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THE C^*-ALGEBRAS ASSOCIATED WITH MINIMAL HOMEOMORPHISMS OF THE CANTOR SET

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We investigate the structure of the C^*-algebras associated with minimal homeomorphisms of the Cantor set via the crossed product construction. These C^*-algebras exhibit many of the same properties as approximately finite dimensional (or AF) C^*-algebras. Specifically, each non-empty closed subset of the Cantor set is shown to give rise, in a natural way, to an AF-subalgebra of the crossed product and we analyze these subalgebras. Results of Versik show that the crossed product may be embedded into an AF-algebra. We show that this embedding induces an order isomorphism at the level of K_0-groups. We examine examples arising from the theory of interval exchange transformations.

1. Preliminaries. We begin with an introduction to some terminology and notation, and a description of the results.

Throughout, we will let X denote the Cantor set. That is, X is a totally disconnected compact metrizable space with no isolated points. Generally, for any compact Hausdorff space, Z, we let $C(Z)$ denote the C^*-algebra of continuous complex-valued functions on Z.

We say a subset E of X is clopen if it is both open and closed. We let χ_E denote the characteristic function of E, which will be continuous if E is clopen. A partition, $\mathscr{P}$, of X we define to be a finite collection of pairwise disjoint clopen sets whose union is all of X. If $\mathscr{P}$ is a partition of X, we let $\mathscr{C}(\mathscr{P}) = \operatorname{span}\{\chi_E | E \in \mathscr{P}\}$. $\mathscr{C}(\mathscr{P})$ may be viewed as those functions in $C(X)$ which are constant on each element of $\mathscr{P}$. The fact that X is totally disconnected implies that any function in $C(X)$ may be approximated by one in some $\mathscr{C}(\mathscr{P})$. Given two partitions $\mathscr{P}_1$ and $\mathscr{P}_2$, of X, we say $\mathscr{P}_2$ is finer than $\mathscr{P}_1$ and write $\mathscr{P}_2 \geq \mathscr{P}_1$, if each element of $\mathscr{P}_2$ is contained in a single element of $\mathscr{P}_1$. This is clearly equivalent to the condition that $\mathscr{C}(\mathscr{P}_1) \subset \mathscr{C}(\mathscr{P}_2)$. Given two partitions $\mathscr{P}_1$ and $\mathscr{P}_2$, we define the partition $\mathscr{P}_1 \vee \mathscr{P}_2$ to be $\{E \cap F | E \in \mathscr{P}_1, F \in \mathscr{P}_2\}$.

We let φ be a homeomorphism of X which we shall always assume to be minimal. That is, there are no closed φ-invariant sets except for the empty set and X itself. This is equivalent to the condition that, for any point x in X, the set $\{\varphi^n(x) | n \geq 0\}$ is dense in X. We shall refer to

$\{\varphi^n(x) | n \in \mathbf{Z}\}$ as the orbit of x under φ. The sets $\{\varphi^n(x) | n \leq 0\}$ and $\{\varphi^n(x) | n \geq 0\}$ will be called half-orbits. Given two homeomorphisms, φ and ψ, of X, we say that they are (topologically) conjugate and write $\varphi \sim \psi$ if there is a homeomorphism h of X such that $\varphi = h \circ \psi \circ h^{-1}$.

We also regard φ as a $*$-automorphism of $C(X)$ by defining $\varphi(f) = f \circ \varphi^{-1}$, for all f in $C(X)$. This generates an action of the group of integers on $C(X)$ and we shall consider the crossed product C^*-algebra $C(X) \times_\varphi \mathbf{Z}$. This is described completely in Chapter 7 of Pedersen **[7]**. For our purposes, we regard it as the C^*-algebra generated by $C(X)$ and a unitary operator, which we will always denote by u, such that $ufu^* = \varphi(f)$, for all f in $C(X)$. By 5.15 of Zeller-Meyer **[18]**, the minimality of φ implies that this C^*-algebra is simple.

One of our main tools will be K-theory. The standard references are Blackadar **[2]** and Effros **[6]**. We will especially make use of the order structure on K_0. Since most of our algebras will be unital and most $*$-homomorphisms will preserve units, we will use the terms "ordered group" and "order isomorphism" to actually mean "ordered group with order unit" (namely the class of the identity element) and "order isomorphism which preserves order units", respectively.

We note that we may identify $K_0(C(X))$ with $C(X, \mathbf{Z})$, the continuous functions on X taking integer values. Under this correspondence, $K_0(C(X))^+$ is identified with the functions taking non-negative values. Also, we note that $K_1(C(X)) = 0$. The K-theory of $C(X) \times_\varphi \mathbf{Z}$ may be computed with the aid of the Pimsner-Voiculescu six-term exact sequence (see Pimsner and Voiculescu **[10]** or Blackadar **[2]**). We summarize the results in the following theorem.

THEOREM 1.1. *With X and φ as above, we have*

(i) $K_1(C(X) \times_\varphi \mathbf{Z}) \simeq \mathbf{Z}$ *and is generated by* $[u]$.

(ii) $K_0(C(X) \times_\varphi \mathbf{Z}) \simeq C(X, \mathbf{Z})/\mathrm{Im}(\mathrm{id} - \varphi_*)$, *where* $\mathrm{id} - \varphi_*$ *is considered as an endomorphism of $C(X, \mathbf{Z})$. More precisely, the inclusion of $C(X)$ into $C(X) \times_\varphi \mathbf{Z}$ is a surjection at the level of K_0 whose kernel is* $\mathrm{Im}(\mathrm{id} - \varphi_*)$.

We also mention here that the "non-stable K-theory" (see Rieffel **[14]**) of $C(X) \times_\varphi \mathbf{Z}$ has been computed by the author **[11]**.

A C^*-algebra is called AF (or approximately finite dimensional) if it is the closure of the union of an increasing sequence of finite dimensional C^*-subalgebras (see Blackadar or Effros). The K-theory of AF-algebras plays a major rôle in the theory. Elliott's theorem states that two unital AF-algebras are $*$-isomorphic if and only if their K_0-groups are order isomorphic (see Blackadar [**2**]).

Here, we show that there is a close relation between AF-algebras and our C^*-algebras $C(X) \times_\varphi \mathbf{Z}$. In §3, we show that for each non-empty closed subset $Y \subset X$, the C^*-subalgebra of $C(X) \times_\varphi \mathbf{Z}$ generated by $C(X)$ and $uC_0(X - Y)$ is an AF-algebra. We denote this subalgebra by A_Y. We give a (partial) description of the K-theory of such a C^*-subalgebra in §4. In particular, if the closed set Y is a single point, the inclusion induces an order isomorphism between the K_0-group of A_Y and that of $C(X) \times_\varphi \mathbf{Z}$. The analysis of Stratila and Voiculescu [**15**] of "diagonal" subalgebras in an AF-algebra lends itself very well to our situation. We examine this in §5. The results allow us to compute the ideal structure of A_Y, in particular. In §6, we show that $C(X) \times_\varphi \mathbf{Z}$ may be embedded into an AF-algebra so that the map induced at the level of K_0 is an order isomorphism. This embedding has already been obtained by Versik [**16**], but since the results of [**16**] are entirely in measure theoretic terms rather than topological terms and since Versik does not compute the map of K_0-groups, we include a proof of this here.

We provide some general examples in §2, and in §7, we conclude with some specific examples of interest, mention some consequences of our results and state some open problems.

2. Examples. Here we present two classes of examples of minimal homeomorphisms of the Cantor set. The first class consists of what are commonly called "odometers" (for reasons which will be obvious). The C^*-algebras arising as the crossed products are the Bunce-Deddens C^*-algebras which have also appeared in many other guises (see 10.11.4 of Blackadar [**2**]).

Many of our results here are already known. Indeed, our AF-subalgebras A_Y, in the case that the closed set Y is a single point, are actually UHF-algebras (see 6.4 of Pedersen [**7**]) and the containment of A_Y in $C(X) \times_\varphi \mathbf{Z}$ appears in the original work of Bunce and Deddens [**3**]. The embedding of Theorem 6.7 in this case was also obtained independently by K. Schmidt and C. Skau.

The second class of examples are obtained from interval exchange transformations. Interval exchange transformations are usually regarded as automorphisms of the Lebesgue space $(L^2(0, 1), \mathscr{B}, \lambda)$, where $\mathscr{B}$ denotes the σ-algebra of Borel subsets of $(0, 1)$ and λ denotes Lebesgue measure. They are bijections of $[0, 1)$ which are "piecewise translations". (We will restrict our attention to "minimal" ones.)

It is not difficult to view these as minimal homeomorphisms of Cantor sets, as we shall show. We will also compute the K-theory of the crossed product C^*-algebras in this situation. In §7, we will consider some specific examples.

Odometers. Let $\{n_i\}_{i=1}^{\infty}$ be a sequence of integers, each greater than or equal to 2. Let $X = X_i\{0, 1, \ldots, n_i - 1\}$. The homeomorphism φ is addition of $(1, 0, 0, 0, \ldots)$ with carry over to the right. In fact, X can be given the structure of a compact abelian group so that φ is just addition of $(1, 0, \ldots)$. With this point of view, the results of Riedel **[13]** are applicable. From **[13]**, we have

$$K_0(C(X) \times_{\varphi} \mathbf{Z}) \simeq \{k/(n_1 n_2 \cdots n_m) | k \in \mathbf{Z},\ m \in \mathbf{Z}^+\}$$

with order structure that inherited from $\mathbf{R}$.

Let φ and ψ be the odometers associated with the sequences $\{n_i\}$ and $\{m_i\}$ respectively. Then φ and ψ are topologically conjugate if and only if the supernatural numbers (see 6.4.8 of Pedersen **[7]**) $\prod_i n_i$ and $\prod_i m_i$ are equal; that is, if n is an integer which divides $n_1 n_2 \cdots n_k$, for some k, then for some l, n divides $m_1 m_2 \cdots m_l$, and vice versa. This, in turn, is true if and only if there is an order isomorphism between the K_0-groups of the associated crossed product C^*-algebras.

Interval exchange transformations. For a more complete discussion of interval exchange transformations, we refer the reader to Chapter 5 of Cornfield, Fomin and Sinai **[4]**. For the most part, we will adopt the notation of **[4]**. Let $M = [0, 1)$ with Lebesgue measure λ. For two intervals E and F in M, we write $E < F$ if each element of E is less than each element of F.

Choose $0 = x_0 < x_1 < x_2 < \cdots < x_{r-1} < x_r = 1$, and let $\Delta_i = [x_{i-1}, x_i)$, for each $i = 1, \ldots, r$. Let $\pi \in S_r$, the permutation group of $\{1, \ldots, r\}$. From this data, we define $T: M \to M$ by $Tx = x + \alpha_i$, for $x \in \Delta_i$, where the α_i are uniquely determined so that $T(\Delta_{\pi(1)}) < T(\Delta_{\pi(2)}) < \cdots < T(\Delta_{\pi(r)})$ and so that T is bijective.

We will assume that the transformation T is minimal in the sense that the orbit under T of any point in M is dense in M.

We wish to construct a Cantor set X and a minimal homeomorphism φ of X so that M is a dense subset of X and $T = \varphi|M$. Let $\mathscr{L}^{\infty}(T) = \{T^n(x_i) | 0 \leq i < r \text{ and } n \in \mathbf{Z}\}$. Our hypothesis of minimality of φ implies that $\mathscr{L}^{\infty}(T)$ is dense in M. To obtain X from M, each point y in $\mathscr{L}^{\infty}(T)$ is replaced by two points y^- and y^+ and we also include the point 1. The space X inherits an order structure from M an obvious way (i.e. if $y_1 \leq y_2$ are in $\mathscr{L}^{\infty}(T)$, then $y_1^{\pm} \leq y_2^{\pm}$) and we set $y^- < y^+$ for all y in $\mathscr{L}^\infty(T)$. The order topology on X obtained makes X a Cantor set. (What we have done amounts to inserting "Cantor gaps" at the points of $\mathscr{L}^\infty(T)$.) We then define $\varphi: X \to X$ in an obvious way so that φ is a homeomorphism of X and $\varphi = T$ on $M - \mathscr{L}^\infty(T) \subset X$. As an alternative we could define X as the spectrum of the C^*-algebra generated by all L^∞ functions (acting on $L^2(0,1)$) which are continuous except for jump discontinuities at finitely many points, all of which are in $\mathscr{L}^\infty(T)$. We will also view the elements of $C(X, \mathbf{Z})$ as functions on M in a similar fashion. We remark that such functions have at most finitely many jump discontinuities.

We note that $C(X) \times_\varphi \mathbf{Z}$ may be viewed as the C^*-algebra of operators on $L^2(0,1)$ generated by $\chi_{\Delta_1}, \dots, \chi_{\Delta_r}$ and the unitary operator $u\xi = \xi \circ T^{-1}$ for $\xi \in L^2(0,1)$.

The K-theory of these C^*-algebras may be computed easily with the aid of the Pimsner-Voiculescu exact sequence. We do not state any results here about the order on K_0, but in §7 we will deal with some explicit examples completely.

THEOREM 2.1. *Let $r, x_0, x_1, \dots, x_r$ and π as above be such that the interval exchange transformation T is minimal and is such that the orbits under T of the points $x_1, \dots, x_{r-1}$ are pairwise disjoint. Let φ be as above. The map $\gamma: \mathbf{Z}^r \to K_0(C(X) \times_\varphi \mathbf{Z})$ defined by*

$$\gamma(k_1, \dots, k_r) = \sum_j k_j [\chi_{\Delta_j}]$$

is an isomorphism of abelian groups.

The proof of the theorem follows easily from the following two lemmas and the Pimsner-Voiculescu exact sequence.

LEMMA 2.2. *With the same hypotheses as in 2.1, define $\tilde{\gamma}: \mathbf{Z}^r \to C(X, \mathbf{Z})$ by*

$$\tilde{\gamma}(k_1, \dots, k_r) = \sum_j k_j \chi_{\Delta_j}.$$

Then $\operatorname{Im} \tilde{\gamma} \cap \operatorname{Im}(\mathrm{id} - \varphi_*) = 0$.

Proof. Suppose that $h = \sum k_j \chi_{\Delta_j} = f - f \circ \varphi^{-1}$, for some f in $C(X, \mathbf{Z})$. Our hypothesis implies that each point of $\mathscr{L}^\infty(T)$ has a unique representation as $T^m(x_i)$. If f is continuous at each point of $\mathscr{L}^\infty(T)$ then it is constant and so $h = 0$, as desired. Suppose this is not the case. Then the set of discontinuities of f is finite and we choose $T^m(x_i)$ where f has a discontinuity and so that m is minimum among such points. Then f must be continuous at $T^{m-1}(x_i)$. So $h = f - f \circ \varphi^{-1}$ has a discontinuity at $T^m(x_i)$. From this we see that $m = 0$. We conclude that if f has a discontinuity at $T^n(x_j)$, then $n \geq 0$. A similar argument (reversing the rôles of f and $f \circ \varphi^{-1}$) shows that if f is discontinuous at $T^n(x_j)$, then $n \leq -1$. We conclude that f has no discontinuities, so that f is constant and $h = 0$ as desired. □

LEMMA 2.3. *Let $m, n \in \mathbf{Z}$ and $1 \leq i, j \leq r$ be such that $E = [T^m(x_{i-1}), T^n(x_{j-1}))$ is non-emtpy. Then $[\chi_E] \in \operatorname{Im} \gamma$.*

Proof. It suffices to consider $E = [0, T^n(x_i))$, with $n \geq 0$. We proceed by induction on n (and for all i). The result is clear for $n = 0$. Assume it is true for n and for all i, let us consider $E = [0, T^{n+1}(x_j))$. Choose k such that $T^{n+1}(x_j) \in T(\Delta_k)$, so $E = [0, Tx_{k-1}] \cup [Tx_{k-1}, T^{n+1}(x_j))$. Now we have

$$[\chi_{[0,Tx_{k-1})}] = \sum [\chi_{T\Delta_l}] = \sum [u\chi_{\Delta_l}u^*] = \sum [\chi_{\Delta_l}] \in \operatorname{Im} \gamma,$$

where the sum is over l such that $\sigma(l) < \sigma(k)$. Also,

$$[\chi_{[Tx_{k-1},T^{n+1}x_j)}] = [u\chi_{[x_{k-1},T^n x_j)}u^*] = [\chi_{[x_{k-1},T^n x_j)}] \in \operatorname{Im} \gamma,$$

by induction hypothesis. Thus $[\chi_E] \in \operatorname{Im} \gamma$. □

3. AF-subalgebras of $C(X) \times_\varphi \mathbf{Z}$. The principal aim of the section is to show that each non-empty closed subset, Y, in X in a natural way gives rise to an AF C^*-subalgebra, A_Y, of $C(X) \times_\varphi \mathbf{Z}$. Specifically, A_Y is the C^*-algebra generated by $C(X)$ and $uC_0(X - Y)$. Here, we use $C_0(X-Y)$ to denote the ideal in $C(X)$ of all functions which vanish on Y. The first step is to show that a partition, $\mathscr{P}$, of X and a non-empty clopen subset, Y, of X gives rise to a finite-dimensional C^*-subalgebra in the following way.

LEMMA 3.1. *Let $\mathscr{P}$ be a partition of X and let Y be a non-empty clopen subset of X. Then the C^*-subalgebra of $C(X) \times_\varphi \mathbf{Z}$ generated by $\mathscr{C}(\mathscr{P})$ and $u\chi_{X-Y}$ is finite dimensional.*

REMARK. The basic idea of the proof is the approximation technique developed by Versik in [17].

Proof. We begin by defining $\lambda \colon Y \to \mathbf{Z}$ by

$$\lambda(y) = \inf\{n \geq 1 \mid \varphi^n(y) \in Y\}, \qquad y \in Y.$$

Notice that since φ is minimal and Y is open, there is, for each point y, a positive integer n such that $\varphi^n(y) \in Y$, so λ is well-defined.

It is straightforward to verify that λ is upper (lower) semi-continuous because Y is open (closed), and so λ is continuous. Then because Y is compact, $\lambda(Y)$ is finite. Let us suppose that $\lambda(Y) = \{J_1, J_2, \dots, J_K\}$, with $J_1 < \cdots < J_K$.

For $k = 1, 2, \dots, K$ and $j = 1, 2, \dots, J_k$, define the clopen set $Y(k, j) = \varphi^j(\lambda^{-1}(J_k))$. Then it follows at once from the definitions that the following properties hold:

(1) $\bigcup_{k=1}^K Y(k, 1) = \varphi(Y)$;

(2) $\varphi(Y(k, j)) = Y(k, j+1)$, for $1 \leq j < J_k$;

(3) $\bigcup_{k=1}^K Y(k, J_k) = Y$.

(Note however that $\varphi(Y(k, J_k))$ is *not* $Y(k, 1)$.) This implies that the union of all $Y(k, j)$ is invariant under φ. It is also clearly closed and so, by minimality, must be all of X.

We shall refer to $\{Y(k, j) | j = 1, \dots, J_k\}$ as a tower of height J_k.

Now we argue that we can make the partition we have constructed above finer than the given one $\mathscr{P}$, without changing its essential structure (namely, properties 1–3 above). Suppose $Z \in \mathscr{P}$ and suppose Z meets some $Y(k, j)$ but does not contain it. Divide $Y(k, j)$ into two clopen sets $Y(k, j) \cap Z$ and $Y(k, j) \cap (X - Z)$. Unfortunately, this "disrupts" the entire kth tower, so we form $Y(k, i)' = \varphi^{i-j}(Y(k, j) \cap Z)$ and $Y(k, i)'' = \varphi^{i-j}(Y(k, j) \cap (X - Z))$, for each $i = 1, \dots, J_k$. Thus the kth tower breaks into two separate towers (both of height J_k) with $Y(k, j)' \subset Z$ and $Y(k, j)''$ disjoint from Z. We repeat this for all Z in $\mathscr{P}$ and all (k, j) (which will be a finite process). We then obtain a new K and new clopen sets $Y(k, j)$ (neither will be given a new notation) which satisfy conditions 1–3 above and such that the partition $\mathscr{P}' = \{Y(k, j) | k = 1, \dots, K, \ j = 1, \dots, J_k\}$ is finer than $\mathscr{P}$.

We are now prepared to define a finite dimensional C^*-subalgebra of $C(X) \times_\varphi \mathbf{Z}$. In fact, it will be $*$-isomorphic to

$$M_{J_1} \oplus M_{J_2} \oplus \cdots \oplus M_{J_K}.$$

To do this, it suffices to define matrix units $e_{ij}^{(k)}$ for all $k = 1, \dots, K$ and $i, j = 1, \dots, J_k$. Let

$$e_{ij}^{(k)} = u^{i-j} \chi_{Y(k,j)} = \chi_{Y(k,i)} u^{i-j}.$$

It is routine to check that for fixed k, $\{e_{ij}^{(k)}\}$ forms a complete system of matrix units for M_{J_k}, that the projections

$$p_k = \sum_{i=1}^{J_k} e_{ii}^{(k)}$$

are pairwise orthogonal and sum to the identity. We also note that $\operatorname{span}\{e_{ii}^{(k)} | k = 1, \ldots, K,\ i = 1, \ldots, J_k\} = \mathscr{C}(\mathscr{P}') \supset \mathscr{C}(\mathscr{P})$ and that

$$u\chi_{X-Y} = \sum_{k=1}^{K} \sum_{i=2}^{J_k} e_{i\,i-1}^{(k)}.$$

The C^*-algebra generated by $\mathscr{C}(\mathscr{P})$ and $u\chi_{X-Y}$ is contained in the finite-dimensional algebra we have just described and therefore must itself be finite-dimensional. □

We will denote the C^*-algebra generated by $\mathscr{C}(\mathscr{P})$ and $u\chi_{X-Y}$ by $A(Y, \mathscr{P})$.

LEMMA 3.2. *Let Y_1 and Y_2 be two non-empty clopen subsets of X and let $\mathscr{P}_1$ and $\mathscr{P}_2$ be two partitions of X. If $\mathscr{P}_1 \leq \mathscr{P}_2$, $\chi_{Y_1} \in \mathscr{C}(\mathscr{P}_2)$ and $Y_1 \supset Y_2$, then $A(Y_1, \mathscr{P}_1) \subset A(Y_2, \mathscr{P}_2)$.*

Proof. Clearly $\mathscr{C}(\mathscr{P}_1) \subset \mathscr{C}(\mathscr{P}_2)$ and, since $Y_2 \subset Y_1$,

$$u\chi_{X-Y_1} = u\chi_{X-Y_2}\chi_{X-Y_1} \in A(Y_2, \mathscr{P}_2). \qquad \square$$

THEOREM 3.3. *Let Y be a non-empty closed subset of X. Then A_Y, the C^*-subalgebra of $C(X) \times_\varphi \mathbf{Z}$ generated by $C(X)$ and $uC_0(X - Y)$, is an AF-algebra.*

Proof. We begin by selecting an increasing sequence of partitions of X, $\mathscr{P}_1 \leq \mathscr{P}_2 \leq \cdots$, whose union generates the topology of X. We also choose a decreasing sequence of clopen subsets of X, $Y_1 \supset Y_2 \supset \cdots$, whose intersection is Y. We will inductively define partitions, $\mathscr{P}'_n$, and finite dimensional subalgebras, $A_n = A(Y_n, \mathscr{P}'_n)$, for each positive integer n. Let $\mathscr{P}'_1 = \mathscr{P}_1$ and $A_1 = A(Y_1, \mathscr{P}_1)$. Now assume that we have defined $\mathscr{P}'_n$ and $A_n = A(Y_n, \mathscr{P}'_n)$. We let $\mathscr{P}'_{n+1} = \mathscr{P}'_n \vee \mathscr{P}_{n+1} \vee \{Y_n, X - Y_n\}$. Then we have $\mathscr{P}'_{n+1} \geq \mathscr{P}_{n+1}, \mathscr{P}'_{n+1} \geq \mathscr{P}'_n$ and $\chi_{X-Y_n} \in \mathscr{C}(\mathscr{P}'_{n+1})$. Let $A_{n+1} = A(Y_{n+1}, \mathscr{P}'_{n+1})$.

We claim that the A_n's form a nested sequence of finite dimensional subalgebras of A_Y whose union is dense in A_Y. First of all, $\mathscr{C}(\mathscr{P}'_n) \subset C(X)$ and $u\chi_{X-Y_n} \in uC_0(X - Y)$, since $Y \subset Y_n$, so $A_n \subset A_Y$. From the properties of $\mathscr{P}'_{n+1}$ as described in the last paragraph and Lemma 3.2, we see that $A_n \subset A_{n+1}$, for all n. Since the union of the $\mathscr{P}_n$'s generates the topology of X and $\mathscr{C}(\mathscr{P}_n) \subset \mathscr{C}(\mathscr{P}'_n) \subset A_n$, we know that $C(X) \subset (\bigcup_n A_n)^-$. As Y is the intersection of the Y_n's, it is clear that $uC_0(X - Y) \subset (\bigcup_n A_n)^-$. □

4. The K-theory of the AF-subalgebras. Our objective in this section is to compute the K_0-groups of the AF-subalgebra A_Y which we constructed in the last section, since this group, with its complete order structure, determines the isomorphism class of A_Y. We fix an inductive sequence $A_n = A(Y_n, \mathscr{P}_n)$ for A_Y as constructed in the last section. Our result is the following theorem.

THEOREM 4.1. *Let Y be a non-empty closed subset of X. Let i denote the inclusion map of A_Y in $C(X) \times_\varphi \mathbf{Z}$. Then there is an exact sequence*

$$0 \to \mathbf{Z} \xrightarrow{\alpha} C(Y, \mathbf{Z}) \xrightarrow{\beta} K_0(A_Y) \xrightarrow{i_*} K_0(C(X) \times_\varphi \mathbf{Z}) \to 0$$

where α is the map taking $n \in \mathbf{Z}$ to the constant function n and β is described in the proof.

Moreover, for every $a \in K_0(C(X) \times_\varphi \mathbf{Z})^+$, there is $b \in K_0(A_Y)^+$ such that $i_*(b) = a$.

In particular, if Y is a single point, i_* is an isomorphism of ordered groups.

REMARKS. Before beginning the proof, we point out the following. First, this description of $K_0(A_Y)$ is not complete, especially that of $K_0(A_Y)^+$. In fact, it is interesting to note that, up to splitting of the above sequence (which is irrelevant in the case $K_0(C(X) \times_\varphi \mathbf{Z})$ is free abelian), the relation between $K_0(A_Y)$ and $K_0(C(X) \times_\varphi \mathbf{Z})$, as abelian groups, depends only on the topology of Y and not on the dynamics of φ. This is not the case for the order structure of $K_0(A_Y)$ as we shall see in the next section. (As a simple example, consider the case when Y is two points. Then the group structure of $K_0(A_Y)$ does not depend on whether the points lie in the same φ-orbit, while we shall see later that the order structure certainly does.)

Also, since Y is a closed subset of a totally disconnected space, it is itself disconnected and so $C(Y, \mathbf{Z}) \simeq K_0(C(Y))$, as ordered abelian groups.

Finally, we notice that the result is really in terms of comparing $K_0(A_Y)$ with $K_0(C(X) \times_\varphi \mathbf{Z})$ via i_*. We are assuming that the latter can be computed by the Pimsner-Voiculescu sequence.

We will need the following lemma in the proof of 4.1.

LEMMA 4.2. *Let p be a projection in $C(X) \cap A_n$ and suppose that $p = 0$ on Y_n. Then $\varphi(p) \in C(X) \cap A_n$ and $[\varphi(p)] = [p]$ in $K_0(A_n)$.*

Proof. Let $v = u\chi_{X-Y_n}p$. Then $v \in A_n$ and since $p = 0$ on Y_n, $\chi_{X-Y_n}p = p$, $v^*v = p$ and $vv^* = upu^* = \varphi(p)$. □

Proof of 4.1. We begin by showing the final part of the theorem; that is, for every a in $K_0(C(X) \times_\varphi \mathbf{Z})^+$ there is b in $K_0(A_Y)^+$ such that $i_*(b) = a$. From this it also follows that i_* is surjective. Consider the following commutative diagram:

$$\begin{array}{ccc} & C(X) & \\ {}^{i_1}\swarrow & & \searrow^{i_2} \\ A_Y & \xrightarrow[i]{} & C(X) \times_\varphi \mathbf{Z} \end{array}$$

From Corollary 2.4 of **[11]**, there is c in $K_0(C(X))^+$ such that $(i_2)_*(c) = a$. Letting $b = (i_1)_*(c)$ gives the conclusion.

We now construct the map $\beta: C(Y, \mathbf{Z}) \to K_0(A_Y)$. Let $f \in C(Y, \mathbf{Z})$. Choose $g \in C(X, \mathbf{Z})$ such that $g|Y = f$. Define

$$\beta(f) = (i_1)_*(g - \varphi_*(g)).$$

To see that β is well-defined, suppose that g and g' are in $C(X, \mathbf{Z})$ and $g|Y = g'|Y = f$. Then we may choose n sufficiently large so that $g, g' \in C(X) \cap A_n$ and so that $g|Y_n = g'|Y_n$. We can write $g - g'$ as a linear combination of characteristic functions in $C(X) \cap A_n$ each of which is zero on Y_n. So by Lemma 4.2, $[g - g'] = [\varphi(g - g')]$ in $K_0(A_n)$, which implies that $(i_1)_*(g - \varphi_*(g)) = (i_1)_*(g' - \varphi(g'))$ in $K_0(A_Y)$.

The exactness of the sequence at $\mathbf{Z}$ is clear and we have already shown exactness at $K_0(C(X) \times_\varphi \mathbf{Z})$. Let us consider exactness $C(Y, \mathbf{Z})$. It is easy to check that $\mathrm{Im}(\alpha) \subset \ker(\beta)$. On the other hand, suppose that $f \in \ker(\beta)$. We wish to show that f is constant. Let $g \in C(X, \mathbf{Z})$ be such that $g|Y = f$. We may choose n sufficiently large so that g and $\varphi(g)$ are in $A_n \cap C(X)$ and so that $[g] = [\varphi(g)]$ in $K_0(A_n)$. From Lemma 4.2, we may replace g by $g\chi_{Y_n}$ without changing this and so we may assume that $g = 0$ on $X - Y_n$. Let $K, J_1, \ldots, J_K$ and $Y(k, j)$ be as in §3 for the finite dimensional algebra A_n. So $A_n \simeq M_{J_1} \oplus \cdots \oplus M_{J_K}$ and $K_0(A_n)$ is isomorphic (via the trace on each matrix summand) to $\bigoplus_k \mathbf{Z}$. The hypothesis that $[g] = [\varphi(g)]$ and our reduction to the case that $g = 0$ on $X - Y_n$ then imply that $g(Y(k, J_k)) = g \circ \varphi^{-1}(Y(k, 1))$, for all k. Pick an integer m in $g(Y_n)$ and define $Z = \bigcup Y(k, j)$, where the union is taken over all (k, j) such that $g(Y(k, J_k)) = m$. From the condition that $g(Y(k, J_k)) = g \circ \varphi^{-1}(Y(k, 1))$ it follows that the set Z is invariant under φ. Clearly Z is closed, so the minimality of φ implies that $Z = X$. So $g(Y(k, J_k)) = m$ for all k, so $f = g|Y = m$.

For exactness at $K_0(A_Y)$, the inclusion $\mathrm{Im}(\beta) \subset \ker(i_*)$ follows from the fact that $\ker((i_2)_*) = \mathrm{Im}(\mathrm{id} - \varphi_*)$ obtained in the Pimsner-Voiculescu exact sequence and our definition of β. As for the reverse inclusion, suppose that $a \in \ker(i_*)$. We may find $g \in C(X, \mathbf{Z})$ such that $(i_1)_*(g) = a$. Then $(i_2)_*(g) = i_*(a) = 0$, which implies that there is h in $C(X, \mathbf{Z})$ such that $g = h - \varphi_*(h)$, again using the fact that $\ker((i_2)_*) = \mathrm{Im}(\mathrm{id} - \varphi_*)$. Then let $f = h|Y \in C(Y, \mathbf{Z})$. It is immediate that $\beta(f) = a$ as desired. □

5. Further analysis of the AF-subalgebras. In this section we apply the analysis of Stratila and Voiculescu **[15]** to our AF-subalgebras A_Y. The idea (roughly speaking) is to consider a maximal abelian subalgebra (masa) of the AF-algebra and look at the group of unitaries in the AF-algebra which normalize the given subalgebra. In our case the masa we will use is $C(X)$ and the unitaries which normalize it may be written explicitly in terms of u (see Lemma 5.1). This analysis will provide us with description of the ideal structure of A_Y and also give information regarding the correspondence between invariant measures on X, traces on the C^*-algebras and states on their K_0-groups.

We begin with some definitions and notation. For a unital C^*-algebra B, let $U(B)$ denote the unitary group of B and for a C^*-subalgebra $C \subset B$, let $\mathscr{N}(C, B)$ denote the normalizer of C in $U(B)$; i.e.

$$\mathscr{N}(C, B) = \{v \in U(B) | vCv^* = C\}.$$

We use $\mathscr{C}(C, B)$ to denote the centralizer of C in B; i.e.

$$\mathscr{C}(C, B) = \{v \in \mathscr{N}(C, B) | vc = cv \text{ for all } c \in C\}.$$

We note that if C is a masa in B, then $\mathscr{C}(C, B) = U(C)$.

The group $\mathscr{N}(C(X), C(X) \times_\varphi \mathbf{Z})$ acts on $C(X)$ as $*$-automorphisms. Each w in $\mathscr{N}(C(X), C(X) \times_\varphi \mathbf{Z})$ induces the automorphism $\mathrm{ad}\, w(f) = wfw^*$, for all $f \in C(X)$. Therefore, $\mathscr{N}(C(X), C(X) \times_\varphi \mathbf{Z})$ acts on X as homeomorphisms. By definition, $\mathscr{C}(C(X), C(X) \times_\varphi \mathbf{Z}) = U(C(X))$ acts trivially and so we obtain an action of the quotient group $\mathscr{N}(C(X), C(X) \times_\varphi \mathbf{Z})/U(C(X))$ on X. We let Γ denote this quotient group.

Since A_Y is a unital subalgebra of $C(X) \times_\varphi \mathbf{Z}$, we see that $\mathcal{N}(C(X), A_Y)$ is a subgroup of $\mathcal{N}(C(X), C(X) \times_\varphi \mathbf{Z})$. We let Γ_Y denote the quotient group $\mathcal{N}(C(X), A_Y)/U(C(X))$, and note that $\Gamma_Y \subset \Gamma$.

Let $\hat{\varphi}$ denote the dual action of the circle group $\mathbf{T}$ on $C(X) \times_\varphi \mathbf{Z}$ (see 7.8.3 of Pedersen [7]). We obtain a conditional expectation $E: C(X) \times_\varphi \mathbf{Z} \to C(X)$ defined by

$$E(a) = \int_{\mathbf{T}} \hat{\varphi}_z(a)\, dz, \quad a \in C(X) \times_\varphi \mathbf{Z},$$

where dz denotes normalized Haar measure on $\mathbf{T}$. Also define, for each integer n, $E_n: C(X) \times_\varphi \mathbf{Z} \to C(X)$ by $E_n(a) = E(au^{-n})$. Note that if $f \in C(X)$ then $\hat{\varphi}_z(f) = f$ for all $z \in \mathbf{T}$ and $E(f) = f$. Also $\hat{\varphi}_z(u) = zu$, for all $z \in \mathbf{T}$. This implies that, for any non-empty closed subset $Y \subset X$, A_Y is invariant under $\hat{\varphi}$.

LEMMA 5.1. *If $v \in \mathcal{N}(C(X), C(X) \times_\varphi \mathbf{Z})$, then*

$$v = f \sum_{n \in \mathbf{Z}} p_n u^n,$$

where $f \in U(C(X))$, each p_n is a projection in $C(X)$ with only finitely many p_n different from 0, $p_n p_m = 0$ for $n \neq m$, and

$$\sum_n p_n = \sum_n \varphi^{-n}(p_n) = 1.$$

Moreover this decomposition is unique.

Proof. Let $p_n = |E_n(v)|$, for each $n \in \mathbf{Z}$. Let $X_n \subset X$ denote the support of $p_n \in C(X)$. Choose $x \in X$ arbitrarily and consider the irreducible representation π_x of $C(X) \times_\varphi \mathbf{Z}$ on the Hilbert space $l^2(\mathbf{Z})$ defined as follows. For each integer i, let ξ_i denote the element of $l^2(\mathbf{Z})$ having value 1 at i and 0 elsewhere. So $\{\xi_i\}_{i \in \mathbf{Z}}$ is the usual basis for $l^2(\mathbf{Z})$. Then for $i \in \mathbf{Z}$ and $f \in C(X)$, $\pi_x(f)\xi_i = f(\varphi^i(x))\xi_i$, and $\pi_x(u)\xi_i = \xi_{i+1}$. (See 7.7.1 of Pedersen [7].)

For each $z \in \mathbf{T}$, define the unitary operator u_z on $l^2(\mathbf{Z})$ by $u_z \xi_i = z^i \xi_i$. Then

$$\pi_x(\hat{\varphi}_z(a)) = u_z \pi_x(a) u_z^*, \qquad z \in \mathbf{T},\ a \in C(X) \times_\varphi \mathbf{Z}.$$

Since $\pi_x(v)$ normalizes $\pi_x(C(X))$, it also normalizes $\pi_x(C(X))''$ which is equal to $l^\infty(\mathbf{Z})$ (acting as multiplication operators on $l^2(\mathbf{Z})$). This implies that there is a unitary diagonal operator $\lambda = (\lambda_i)_{i=-\infty}^{\infty} \in l^\infty(\mathbf{Z})$, and a permutation σ of $\mathbf{Z}$ such that

$$\pi_x(v)\xi_i = \lambda_i \xi_{\sigma(i)}, \qquad i \in \mathbf{Z}.$$

Then, for $n, i \in \mathbf{Z}$, we have

$$\begin{aligned}\pi_x(E_n(v))\xi_i &= \int_{\mathbf{T}} \pi_x(\hat{\varphi}_z(vu^{-n}))\xi_i \, dz \\ &= \int u_z \pi_x(v)\pi_x(u^{-n})u_z^* \xi_i \, dz \\ &= \int z^{\sigma(i-n)-i}\lambda_{i-n}\xi_{\sigma(i-n)} \, dz \\ &= \begin{cases} \lambda_{i-n}\xi_i & \text{if } \sigma(i-n) = i, \\ 0 & \text{if } \sigma(i-n) \neq i. \end{cases}\end{aligned}$$

From this we conclude that

$$\pi_x(p_n)\xi_i = \begin{cases} \xi_i & \text{if } \sigma(i-n) = i, \\ 0 & \text{if } \sigma(i-n) \neq i. \end{cases}$$

From this we see that $p_n p_m = 0$, for $n \neq m$ and that p_n is a projection and so its support, X_n, is clopen, for all n. We also see that there is an n such that $p_n(x) = \langle \pi_x(p_n)\xi_0, \xi_0 \rangle = 1$. Since x was arbitrary, the union of all X_n is all of X. By compactness (and the fact that the X_n are pairwise disjoint), we see that all but finitely many X_n are empty. We now have that all but finitely many p_n are zero, that they are mutually orthogonal and that their sum is 1. It follows easily from the fact $v^*v = 1$ that the sum of $\varphi^{-n}(p_n)$ is 1. Finally, let v_0 be the sum of $p_n u^n$. Then v_0 is a unitary in $C(X) \times_\varphi \mathbf{Z}$ and vv_0^* is a unitary in $C(X) \times_\varphi \mathbf{Z}$ whose image under π_x is $\lambda \in \pi_x(C(X))''$. Since $C(X)$ is maximal abelian, we conclude that $\lambda = \pi_x(f)$ for some unitary $f \in C(X)$. □

We now wish to describe Γ and its action on X in a more convenient form.

Endow $C(X, \mathbf{Z})$ with the following associative product

$$\eta \cdot \nu(x) = \eta(x) + \nu(\varphi^{-\eta(x)}(x)),$$

for $\eta, \nu \in C(X, \mathbf{Z})$ and $x \in X$. Then $C(X, \mathbf{Z})$ becomes a semigroup with identity ($\eta = 0$). We let G denote the group of invertible elements $C(X, \mathbf{Z})$. We may define an action of G on X by

$$\eta \cdot x = \varphi^{-\eta(x)}(x)$$

for $\eta \in G$ and $x \in X$. We note that each element of G can be written in the form $\sum m p_m$, where each p_m is a projection in $C(X)$ with $p_m = 0$ for all but finitely many m, $p_m p_n = 0$ for $n \neq m$ and with the sum of the p_m's equal to 1. This representation of the elements of G is unique.

THEOREM 5.2. *The map sending*

$$w = f\sum p_m u^m \in \mathcal{N}(C(X), C(X)\times_\varphi \mathbf{Z})$$

(as in Lemma 5.1) to $\eta_w = \sum m p_m \in C(X, \mathbf{Z})$, *induces an isomorphism between the groups* Γ *and* G. *Moreover, we have*

$$(wfw^*)(x) = f(\eta_w \cdot x)$$

for all $f \in C(X)$ *and* $x \in X$.

The proof is completely routine and so we omit it. From now on, we will identify the groups G and Γ (with their actions on X) and work either with unitaries $w = \sum p_m u^m$ or the functions $\sum m p_m$ interchangeably.

We note that the short exact sequence of groups

$$1 \to U(C(X)) \to \mathcal{N}(C(X), C(X)\times_\varphi \mathbf{Z}) \to \Gamma \to 1$$

has a splitting, namely $\eta = \sum m p_m \to w = \sum p_m u^m$.

COROLLARY 5.3. *For each* x *in* X, *the* Γ*-orbit of* x *coincides with the* φ*-orbit of* x; *i.e.*

$$\Gamma\cdot x = \{\varphi^j(x) \mid j \in \mathbf{Z}\}.$$

The proof is trivial at this point, so we omit it.

We now turn our attention to A_Y, where Y is a fixed closed nonempty subset of X.

We fix an increasing sequence of finite dimensional subalgebras $A_n = A(Y_n, \mathcal{P}_n) \subset A_Y$ as in §3. For each positive integer n, define $\lambda_n^+, \lambda_n^- : X \to \mathbf{Z}$ by

$$\lambda_n^+(x) = \inf\{m \geq 0 \mid \varphi^m(x) \in Y_n\},$$
$$\lambda_n^-(x) = \sup\{m \leq 0 \mid \varphi^{m-1}(x) \in Y_n\},$$

for $x \in X$. Just as for the function λ of Lemma 3.1, λ_n^+ and λ_n^- are both well-defined and continuous. Now define $\lambda^+ : X \to \mathbf{Z}\cup\{+\infty\}$ and $\lambda^- : X \to \mathbf{Z}\cup\{-\infty\}$ by $\lambda^+(x) = \sup_n \lambda_n^+(x)$ and $\lambda^-(x) = \inf_n \lambda_n^-(x)$, for all $x \in X$. Notice that

$$\lambda^+(x) = \inf(\{m \geq 0 \mid \varphi^m(x) \in Y\} \cup \{+\infty\}),$$
$$\lambda^-(x) = \sup(\{m \leq 0 \mid \varphi^{m-1}(x) \in Y\} \cup \{-\infty\})$$

since Y is the intersection of the Y_n's. In particular, λ^+ and λ^- are independent of the choice of inductive sequence. Note that λ^+ and λ^- both depend on Y, but as we will hold Y fixed we will omit this in our notation.

We may completely describe Γ_Y in terms of λ^+ and λ^- as follows.

THEOREM 5.4. *Let $\eta = \sum m p_m$ be an element of Γ. Then η is in Γ_Y if and only if $\lambda^- \leq -\eta \leq \lambda^+$.*

Proof. We begin by assuming that $\eta \in \Gamma_Y$ and show that $\lambda^- \leq -\eta \leq \lambda^+$. For some n, the unitary $w = \sum p_m u^m \in A_n$ and $\eta \in C(X) \cap A_n$. We let $K, J_1, \dots, J_K, Y(k,j)$ and $e_{ij}^{(k)}$ all be as in Lemma 3.1 for A_n. For each $k = 1, \dots, K$, there is a permutation σ_k of $\{1, \dots, J_k\}$ such that

$$w = \sum_k \sum_j e_{j\ \sigma_k(j)}^{(k)}.$$

Now $e_{j\ \sigma_k(j)}^{(k)} = \chi_{Y(k,j)} u^{j-\sigma_k(j)}$, so $\eta(Y(k,j)) = j - \sigma_k(j)$. From properties 1–3 of the sets $Y(k,j)$ (in 3.1), we see that

$$\lambda_n^+(Y(k,j)) = J_k - j \quad \text{and} \quad \lambda_n^-(Y(k,j)) = 1 - j.$$

Then since $1 \leq \sigma_k(j) \leq J_k$, we have $\lambda_n^- \leq -\eta \leq \lambda_n^+$ and the conclusion follows.

As for the converse, let us now suppose that $\lambda^- \leq -\eta \leq \lambda^+$. A standard argument using the continuity of $\lambda_n^\pm$ and η and the compactness of X shows that for sufficiently large n, $\lambda_n^- \leq -\eta \leq \lambda_n^+$. Also choose n large enough so that $\eta \in C(X) \cap A_n$. Fix an integer m and let $E \subset X$ be the (clopen) support of p_{-m}; so $-\eta(E) = m$. We wish to see that $p_{-m}u^{-m} \in A_n$. In the case $m = 0$, this is immediate since $\eta \in C(X) \cap A_n$.

Let us consider the case $m > 0$. The hypothesis that $-\eta \leq \lambda_n^+$ implies that E does not meet $Y_n \cup \varphi^{-1}(Y_n) \cup \cdots \cup \varphi^{-m+1}(Y_n)$ so that

$$\begin{aligned} p_{-m}u^{-m} &= \chi_E u^{-m} = \chi_E \chi_{X - Y_n - \cdots - \varphi^{-m+1}(Y_n)} u^{-m} \\ &= \chi_E((u\chi_{X-Y_n})^*)^m \in A_n. \end{aligned}$$

Similarly if $m < 0$, we have $\lambda_n^- \leq -\eta$ implying that

$$p_{-m}u^{-m} = \chi_E(u\chi_{X-Y_n})^{-m} \in A_n.$$

We conclude that $\sum p_{-m}u^{-m} \in A_n \subset A_Y$, and so $\eta \in \Gamma_Y$, as desired. □

REMARK. Let us pause for a moment and give a heuristic description of the dynamics of Γ_Y acting on X. If we think of u as an operator which moves the points of X as φ does, then $uC_0(X - Y) \subset A_Y$ is a collection of operators which will move all the points of $X - Y$ as φ does. This rough idea is correct and stated precisely in the following fashion.

COROLLARY 5.5. *Let x be a point of X. Then the orbit of x under Γ_Y is*

$$\Gamma_Y \cdot x = \{\varphi^j(x) | j \in \mathbf{Z},\ \lambda^-(x) \leq j \leq \lambda^+(x)\}.$$

Proof. The containment of $\Gamma_Y \cdot x$ in the set above is immediate from Theorem 5.4.

We suppose that $j \in \mathbf{Z}$ and $\lambda^-(x) \leq j \leq \lambda^+(x)$. We wish to exhibit a unitary w in A_Y which, as an element of Γ_Y, carries x to $\varphi^j(x)$. First, let us consider the case $j \geq 0$. By definition, there is a positive integer n such that $\lambda_n^+(x) \geq j$. Let $\mathscr{P}$ be the partition of X so that $\mathscr{C}(\mathscr{P}) = A_n \cap C(X)$. Let E be the unique element of $\mathscr{P}$ containing x. Since $\lambda_n^+ \in \mathscr{C}(\mathscr{P}), \lambda_n^+(E) = \lambda_n^+(x) \geq j$. As in the proof of Theorem 5.4, this implies that $F = \varphi^{-j}(E)$ is in $\mathscr{P}$ and that $w = \chi_E u^{-j} + \chi_F u^j + (1 - \chi_E - \chi_F)$ is a unitary in $A_n \subset A_Y$. This unitary corresponds to $\eta = -j\chi_E + j\chi_F$ in Γ_Y, and $\eta \cdot x = \varphi^{-\eta(x)}(x) = \varphi^j(x)$, as desired.

The case $j \leq 0$ is similar. □

COROLLARY 5.6. *Let Y be a non-empty closed subset of X. The C^*-algebra A_Y is simple if and only if $Y \cap \varphi^j(Y)$ is empty for all $j \neq 0$; that is, Y meets each φ-orbit at most once.*

Proof. Let us first suppose that there is a point x in $Y \cap \varphi^j(Y)$, for some $j \neq 0$. Without loss of generality, we may assume that $j < 0$. Then one may easily compute that $\lambda^+(x) \leq 0$ and $\lambda^-(x) \geq 1 - j$. Then $\Gamma_Y \cdot x$ is finite, by Theorem 5.5, and therefore closed. By I.2.4 of Stratila-Voiculescu, there is a bijective correspondence between the closed Γ_Y-invariant subsets of X and ideals in A_Y, and so A_Y is not simple.

As for the converse, let x be any point of X. The condition that Y meet each φ-orbit at most once guarantees that either $\lambda^+(x) = +\infty$ or $\lambda^-(x) = -\infty$. Therefore, $\Gamma_Y \cdot x$ contains an entire φ-half-orbit and is therefore dense in X by our minimality hypothesis on φ. From this we conclude that there are no non-trivial closed Γ_Y-invariant subsets of X, so again by I.2.4 of **[15]**, A_Y is simple. □

REMARK. Corollary 5.5 and I.2.4 of Stratila-Voiculescu together will yield a complete description of the ideal structure of A_Y in specific cases. For example, if y is some fixed point in X, j is some positive integer and we let $Y = \{y, \varphi^j(y)\}$, then there is a unique non-trivial ideal $\mathscr{J}$ in A_Y and the quotient $A_Y/\mathscr{J}$ is $*$-isomorphic to M_j.

By a probability measure on X, we mean a normalized, finite, positive, regular Borel measure on X. A state on an ordered group G, with order unit g, is a group homomorphism $\rho: G \to \mathbf{R}$ such that $\rho(G^+) \subset [0, \infty)$ and $\rho(g) = 1$.

COROLLARY 5.7. *Let Y be a non-empty closed subset of X such that $Y \cap \varphi^j(Y)$ is empty for all $j \neq 0$. Then there is a bijective correspondence between each of the following:*

(i) *φ-invariant probability measures on X,*
(ii) *tracial states on $C(X) \times_\varphi \mathbf{Z}$,*
(iii) *states on $K_0(C(X) \times_\varphi \mathbf{Z})$,*
(iv) *Γ_Y-invariant probability measures on X,*
(v) *tracial states on A_Y,*
(vi) *states on $K_0(A_Y)$.*

Proof. The correspondence between (i) and (ii) is given by 9.6 of Zeller-Meyer [**18**]. In a similar fashion, I.3.2 of Stratila and Voiculescu shows the correspondence between (iv) and (v). Since Y meets each φ-orbit at most once, A_Y is simple and in this case the correspondence between (v) and (vi) was shown by Blackadar (see p. 58 of [**2**]).

We sketch a proof of the correspondence between (i) and (iii). If ρ is a state on $K_0(C(X) \times_\varphi \mathbf{Z})$, then $\rho \circ (i_2)_*$ is a state on $C(X, \mathbf{Z})$ which arises from a probability measure on X. Since $\rho \circ (i_2)_*$ kills $\mathrm{Im}(\mathrm{id} - \varphi_*)$, this measure is invariant on the clopen subsets of X and therefore on all Borel subsets as well.

We now examine the correspondence between (i) and (iv). It is clear from the results above that each φ-invariant measure is also Γ-invariant. To prove the result, it suffices to show that each Γ-invariant measure, μ, is also φ-invariant.

To begin, we wish to show that $\mu(\varphi^j(Y)) = 0$, for all j. Fix $j \geq 1$. It is easy to compute $\lambda^+|\varphi^j(Y) = +\infty$. Therefore, by Corollary 5.5 and a simple argument using the compactness of Y, for each $i \geq j$ there is an element of Γ_Y which carries $\varphi^j(Y)$ to $\varphi^i(Y)$. We conclude that the sets $\varphi^i(Y)$ are pairwise disjoint (by hypothesis) and all have the same μ-measure (by the Γ_Y-invariance of μ). Since μ is finite, $\mu(\varphi^j(Y)) = 0$. The case $j \leq 0$ is similar.

Now we let Z be an arbitrary Borel subset of X and we wish to show that $\mu(Z) = \mu(\varphi(Z))$. Let ε be positive. Since μ is regular, we may find a clopen set Y' containing Y with $\mu(Y') < \varepsilon$ and such that $\mu(\varphi(Y')) < \varepsilon$. Our results imply that there is an element η of Γ_Y such that $\eta \cdot (Z - Y') = \varphi(Z - Y')$. So we have

$$|\mu(Z) - \mu(\varphi(Z))| \leq |\mu(Z - Y') - \mu(\varphi(Z - Y'))| + \mu(Y') + \mu(\varphi(Y')) < 2\varepsilon.$$

Since ε was arbitrary, the conclusion follows. □

REMARK. We may now conclude the remark after Theorem 4.4 by pointing out the following example. Suppose that Y is two points. We see from Corollary 5.5 that A_Y is simple if and only if these points lie in distinct orbits. The isomorphism class of A_Y, and also the order structure on its K_0-group, depend on more than Y as a topological space.

6. Embedding $C(X) \times_\varphi \mathbf{Z}$ into an AF-algebra. The technique of approximation which we have used was first developed by Versik in [**16**] and [**17**] to obtain embeddings of certain algebras into AF-algebras. In particular, the results of [**16**] imply that our C^*-algebras, $C(X) \times_\varphi \mathbf{Z}$, may be embedded into AF-algebras. Moreover, although this is not mentioned in [**16**], the embedding induces an order isomorphism at the level of K_0. (This is an improvement on the result of Pimsner [**8**] which also shows that $C(X) \times_\varphi \mathbf{Z}$ may be embedded into an AF-algebra. Pimsner treats a much more general situation and his technique, in our case, will produce an AF-algebra which is much too large.)

We will construct the embedding of [**16**] here. Our technique is basically the same as Versik's, but we will show that the embedding induces an order at the level of K_0.

Fix a point y in X. Choose a decreasing sequence of clopen sets $\{Y_n\}_{n\in\mathbf{Z}}$ whose intersection is $\{y\}$. Also choose an increasing sequence of partitions $\{\mathscr{P}_n\}_{n\in\mathbf{Z}}$ whose union generates the topology on X.

For each integer $n \geq 0$, we construct a finite dimensional C^*-algebra A_n. We begin with $A_0 = \mathbf{C}$ and, assuming we have a clopen set Z_n, a partition $\mathscr{P}'_n$ and $A_n = A(Z_n, \mathscr{P}'_n)$, we define A_{n+1} as follows. Let $\mathscr{P}''_n$ be the partition of X so that $\mathscr{C}(\mathscr{P}''_n) = C(X) \cap A_n$. Choose Z_{n+1} a clopen subset of X containing $\{y\}$ such that $Z_{n+1} \subset Y_{n+1}$ and, for $j = 0, 1, \ldots, 2^{n+1}$, the sets $\varphi^j(Z_{n+1})$ are pairwise disjoint and each is contained in a single element of $\mathscr{P}''_n$. Then let $\mathscr{P}'_{n+1} = \mathscr{P}''_n \vee \mathscr{P}_{n+1} \vee \{Z_n, X - Z_n\}$ and let $A_{n+1} = A(Z_{n+1}, \mathscr{P}'_{n+1})$.

From Lemma 3.2, $A_n \subset A_{n+1}$ for all n and $\bigcup A_n$ is dense in $A_{\{y\}}$. Fix n for the moment and let $\{e^{(k)}_{ij} | 1 \leq k \leq K,\ 1 \leq j \leq J_k\}$ be the system of matrix units for A_n as in 3.2. (Of course, these parameters all depend on n, but we will suppress this in the notation.) Define the unitary $v_n \in A_n$ by

$$\begin{aligned} v_n &= \sum_{k=1}^{K} \left[e_{1J_k}^{(k)} + \sum_{i=2}^{J_k} e_{ii-1}^{(k)} \right] \\ &= \sum_k e_{11}^{(k)} u^{1-J_k} + u\chi_{X-Z_n}. \end{aligned}$$

The basic properties of v_n are summarized in the following lemma. The proof is straightforward so we omit it.

LEMMA 6.1. *The unitary operator $v_n \in A_n$ satisfies*
(i) $v_n \chi_{X-Z_n} = u\chi_{X-Z_n}$,
(ii) $v_n \chi_{Z_n} v_n^* = u\chi_{Z_n} u^* = \chi_{\varphi(Z_n)}$,
(iii) *if $f \in C(X) \cap A_n$ and f is constant on Z_n, then $v_n f v_n^* = \varphi(f)$.*

As a consequence we also obtain the following.

LEMMA 6.2. *If $f \in C(X) \cap A_n$, then $v_{n+1} f v_{n+1}^* = \varphi(f)$.*

Proof. First, $A_n \subset A_{n+1}$ so that $f \in C(X) \cap A_{n+1}$. Recall from our construction of the sets Z_n that Z_{n+1} is contained in a single element of $\mathscr{P}_n''$, where $\mathscr{C}(\mathscr{P}_n'') = C(X) \cap A_n$. This implies that every function in $C(X) \cap A_n$ is constant on Z_{n+1}. The result follows from part (iii) of Lemma 6.1. □

We now define a unitary $w_n \in A_{n+1}$, for each integer $n \geq 1$. Consider $v_{n+1} v_n^* \in A_{n+1}$. Using the fact that $Z_n \subset Z_{n+1}$ and repeated use of (i) and (ii) of 6.1, we obtain

$$\begin{aligned} \chi_{X-\varphi(Z_n)} v_{n+1} v_n^* &= v_{n+1} v_n^* \chi_{x-\varphi(Z_n)} = \chi_{x-\varphi(Z_n)}, \\ \chi_{\varphi(Z_n)} v_{n+1} v_n^* &= v_{n+1} v_n^* \chi_{\varphi(Z_n)}. \end{aligned}$$

Since A_{n+1} is a finite dimensional C^*-algebra, we may apply a simple spectral argument to show that there is a unitary $z \in A_{n+1}$ such that

$$\begin{aligned} z^{2^n} &= v_{n+1} v_n^*, \\ z\chi_{X-\varphi(Z_n)} &= \chi_{X-\varphi(Z_n)} z = \chi_{X-\varphi(Z_n)}, \\ z\chi_{\varphi(Z_n)} &= \chi_{\varphi(Z_n)} z \quad \text{and} \quad \|z - 1\| < \pi 2^{-n}. \end{aligned}$$

Define

$$w_n = z^{2^n}(uz^{2^n-1}u^{-1})(u^2 z^{2^n-2} u^{-2}) \cdots (u^{2^n-1} z u^{1-2^n}).$$

Since the sets $\varphi^j(Z_n)$ are pairwise disjoint for $j = 1, \dots, 2^n$, we have

$$\chi_{\varphi^j(Z_n)} w_n = w_n \chi_{\varphi^j(Z_n)} = \chi_{\varphi^j(Z_n)}(u^{j-1} z^{2^n-j+1} u^{1-j}).$$

Let $Y = X - (\varphi(Z_n) \cup \dots \cup \varphi^{2^n}(Z_n))$. We also have

$$\chi_Y w_n = w_n \chi_Y = \chi_Y.$$

LEMMA 6.3. $\|w_n v_{n+1} w_n^* - v_n\| < \pi 2^{-n}$.

Proof. We shall actually show that $\|v_n w_n v_{n+1}^* - w_n\| < \pi 2^{-n}$, from which the conclusion follows. The proof makes repeated use of Lemma 6.1 (i) and (ii).

Each $\chi_{\varphi^j(Z_n)}$ commutes with $v_n w_n v_{n+1}^* - w_n$, for $j = 1, \dots, 2^n$, and so does χ_Y. Therefore, it suffices for us to show that

$$\|(v_n w_n v_{n+1}^* - w_n)\chi_E\| < \pi 2^{-n},$$

for $E = \varphi^j(Z_n)$ and for $E = Y$. First of all, for $j = 1$, we have

$$\begin{aligned}\|(v_n w_n v_{n+1}^* - w_n)\chi_{\varphi(Z_n)}\| &= \|v_n w_n \chi_{Z_n} v_{n+1}^* - \chi_{\varphi(Z_n)} z^{2^n}\| \\ &= \|v_n \chi_{Z_n} v_{n+1}^* - \chi_{\varphi(Z_n)} z^{2^n}\| \\ &= \|\chi_{\varphi(Z_n)}(v_n v_{n+1}^* - z^{2^n})\| = 0.\end{aligned}$$

Secondly, for each $j = 2, \dots, 2^n$, let $Z = \varphi^j(Z_n)$ and $A' = \varphi^{j-1}(Z_n)$,

$$\begin{aligned}v_n w_n v_{n+1}^* \chi_Z &= v_n w_n \chi_{A'} u^* = v_n \chi_{A'} u^{j-2} z^{2^n-j+2} u^{2-j} u^* \\ &= \chi_Z u^{j-1} z^{2^n-j+2} u^{1-j}\end{aligned}$$

and

$$w_n \chi_Z = \chi_Z u^{j-1} z^{2^n-j+1} u^{1-j}$$

so then we have

$$\|(v_n w_n v_{n+1}^* - w_n)\chi_Z\| \le \|z - 1\| < \pi 2^{-n}.$$

Finally, we have

$$\begin{aligned}\|(v_n w_n v_{n+1}^* - w_n)\chi_Y\| &= \|v_n w_n \chi_{\varphi^{-1}(Y)} u^* - \chi_Y\| \\ &= \|v_n \chi_{\varphi^{-1}(Y)} u^* - \chi_Y\| = 0.\end{aligned}$$

This completes the proof. □

LEMMA 6.4. *For all $n \ge 2$, w_n commutes with $C(X) \cap A_{n-1}$.*

Proof. We observe that since w_n commutes with $\chi_{\varphi^j(Z_n)}$, for $j = 0, \dots, 2^n$, and because $\chi_Y w_n = \chi_Y$, we may write

$$w_n = \sum_{j=0}^{2^n} (\chi_{\varphi^j(Z_n)} w_n \chi_{\varphi^j(Z_n)}) + \chi_Y,$$

where Y is as before.

It is clear that χ_Y commutes with all of $C(X)$. Since Z_n was chosen so that each set $\varphi^j(Z_n)$ is contained in a single element of $\mathscr{P}''_{n-1}$, each $\chi_{\varphi^j(Z_n)} w_n \chi_{\varphi^j(Z_n)}$ commutes with $C(X) \cap A_n$. □

For each positive integer n, we define $\alpha_n: A_{\{y\}} \to A_{\{y\}}$ by $\alpha_n = \mathrm{ad}(w_1 w_2 \cdots w_n)$.

LEMMA 6.5. (i) *For all $f \in C(X)$, $\lim_n \alpha_n(f)$ exists.*
(ii) $\lim_n \alpha_{n-1}(v_n)$ *exists.*

Proof. For both parts we will show that the sequences in question are Cauchy.

(i) Let $\varepsilon > 0$ be arbitrary. There is a positive integer m and a function g in $C(X) \cap A_m$ such that $\|f - g\| < \varepsilon$. Then for all $n \geq m + 1, w_n$ commutes with g, by Lemma 6.4. So, for all $l, k \geq m$, we have $\alpha_k(g) = \alpha_l(g)$ and

$$\begin{aligned} \|\alpha_k(f) - \alpha_l(f)\| &\leq \|\alpha_k(f) - \alpha_k(g)\| + \|\alpha_k(g) - \alpha_l(g)\| \\ &\quad + \|\alpha_l(g) - \alpha_l(f)\| \\ &< \varepsilon + 0 + \varepsilon = 2\varepsilon. \end{aligned}$$

This completes the proof of part (i).

(ii) Follows immediately from the following inequality

$$\|\alpha_{n-1}(v_n) - \alpha_n(v_{n+1})\| = \|v_n - w_n v_{n+1} w_n^*\| \leq \pi 2^{-n},$$

by Lemma 6.3. □

For all $f \in C(X)$, we define $\alpha(f) = \lim_n \alpha_n(f) \in A_{\{y\}}$ and we define the unitary $v = \lim_n \alpha_{n-1}(v_n) \in A_{\{y\}}$.

LEMMA 6.6. *For all $f \in C(X)$, $v\alpha(f)v^* = \alpha(\varphi(f))$.*

Proof. It suffices to show that, for any integer m and any $f \in C(X) \cap A_m$, the result is true.

$$\begin{aligned} v\alpha(f)v^* &= \lim_n \alpha_{n-1}(v_n)\alpha_{n-1}(f)\alpha_{n-1}(v_n^*) \\ &= \lim_n \alpha_{n-1}(v_n f v_n^*). \end{aligned}$$

By Lemma 6.2, if $n \geq m + 1$, then $v_n f v_n^* = \varphi(f)$ and so the limit is just $\alpha(\varphi(f))$ as desired. □

THEOREM 6.7 (*Versik*). *Let y be any point of X. There is a unital embedding*

$$\tilde{\alpha}: C(X) \times_{\varphi} \mathbf{Z} \to A_{\{y\}},$$

such that $\tilde{\alpha}_*: K_0(C(X) \times_{\varphi} \mathbf{Z}) \to K_0(A_{\{y\}})$ is an isomorphism of ordered groups.

Proof. The embedding is defined by $\tilde{\alpha}(f) = \alpha(f)$ for all $f \in C(X)$ and $\tilde{\alpha}(u) = v$. This map is injective because $C(X) \times_{\varphi} \mathbf{Z}$ is simple.

We now wish to show that $\tilde{\alpha}_*$ is an order isomorphism. Recall our earlier commutative diagram

$$\begin{tikzcd} & C(X) \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\ A_{\{y\}} \arrow[rr, "i"'] & & C(X) \times_{\varphi} \mathbf{Z} \end{tikzcd}$$

We add to this

$$\begin{tikzcd} & C(X) \arrow[dl, "i_1"'] \arrow[d, "i_2"] \arrow[dr, "\alpha"] & \\ A_{\{y\}} \arrow[r, "i"'] & C(X) \times_{\varphi} \mathbf{Z} \arrow[r, "\tilde{\alpha}"'] & A_{\{y\}} \end{tikzcd}$$

We note that each $\alpha_n: C(X) \to A_{\{y\}}$ is unitarily equivalent to i_1 and so $(\alpha_n)_* = (i_1)_*$. Then, since $\alpha = \lim \alpha_n$ and because of the homotopy invariance of K-theory, we have that $\alpha_* = (i_1)_*$. We obtain the following commutative diagram.

$$\begin{tikzcd} & K_0(C(X)) \arrow[dl, "(i_1)_*"'] \arrow[d, "(i_2)_*"] \arrow[dr, "\alpha_* = (i_1)_*"] & \\ K_0(A_{\{y\}}) \arrow[r, "i_*"'] & K_0(C(X) \times_{\varphi} \mathbf{Z}) \arrow[r, "(\tilde{\alpha})_*"'] & K_0(A_{\{y\}}) \end{tikzcd}$$

Since i_* is an order isomorphism (Theorem 4.1) and since

$$(i_2)_*(K_0(C(X))^+) = K_0(C(X) \times_{\varphi} \mathbf{Z})^+$$

as noted in §4, routine arguments then show that $(\tilde{\alpha})_* = (i_*)^{-1}$ is an order isomorphism. □

REMARK. Unlike our embeddings $A_{\{y\}}$ into $C(X) \times_{\varphi} \mathbf{Z}$, the image of $C(X)$ under $\tilde{\alpha}$ is not a Cartan subalgebra or "standard diagonal" (as in Stratila-Voiculescu). In fact this situation cannot be improved upon. A result of Archbold and Kumjian **[1]** states that if C is a Cartan subalgebra of an AF-algebra A, and B is any C^*-algebra such that $C \subset B \subset A$, then B is AF. In our situation, $C(X) \times_{\varphi} \mathbf{Z}$ is certainly not AF.

7. Further examples and concluding remarks. We conclude by presenting some open problems, some related results and some specific examples as illustrations.

Our examples all arise from interval exchanges as in §2, and we use the same terminology and notation as there.

EXAMPLE 1. Let θ be an irrational number between 0 and 1. Let φ_1 be the homeomorphism induced from the following data: $r = 2$, $x_0 = 0, x_1 = 1 - \theta, x_2 = 1$ and $\pi = (1\ 2) \in S_2$. This example is closely linked with irrational rotation, R_θ, on the circle S^1. (In fact, as measure preserving transformations, they are the same.) It is also easy to see that there is a continuous surjection $q: X \to S^1$ such that $q \circ \varphi = R_\theta \circ q$. This implies that there is an embedding of A_θ, the irrational rotation C^*-algebra (see Rieffel **[9]**), into $C(X) \times_\varphi \mathbf{Z}$. Indeed, this crossed product C^*-algebra was constructed by Cuntz in 2.5 of **[5]** for the purpose of containing A_θ. We remark that if one follows this embedding by that of Theorem 6.7, one obtains the Pimsner-Voiculescu embedding of A_θ into an AF-algebra **[9]**. In this case, φ_1 is the restriction of a Denjoy homeomorphism of the circle (φ, with parameters $\rho(\varphi) = \theta$ and $Q(\varphi) = \{R_\theta^n(0) | n \in \mathbf{Z}\}$) to its unique minimal Cantor set. (See Putnam, Schmidt and Skau **[12]** for details and notation.)

EXAMPLE 2. Let θ and γ be irrational numbers with $0 < \gamma < 1 - \theta < 1$, and such that $\{1, \theta, \gamma\}$ is linearly independent over the rational numbers. Let φ_2 be the homeomorphism of X induced by the following data: $r = 3, x_0 = 0, x_1 = \gamma, x_2 = 1 - \theta, x_3 = 1$ and $\pi = (1\ 2\ 3) \in S_3$. Notice that we obtain the same transformation of $[0, 1)$ as in Example 1, but we have built a "different" Cantor set. The homeomorphism φ_2 is also the restriction of a Denjoy homeomorphism of the circle (φ, with parameters $\rho(\varphi) = \theta$ and $Q(\varphi) = \{R_\theta^n(0), R_\theta^n(\gamma) | n \in \mathbf{Z}\}$) to its unique minimal Cantor set. The condition of linear independence over the rationals implies that the hypotheses of Theorem 2.1 are satisfied (less will do).

EXAMPLE 3. Let $\theta, \gamma, r, x_0, x_1, x_2, x_3$ be as in Example 2. Let $\pi = (1\ 3) \in S_3$. Let φ_3 be the homeomorphism obtained from this data. Again this satisfies the conditions of 2.1.

It is clear from the definition of crossed product C^*-algebra that if φ and ψ are conjugate, or if φ and ψ^{-1} are conjugate, then

$C(X) \times_{\varphi} \mathbf{Z}$ and $C(X) \times_{\psi} \mathbf{Z}$ are $*$-isomorphic. It has been conjectured by K. Schmidt, C. Skau and myself that, for minimal homeomorphisms of the Cantor set, the converse is also true.

The homeomorphisms φ_2 and φ_3 as described above are not conjugate. However, from Theorem 2.1 (and a simple argument regarding the positive cone) we have that

$$K_0(C(X) \times_{\varphi_2} \mathbf{Z}) \simeq \mathbf{Z} + \theta\mathbf{Z} + \gamma\mathbf{Z} \simeq K_0(C(X) \times_{\varphi_3} \mathbf{Z})$$

where the isomorphisms are order isomorphisms, and the order structure on $\mathbf{Z} + \theta\mathbf{Z} + \gamma\mathbf{Z}$ is that inherited from $\mathbf{R}$. I do not know whether the C^*-algebras themselves are $*$-isomorphic.

Let us consider Example 2 for a moment. As in 1, there is a surjection $q: X \to S^1$ such that $q \circ \varphi_2 = R_\theta \circ q$, while it can be shown that there is no q' such that $q' \circ \varphi_2 = R_\gamma \circ q'$. For this reason, Schmidt, Skau and I conjectured that while there was an embedding of A_θ into $C(X) \times_{\varphi_2} \mathbf{Z}$, there was no embedding of A_γ. However, if we let A and B be the AF-algebras whose K_0-groups are $\mathbf{Z} + \theta\mathbf{Z} + \gamma\mathbf{Z}$ and $\mathbf{Z} + \gamma\mathbf{Z}$, respectively, (as ordered groups) then A_γ may be embedded into B (Pimsner-Voiculescu), B may be embedded into A and A is $*$-isomorphic to a C^*-subalgebra of $C(X) \times_{\varphi_2} \mathbf{Z}$ (by Theorem 4.1). Thus there is indeed an embedding of A_γ into $C(X) \times_{\varphi_2} \mathbf{Z}$. However, since this map goes through AF-algebras, it induces the zero map at the level of K_1. The following question seems reasonable:

If there is an embedding $\rho: A_\alpha \to C(X) \times_{\varphi_2} \mathbf{Z}$ such that $\rho_*: K_1(A_\alpha) \to K_1(C(X) \times_{\varphi_2} \mathbf{Z})$ is surjective, then $\alpha = \theta$ or $1 - \theta$.

We also note that by similar arguments to those above there are plenty of $*$-homomorphisms between $C(X) \times_{\varphi_2} \mathbf{Z}$ and $C(X) \times_{\varphi_3} \mathbf{Z}$ which will all induce the zero map at the level of K_1 but be order isomorphisms on K_0.

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