supported by NSERC Contract A7357.

SYRACUSE UNIVERSITY
SYRACUSE, NY 13244-1150

AND

UNIVERSITY OF ALBERTA
EDMONTON, ALBERTA, CANADA T6G 2G1