PACIFIC JOURNAL OF MATHEMATICS
Vol. 143, No. 1, 1990

FIXED POINTS FOR ORIENTATION PRESERVING HOMEOMORPHISMS OF THE PLANE WHICH INTERCHANGE TWO POINTS

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Suppose h is an orientation preserving homeomorphism of the plane which interchanges two points p and q. If A is an arc from p to q, then h has a fixed point in one of the bounded complementary domains of $A \cup h(A)$.

1. Introduction. Brouwer's Lemma [2], one version of which is that each orientation preserving homeomorphism of the plane with a periodic point has a fixed point, has had much attention in the last few years. It has played a central role in some work of Fathi [7], Franks [**8**, **9**], Pelikan and Slaminka [**11**], Slaminka [**12**] and the author [**3**, **4**].

An interesting special case is when the periodic point has period two. Indeed, this case is at the heart of Fathi's argument in [7], and his proof of Brouwer's lemma requires a separate proof of this case. The purpose of this note is to show that this result follows from a particularly simple and elegant application of the notion of index of a homeomorphism along an arc. Furthermore, we get constructive information about the location of the fixed point. Our proof both simplifies and strengthens a result of Galliardo and Kottman [**10**].

In a final section we illustrate some techniques which can be used to locate fixed points more precisely.

2. The index. Let f, g be maps of the interval [01] into the plane such that $f(t)$ is distinct from $g(t)$ for each t in [01]. Then index (f, g) is defined to be the total winding number of the vector $g(t)-f(t)$ as t runs from 0 to 1. For example, in Figure 1 this vector makes a total of 1 and 1/2 turns in the clockwise (i.e., negative) direction, so the index is $-(1+1/2)$. The reader who wishes a more precise definition of index and its properties should consult [**5**] and [**6**].

If f and f' are two maps of [01] into the plane such that $f(1) = f'(0)$ then we denote by $f * f'$ the map of [01] into the plane which is $f(2t)$ on $0 \leq t \leq 1/2$, and $f'(2t-1)$ on $1/2 \leq t \leq 1$. Clearly, if index(f, g) and index(f', g') are defined then index$(f*f', g*g')$ is well defined and equal to index$(f, g)+$ index(f', g').

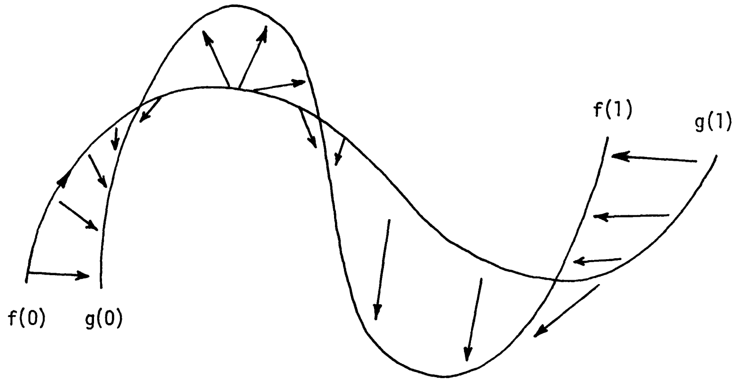

FIGURE 1

3. LEMMA. *Let h be an orientation preserving homeomorphism of the plane and let p, q be distinct points such that $h(p) = q$ and $h(q) = p$. Let f be a path from p to q whose image contains no fixed points of h. Then there exists an integer k such that*

$$\operatorname{index}(f, hf) = \operatorname{index}(hf, hhf) = 1/2 + k.$$

Proof. h interchanges p and q, so the vectors $hf(0) - f(0) = q - p$ and $hf(1) - f(1) = p - q$ point in opposite directions, i.e., $\operatorname{index}(f, hf) = 1/2 + k$. Since h is orientation preserving, there is an isotopy g_s, $0 \leq s \leq 1$, connecting the identity to h. Then $\operatorname{index}(g_s f, g_s hf)$ varies continuously from $\operatorname{index}(f, hf)$ to $\operatorname{index}(hf, hhf)$. On the other hand, for each s, the vectors $g_s hf(0) - g_s f(0) = g_s(q) - g_s(p)$ and $g_s hf(1) - g_s f(1) = g_s(p) - g_s(q)$ point in opposite directions, so, by continuity, $\operatorname{index}(g_s f, g_s hf)$ is constant as s varies from 0 to 1. Hence

$$\operatorname{index}(g_1 f, g_1 hf) = \operatorname{index}(hf, hhf) = 1/2 + k.$$

4. THEOREM. *Let h, p, q, f be as in the Lemma. Then,*

$$\operatorname{index}(f * hf, hf * hhf)$$

is an odd integer, and h has a fixed point in a bounded complementary domain of the image of the loop $f * hf$.

Proof. By the additivity of the index, $\operatorname{index}(f * hf, hf * hhf) = \operatorname{index}(f, hf) + \operatorname{index}(hf, hhf) = 2(1/2 + k)$, which is an odd integer.

Since the image of $f * hf$ is locally connected, the set X consisting of the image of $f * hf$ and the union of its bounded complementary domains is a locally connected continuum ([**13**], p. 112–113). Since X does not separate the plane it is an absolute retract ([**1**]), and hence contractible. If h were fixed point free in each of the bounded complementary domains of the image of the loop $f * hf$, then the loop could be shrunk to a point within X, and index$(f * hf, hf * hhf)$ would be zero, a contradiction.

5. Examples. Let h, p, q, f be as in the Theorem.

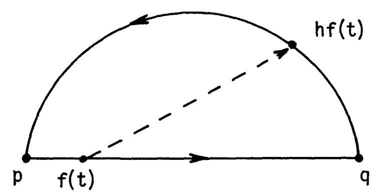

FIGURE 2

In Figure 2 the curve f (more precisely the image of f) is a simple arc from p to q and intersects hf only at the endpoints which h interchanges. Then index$(f * hf, hf * hhf) = 1$, and there is a fixed point h inside the simple closed curve $f * hf$.

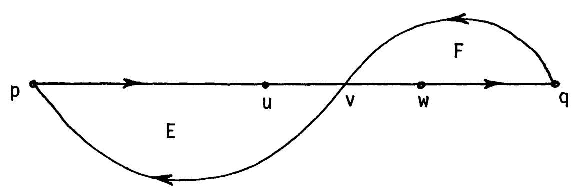

FIGURE 3

In Figure 3, f is again a simple arc and f intersects hf in one other point v. The index (f, hf) is seen by inspection to be $-1/2$ or $+1/2$ depending on whether $h(u) = v$ or $h(w) = v$, respectively. Hence, by the Lemma, index$(f * hf, hf * hhf) = -1$ or $+1$, respectively. Suppose $h(u) = v$. We wish to calculate the index of h "around" each of the domains, E, F; that is, the index of positively oriented simple closed curves lying in and surrounding the fixed point sets of h in E, F respectively. Then

$$\text{index}(f * hf, hf * hhf) = (\text{index of } h \text{ around } F) - (\text{index of } h \text{ around } E) = 1.$$

(Note that $f * hf$ goes around E in the negative direction.) It is not difficult to construct a homeomorphism g of the plane which equals h when restricted to $K = \text{image} f$, and such that g has index 1 around F, and 0 around E. I claim that this ensures that h has the same indicial values around E, F, respectively. The justification for the claim lies in the following Theorem.

THEOREM. *Let h, g be orientation preserving homeomorphisms of the plane and let K be an arc that K contains no fixed points of h, and $h = g$ on K. Let $X = K \cup h(k) = K \cup g(K)$. Then the maps*

$$\frac{x - h(x)}{\|x - h(x)\|} \quad \text{and} \quad \frac{x - g(x)}{\|x - g(x)\|}$$

are homotopic maps of X into the unit circle.

Proof. By a variation of Alexanders Isotopy Theorem ([3], page 38) h is isotopic to g relative to K. Let p_t denote the isotopy ($p_0 = h$, $p_1 = g$, and for each t, $p_t = h$ on K). Since p_t has no fixed points on K it has no fixed points on $p(K)$, so the required homotopy is $(x - p_t(x))/\|x - p_t(x)\|$.

A consequence of this result is that g and h have the same index around each complementary domain of $K \cup h(K)$.

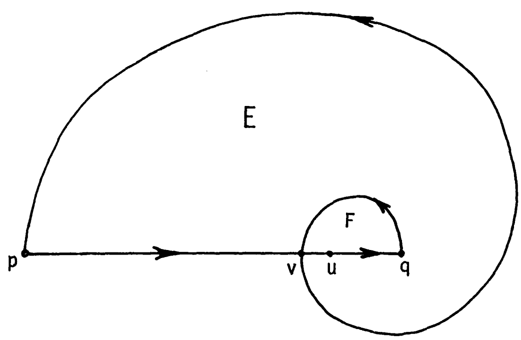

FIGURE 4

In Figure 4, the calculation of the index (f, hf) depends again on the location of $f^{-1}(v)$. Let us suppose it is u, so that $\text{index}(f, hf) = 3/2$ and $\text{index}(f * hf, hf * hhf) = 3$. Notice that $f * hf$ winds twice positively around F and once positively around E, so that

$$(\text{index of } h \text{ around } E) + 2(\text{index of } h \text{ around } F) = 3.$$

With a bit more work than the previous case one can construct a homeomorphism g which equals h on $K =$ image f and which has index 1 around each of E and F. Thus, by the Theorem above, the same is true for h, and h has a fixed point in each of the domains E and F.

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Received June 6, 1988 and, in revised form July 29, 1988.

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