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THE LIMIT OF A SEQUENCE OF SQUARES IN AN ALGEBRA NEED NOT BE A SQUARE

RICHARD ARENS

Dedicated to the memory of Professor Henry A. Dye

Let $A_1 \leftarrow A_2 \leftarrow A_3 \leftarrow \cdots$ be an inverse system of Banach-algebra homomorphisms, and let f be an element of the limit algebra A. Suppose f is the limit of squares, that is,

$$f_1 \leftarrowtail f_2 \leftarrowtail f_3 \leftarrowtail \cdots \leftarrowtail f,$$

where each f_n has a *square root* in its algebra A_n. Does this require that f have a square root in A? Our object is to show that the answer to this as well as some similarly natural questions is 'no'.

1. A topological reformulation. Let Σ be a covering of a topological space X. We will say that X has the Σ-topology if a set F is closed in X if and only if $F \cap S$ is closed for every member S of Σ **[2]**.

In what follows, Σ will be an increasing sequence of compact subsets of X.

Let P be the cartesian product of countably many copies of the unit circle $U(1)$, with the usual compact topology. Let X_p be those sequences $x = (\lambda_1, \lambda_2, \lambda_3, \dots)$ for which the numbers $(\lambda_n)^2$ with $n \geq p$ are all equal to $(\lambda_p)^2$. This set is closed in P and hence is compact.

Let X be the union of these compact spaces X_n. The X_n form a covering Σ for X and thus define a Σ-topology for X. This Σ-topology, and not the topology X inherits from P, will be meant when we say 'the topology of X'.

Let $x = (\lambda_1, \lambda_2, \lambda_3, \dots)$ be a point of P. Define $g_n(x)$ to be λ_n. Then g_n is continuous on P and hence on each X_q and hence on X.

For $x = (\lambda_1, \lambda_2, \lambda_3, \dots)$ in X, supposing it to lie in X_n define $f(x)$ to have the value $(\lambda_n)^2$. On X_n, g_n^2 agrees with f, so f is continuous on that set. Thus f is continuous on X.

Consider a curve $\alpha(t)$ lying in X_n. If it is continuous relative to the topology X_n inherits from P, it is continuous as a curve in X.

THEOREM 1. *f has no continuous square root on X.*

Proof. Suppose g were a continuous square root of f. Then $g(x) = g(y)$ if x and y agree from some point on, because the obvious curve from x to y lies in some X_n. In fact, f is constant on this curve and g is always one of the square roots of that constant, hence constant.

Suppose some directed set x_k lies in some X_n and converges to a point y there, in the topology X_n inherits from P. Then it converges in X to y. An example is the sequence given by $\{x_k\}$ where x_k has k (-1)'s followed by $1, 1, 1, \ldots$, and $y = (-1, -1, -1, \ldots)$. All these points lie in X_1. Now $g(1, 1, 1, \ldots)$ is surely ± 1. It will suffice to consider the case where it is $+1$. Then $g(x_k)$ is also 1, and $g(-1, -1, -1, \ldots)$ is 1 because g is continuous on X_1.

Consider $\alpha(t) = (e^{it}, e^{it}, e^{it}, \ldots)$. Because it lies in X_1, it is a curve connecting $(1, 1, 1, \ldots)$ to $(-1, -1, -1, \ldots)$ as t varies from 0 to π. Now $g(\alpha(t)) = \pm e^{it}$. For $t = 0$, the sign is $+$, so it must be also for $t = \pi$. Thus $g(-1, -1, -1, \ldots)$ is -1. This contradicts the earlier finding that this value is $+1$. Thus there cannot be a continuous square root of our f, proving the theorem.

2. Generalization to powers greater than the second. Let P be as above. Let H be those sequences $x = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ for which the sequence of numbers $(\lambda_n)^n$ ultimately assumes some constant value in $U(1)$. Define $f(x)$ as equal to that limiting value.

Let H_n be the subset of P of those x for which that limiting value has already been attained with $(\lambda_n)^n$. Note that f has a continuous nth root g_n on H_n.

It is natural to ask, does f have a square root h on P? It does not. We take $\alpha(t) = \exp(it/n)$. Let $y = \alpha(2\pi)$, and let x_k look like y for the first k places and then have 1's. All these x_k and the y lie in H_1, so x_k converges to y, and $g(y) = g(1, 1, 1, \ldots)$ which we assume to be 1.

On the other hand, $f(\alpha(t)) = \exp(it)$, and so $g((t)) = \pm\exp(it/2)$, and again the sign cannot change. So $g(y) = -1$. Hence f cannot have a continuous square root. A simple modification shows that it could not have a mth root for any $m \geq 2$. We formulate this.

THEOREM 2. *The space H is the union of a nested sequence of compact sets H_n. Let $A_n = C(H_n, \mathbf{C})$ be the continuous complex valued functions on H_n. Let $A = C(H, \Sigma, \mathbf{C})$ be the functions on H which for each n when restricted to H_n fall into A_n. There is an f in A whose restriction to H_n has a continuous nth root in A_n for every n. But f is not the nth power of any element of A with n greater than* 1.

3. The algebraic version of these theorems. We evidently have here an inverse system

$$A_1 \leftarrow A_2 \leftarrow A_3 \leftarrow \cdots$$

of Banach algebra homomorphisms. This defines an inverse limit algebra A. This A has an element which is not an nth power of any element of A for any n greater than 1. However, its image f_n in A_n is an nth power of an element of A_n for each n.

In our example, the algebras are commutative, and f is invertible.

By way of contrast, in **[1]** it is shown that if f_n has a right inverse in each A_n, then f also has a right inverse in A.

4. The inverse limit of exponentials. This topic is not in the realm of pure algebra, but of topological algebra. We will show that the inverse limit of exponentials is not necessarily an exponential.

Let X_n be the set of bounded complex sequences $(\lambda_1, \lambda_2, \lambda_3, \dots)$ for which λ_m is constant *modulo* $\mathbf{Z}$ from n onward, with bound $B_n \leq n$.

Suppose there were a continuous g such $g(x)$ that differs from each $g_n(x)$ by an integer at most. Then $g(x) = g(y)$ if x and y agree from some point on, because the obvious curve from x to y lies in some X_n. Let $\alpha(\theta)$ be $(\theta, \theta, \theta, \dots)$. Then $g_n(\alpha(\theta)) = \theta$. Therefore $g((\theta)) = \theta + j$, j an integer. This j is constant. It suffices to treat the case where it is 0. Then $g(0, 0, 0, \dots) = 0$ and $g(1, 1, 1, \dots) = 1$.

Let x_k have k zeros and then all 1's. These elements x_k are all in X_1, and so is $(0, 0, 0, \dots)$. The limit of the x_k is $(0, 0, 0, \dots)$. By the x and y argument, the $g(x_k)$ are all equal to 1. Hence $g(0, 0, \dots)$ should be the limit of these 1's, and so be 1.

This contradiction proves the following theorem, expressed in the notation of Theorem 2.

THEOREM 3. *The space H is the union of a nested sequence of compact sets H_n. Let $A_n = C(H_n, \mathbf{C})$ be the continuous complex valued functions on H_n. Let $A = C(H, \Sigma, \mathbf{C})$ be the functions on H which for each n when restricted to H_n fall into A_n. There is an f in A whose restriction to H_n has a continuous logarithm in A_n for every n. But f is not the exponential of any element of A.*

Here we are taking $f(x)$ to be the common value of the numbers $\exp[2\pi i g_n(x)]$ where n is large enough so that $x \in X_n$.

References

[1] Richard Arens, *Dense inverse limit rings*, Michigan Math. J., **5** (1958), 169–182.
[2] R. Brown, *Ten topologies for $X - Y$*, Quart J. Math. Oxford (2), **14** (1963), 303–319.

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University of California
Los Angeles, CA 90024-1555-05