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DIAGONAL STATES ON $\mathcal{O}_2$

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In this paper we investigate certain states on the Cuntz algebra $\mathcal{O}_2$, and the von Neumann algebras obtained from their GNS representations.

Introduction. In this paper we investigate certain states on the Cuntz algebra $\mathcal{O}_2$, and the von Neumann algebras obtained from their GNS representations. The problem we begin with is that of finding different types of factor states on $\mathcal{O}_2$ which extend the trace on Choi's algebra, viewed as a subalgebra of $\mathcal{O}_2$. The existence in general of such factor state extensions was established independently by Longo and Popa (see [**1**]).

The construction of specific examples, and classifying them as to type, has been done by several mathematicians. These examples arise by composing the expectation of $\mathcal{O}_2$ onto the CAR algebra with a factor state on the CAR algebra. Work of Evans and Lance showed that by starting with the trace on the CAR algebra, a type $\mathrm{III}_{1/2}$ factor state extension is obtained (see [**1**]). In [**9**], pure state extensions are constructed by a combinatorial argument. In [**2**], it is shown that uncountably many inequivalent pure state extensions are so obtained, and that they arise from certain pure states on the CAR algebra. Moreover, it is shown that if the Powers III_λ states on the CAR algebra are extended to $\mathcal{O}_2$, they result in factor states of type III_λ if $\lambda^{n+1}+\lambda^n = 1$, some n in $\mathbb{Z}_+$, or III_1, if $\log\lambda$ and $\log(\lambda+1)$ are algebraically independent. In [**11**], a different collection of product states on the CAR algebra is shown to give rise to factor state extensions of type III_λ for all $0 \le \lambda \le 1$.

The techniques in [**2**] and [**11**] rely on the quasi-invariance, under the shift automorphism of [**6**], of an appropriate state or weight on the stabilized CAR algebra. In §1 of the present paper it is shown that arbitrary (infinite) Krieger factors can be obtained from factor state extensions on $\mathcal{O}_2$. The technique is the opposite of the above: namely, we use weights all of whose translates by powers of the shift are disjoint.

The remainder of the paper is a more detailed study of the states on $\mathcal{O}_2$ which factor through product states on the diagonal of the CAR algebra. In §2 we use results of **[11]** to give a characterization of quasi-equivalence of product states on the CAR algebra somewhat different from that of **[10]** and **[3]**. We use these results in §3 to classify the corresponding states on $\mathcal{O}_2$ up to quasi-equivalence. In §4 we apply these results to give a purely dynamical proof of the result of **[2]** on the existence of pure state extensions. In §5 we briefly indicate how the results of **[11]**, computing invariants for the factors obtained from certain diagonal product states on $\mathcal{O}_2$, can be generalized to all such states.

Finally, we wish to thank Masamichi Takesaki for making the crucial suggestion which led to the results of §1.

Preliminaries. We recall some definitions and results from **[11]**. For $n \geq 0$, let $A_n = \bigotimes_{j=-n}^{\infty} M_2^{(j)}$, where $M_2^{(j)} = M_2(\mathbb{C})$, for all j. Let $i_n \colon A_n \to A_{n+1}$ be defined by $i_n(x) = e_{11}^{(-n-1)} \otimes x$, where for $b \in M_2$ we write $b^{(j)}$ for the same element viewed as an element of $M_2^{(j)}$. Let $A = \overline{\bigcup_n A_n}$. For $k \leq l$, and $b_k, b_{k+1}, \ldots, b_l \in M_2(\mathbb{C})$, we write $\bigotimes_{j=k}^{l} b_j^{(j)}$ for the element

$$\left(\bigotimes_{j=-n}^{k-1} e_{11}^{(j)}\right) \otimes \left(\bigotimes_{j=k}^{l} b_j^{(j)}\right) \otimes \left(\bigotimes_{j=l+1}^{\infty} 1^{(j)}\right)$$

in A_n for suitably large n. Let $B_n = \bigotimes_{j=-n}^{n} M_2^{(j)}$, $B = \bigcup_n B_n$. Define $\alpha \in \operatorname{Aut}(A)$ by letting $\alpha(\bigotimes_j x_j^{(j)}) = \bigotimes_j x_{j-1}^{(j)}$, for $x = \bigotimes_j x_j^{(j)}$ in B, and extending to A by linearity and continuity. Let $e_n = 1^{(-n)}$. Then as in **[6]**, we have $A \times_\alpha \mathbb{Z} \cong \mathcal{K} \otimes \mathcal{O}_2$, and $e_0(A \times_\alpha \mathbb{Z})e_0 \cong \mathcal{O}_2$. Note also that A_0 is the CAR algebra, and $A \cong \mathcal{K} \otimes A_0$. We will use the notation

$$(i_0, j_0) \otimes \cdots \otimes (i_n, j_n)$$

for the element $\bigotimes_{k=0}^{n} e_{i_k, j_k}^{(k)}$ in A_0. By a *weight sequence* we mean a sequence $\{t_j\}_{j \geq j_0}$ in $[0, 1]$. Given $\{t_j\}_{j \geq j_0}$, define $\Lambda_j \in M_2(\mathbb{C})$ for $j \in \mathbb{Z}$ by

$$\Lambda_j = \begin{cases} \operatorname{diag}(t_j, 1 - t_j), & j \geq j_0, \\ 1, & j < j_0. \end{cases}$$

Define $f_n \in (A_n)_+^*$ by $f_n(\bigotimes_j x_j^{(j)}) = \prod_{j \geq -n} \operatorname{Tr}(\Lambda_j x_j)$ for $\bigotimes_j x_j^{(j)}$ in $A_n \cap B$, where Tr is the non-normalized trace on $M_2(\mathbb{C})$. We define a weight f on A by $f(x) = \sup f_n(e_n x e_n)$, $x \in A_+$. (The fact that f is additive on A_+ follows from **[11]**, Lemma 1.4.) Then f

is densely defined and lower semi-continuous. If $t_j \in (0, 1)$ for each j, then f is faithful. Let $\tilde{f}$ be the canonical extension of f to a weight on $A \times_\alpha \mathbb{Z}$, and $\tilde{f}_0 = \tilde{f}|(e_0(A \times_\alpha \mathbb{Z})e_0)$. Then the restriction of $\tilde{f}_0$ to Choi's algebra within $\mathcal{O}_2$ is tracial, and hence is a multiple of the unique tracial state on Choi's algebra. The functionals on $\mathcal{O}_2$ so constructed all factor through the conditional expectation $F_0 \colon \mathcal{O}_2 \to A_0$ of **[6]**. In fact, as noted in [2], these functionals factor further, through the conditional expectation E_0 of A_0 onto its diagonal $(= \overline{\operatorname{span}}\{\bigotimes_j x_j^{(j)} \in B \cap A_0 \colon x_j = e_{11} \text{ or } e_{22} \text{ for each } j\})$. It is shown in [2] that any state on $\mathcal{O}_2$ which factors through the diagonal of A_0 extends the trace on Choi's algebra. We will make use of this fact. We remark that it also follows from our results mentioned above, as the pure states on the diagonal arise from weight sequences for which $t_j = 0$ or 1 for each j. We will refer to weights on a C^*-algebra as disjoint, quasi-equivalent, or unitarily equivalent, if their GNS representations are so related. Given a weight f on a C^*-algebra A, we will let π_f denote the representation of A given by the GNS construction applied to f, we will let H_f denote the Hilbert space of this representation, and $\eta_f \colon A \to H_f$ the canonical map of A onto a dense subspace of H_f. If X is a topological space we will let $C(X)$ denote the space of continuous complex-valued functions on X. If A is another topological space, we will let $C(X, A)$ denote the space of continuous functions from X into A. We will let $C_c(X, A)$ denote the space of functions in $C(X, A)$ having compact support. If T is a closed linear operator on a Hilbert space, we will let $\operatorname{polar}(T)$ denote the partial isometry in the polar decomposition of T having the same kernel as T: $T = \operatorname{polar}(T) \cdot |T|$.

1. Krieger factor states on $\mathcal{O}_2$.

LEMMA 1.1. *Let M be an infinite Krieger factor. Then there is a state f_0 on A_0 such that $\pi_{f_0}(A_0)''$ is isomorphic to M, and such that, letting $f = \operatorname{Tr} \otimes f_0$ on A, we have that $f \circ \alpha^k$ is disjoint from f for all $k \neq 0$.*

Proof. Choose a sequence of integers $0 = n_0 < n_1 < n_2 < \cdots$, with $\lim_{k \to \infty}(n_{k+1} - n_k) = \infty$. Let

$$I_1 = \{n_{2k} \colon k = 0, 1, 2, \cdots\},$$
$$I_0 = \{n_{2k-1} \colon k = 1, 2, 3, \cdots\},$$
$$I_2 = \mathbb{N} \setminus (I_0 \cup I_1).$$

For $j \in I_1 \cup I_2$ let μ_j be the measure on $\{1, 2\}$ given by

$$\mu_j = \begin{cases} \delta_1, & j \in I_1, \\ \delta_2, & j \in I_2. \end{cases}$$

By the diagonal of M_2 we mean $\operatorname{span}\{e_{11}, e_{22}\}$. This is isomorphic to the space of continuous functions on the two-point space $\{1, 2\}$, where we may let e_{ii} represent the characteristic function of the clopen set $\{i\}$, for $i = 1, 2$. Let μ' be a measure on $\prod_{j \in I_0}\{1, 2\}^{(j)}$ such that $\pi_{\mu' \circ E_0}(\bigotimes_{j \in I_0} M_2^{(j)})''$ is isomorphic to M (**[12]**), where we let measures on a space X also denote functionals on $C(X)$. Let μ be the measure on $\prod_{j=0}^{\infty}\{1, 2\}^{(j)}$ given by $\mu = (\prod_{j \in I_1 \cup I_2} \mu_j) \times \mu'$. Let $f_0 = \mu \circ E_0$. Then $\pi_{f_0}(A_0)''$ is isomorphic to M, since the portions of μ over I_1 and I_2 yield type I factors, and M is an infinite factor. Let $f = \mathrm{Tr} \otimes f_0$. Then B is contained in the definition domain of f. For $j \in I_1$ let $x_j = (\bigotimes_{p=-j}^{j-1} 1^{(p)}) \otimes e_{11}^{(j)}$. Let $y \in B$. Then for all large enough j, x_j commutes with y. We have then, for all large enough j,

$$\begin{aligned} \|\pi_f(x_j)\eta_f(y) - \eta_f(y)\|^2 &= f\left(y^*(x_j - 1)^2 y\right) \\ &= f(y^* y) f\left((x_j - 1)^2 e_0\right) = 0. \end{aligned}$$

Since $\eta_f(B)$ is dense in H_f, it follows that $\pi_f(x_j)$ tends σ-strongly to the identity. Now let $k \neq 0$. Then $I_1 + k$ has only finitely many elements outside of I_2. Thus for all large enough j,

$$\left\|\pi_f \circ \alpha^k(x_j)\eta_f(y)\right\|^2 = \|\eta_f(y)\|^2 f\left(\alpha^k(x_j) e_0\right) = 0.$$

Thus $\pi_f \circ \alpha^k(x_j)$ tends σ-strongly to zero. It follows that π_f and $\pi_{f \circ \alpha^k}$ are disjoint [7, 5.2.4]. □

LEMMA 1.2. *Let A be a C^*-algebra, let $\alpha \in \mathrm{Aut}(A)$, and let f be a lower semi-continuous weight on A such that f and $f \circ \alpha^k$ are disjoint for $k \neq 0$. Let $\tilde{f}$ be the canonical extension of f to $A \times_\alpha \mathbb{Z}$. Then $\pi_{\tilde{f}}(A \times_\alpha \mathbb{Z})''$ is unitarily equivalent to $\pi_f(A)'' \otimes L(l^2(\mathbb{Z}))$, acting on $H_f \otimes l^2(\mathbb{Z})$.*

Proof. Let $H = H_f \otimes l^2(\mathbb{Z}) \cong l^2(\mathbb{Z}, H_f)$, let $\pi: A \to L(H)$ be given by $\pi(x) = \sum_{n \in \mathbb{Z}} \pi_f \circ \alpha^{-n}(x) \otimes \delta_n$, and let $U = 1 \otimes U_0$, where U_0 is the (rightward) shift on $l^2(\mathbb{Z})$. Then (π, U) is a covariant representation of $(A, \alpha, \mathbb{Z})$. We claim first that for $x \in C_c(\mathbb{Z}, A) \subset A \times_\alpha \mathbb{Z}$, the map $\pi_{\tilde{f}}(x) \mapsto \sum_n \pi(x(n))U^n$ is unitarily implemented. To see this we will need to identify a suitable dense subspace of $H_{\tilde{f}}$. Let

$$L_f = \{x \in A : f(x^*x) < \infty\},$$
$$L_{\tilde{f}} = \left\{x \in A \times_\alpha \mathbb{Z} : \tilde{f}(x^*x) < \infty\right\}.$$

Let $E : A \times_\alpha \mathbb{Z} \to A$ be the canonical conditional expectation, so that $\tilde{f} = f \circ E$. If $\{\hat{\alpha}_t\}_{t \in [0, 2\pi)}$ is the dual action, then E is given by:

$$E(x) = \frac{1}{2\pi} \int_0^{2\pi} \hat{\alpha}_t(x)\, dt.$$

For $x \in A \times_\alpha \mathbb{Z}$, the elements

$$z_j = \frac{1}{2\pi} \int_0^{2\pi} \hat{\alpha}_t(x) e^{-ijt}\, dt$$

belong to $C_c(\mathbb{Z}, A)$, and $z_j(k) = \delta_{jk} x_j$, where the last equation defines the elements $\{x_j\}$ in A. The $\{x_j\}$ uniquely determine x. By Lemma 2.2, Chapter 1, of **[8]**,

$$x = \lim_{n \to \infty} \frac{1}{2\pi} \int_0^{2\pi} \hat{\alpha}_t(x) k_n(t)\, dt,$$

where $\{k_n(t)\}$ is any summability kernel. Choosing $\{k_n\}$ to be Fejer's kernel,

$$k_n(t) = \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n+1}\right) e^{ijt},$$

and letting

$$\sigma_n(x) = \frac{1}{2\pi} \int_0^{2\pi} \hat{\alpha}_t(x) k_n(t)\, dt,$$

a short computation yields $\sigma_n(x) \in C_c(\mathbb{Z}, A)$, and

$$\sigma_n(x)(j) = \begin{cases} \left(1 - \frac{|j|}{n+1}\right) x_j, & |j| \le n, \\ 0, & |j| > n. \end{cases}$$

Another short computation yields

$$\begin{aligned} 0 &\le E((x - \sigma_n(x))^* (x - \sigma_n(x))) \\ &= E(x^*x) - \sum_{j=-n}^{n} \left(1 - \left(\frac{|j|}{n+1}\right)^2\right) \alpha^{-j}(x_j^* x_j). \end{aligned}$$

Hence

$$\tilde{f}(x^*x) \geq \sum_{j=-n}^{n}\left(1-\left(\frac{|j|}{n+1}\right)^2\right) f\circ\alpha^{-j}(x_j^*x_j), \qquad n=0,1,2,\dots.$$

It follows that $x \in L_{\tilde{f}} \Rightarrow x_j \in \alpha^{-j}(L_f)$ for all j. From the above and the lower semi-continuity of f, it follows that

$$\tilde{f}(x^*x) = \lim_{n\to\infty}\sum_{j=-n}^{n}\left(1-\left(\frac{|j|}{n+1}\right)^2\right) f\circ\alpha^{-j}(x_j^*x_j).$$

Hence letting

$$L_0 = \left\{x \in C_c(\mathbb{Z}, A) : x(j) \in \alpha^j\left(L_f\right)\right\},$$

it follows that $\eta_{\tilde{f}}(L_0)$ is dense in $H_{\tilde{f}}$.

Now define $W_0 \colon \eta_{\tilde{f}}(L_0) \to H$ by

$$\left(W_0\eta_{\tilde{f}}(x)\right)(j) = \eta_f\left(\alpha^{-j}(x(j))\right).$$

One easily verifies that W_0 is isometric, and that

$$W_0\pi_{\tilde{f}}(x)\eta_{\tilde{f}}(y) = \left(\sum_n \pi(x(n))U^n\right)W_0\eta_{\tilde{f}}(y),$$

for $x \in C_c(\mathbb{Z}, A)$, $y \in L_0$. Thus W_0 extends by continuity to a unitary operator implementing the desired equivalence.

Now we note that by the disjointness hypothesized for the $\{\pi_f \circ \alpha^k : k \in \mathbb{Z}\}$, it follows (see [7, 5.3]) that

$$\pi(A)'' = \pi_f(A)'' \otimes l^\infty(\mathbb{Z}).$$

Hence

$$(\pi(A) \cup \{U\})'' = \pi_f(A)'' \otimes L\left(l^2(\mathbb{Z})\right). \qquad \square$$

THEOREM 1.3. *For every infinite Krieger factor M, there is a state ρ on O_2, with $\pi_\rho(O_2)''$ isomorphic to M, and such that ρ extends the* (*unique*) *tracial state on Choi's algebra.* $\square$

Proof. Let M be an infinite Krieger factor. Choose a state f_0 on A_0 as in Lemma 1.1. Since M is infinite, $\pi_f(A)''$ is isomorphic to M, where f is as in the statement of Lemma 1.1. Since f is lower semi-continuous, and M infinite, Lemma 1.2 implies that $\pi_{\tilde{f}}(A \times_\alpha \mathbb{Z})''$ is isomorphic to M. Let $\rho = \tilde{f}_0 = \tilde{f}|O_2$, as in the section on preliminaries. Let P be the projection onto $\overline{\eta_{\tilde{f}}(O_2)}$, and let $Q = \pi_{\tilde{f}}(e_0)$. Then $\pi_\rho(O_2)''$ is unitarily equivalent to $PQ\pi_{\tilde{f}}(A \times_\alpha \mathbb{Z})''QP$. The latter is a factor and is infinite since O_2 is infinite. Therefore $\pi_\rho(O_2)''$ is isomorphic to M. As noted in the section on preliminaries, the restriction of ρ to Choi's algebra is tracial. Since f_0 is a state, so is ρ. Thus ρ extends the tracial state on Choi's algebra. □

2. Equivalence of states on the CAR algebra. We will now consider unitary equivalence and quasi-equivalence of the states of O_2 obtained from different weight sequences. First we need to study the corresponding weights on A. We use the results of **[11]** to get a slightly different characterization than that of **[10]** and **[3]**.

Let $\{t_j\}_{j\geq j_0}$ and $\{r_j\}_{j\geq j_1}$ be weight sequences in $(0, 1)$, and let f and g be the corresponding weights on A. In **[11]** it was shown that $\sum^\infty (t_j - r_j)^2 t_j^{-1}(1-t_j)^{-1} < \infty$ implies the unitary equivalence of π_f and π_g. We wish to reformulate this convergence condition so as to obtain an equivalence relation.

Definitions. Let $\{t_j\}_{j\geq j_0}$ and $\{r_j\}_{j\geq j_1}$ be weight sequences in $(0, 1)$, and let f and g be the corresponding weights.

1. We write $\{t_j\}\mathscr{R}\{r_j\}$, or equivalently $f\mathscr{R}g$, if

$$\sum^{\infty} (t_j - r_j)^2 t_j^{-1} (1-t_j)^{-1} < \infty,$$

2. We write $\{t_j\} \sim \{r_j\}$, or equivalently $f \sim g$, if there is a partition $\{I_1, I_2, I_3\}$ of $\mathbb{N}$ (unrelated to the partition constructed in Lemma 1.1) such that

(i) $\sum_{I_1} t_j < \infty$, and $\sum_{I_1} r_j < \infty$,
(ii) $\sum_{I_2} (1-t_j) < \infty$, and $\sum_{I_2}(1-r_j) < \infty$,
(iii) $\sum_{I_3}(t_j - r_j)^2 t_j^{-1}(1-t_j)^{-1} < \infty$, and
$\sum_{I_3}(t_j - r_j)^2 r_j^{-1}(1-r_j)^{-1} < \infty$.

The fact that $\sim$ is an equivalence relation on the weight sequences in $(0, 1)$ follows from Theorem 2.4 below. We remark here that $\sim$ is clearly reflexive and symmetric.

Lemma 2.1. *Let $\{t_j\}$ and $\{r_j\}$ be weight sequences in $(0, 1)$ with $\{t_j\}\mathscr{R}\{r_j\}$. Then $\{t_j\} \sim \{r_j\}$.*

Proof. Choose $p > 1$. Let

$$I_1 = \{j : t_j r_j^{-1} \geq p\}, \qquad I_2 = \{j : (1-t_j)(1-r_j)^{-1} \geq p\},$$

and
$$I_3 = \mathbb{N} \setminus (I_1 \cup I_2).$$

(i):
$$\begin{aligned}\infty &> \sum_{I_1} (r_j - t_j)^2 t_j^{-1} (1-t_j)^{-1} \\ &= \sum_{I_1} \left(1 - \frac{r_j}{t_j}\right)^2 t_j (1-t_j)^{-1} \\ &\geq (1-p^{-1})^2 \sum_{I_1} t_j (1-t_j)^{-1} \\ &\geq (1-p^{-1})^2 \sum_{I_1} t_j.\end{aligned}$$

Hence $\sum_{I_1} t_j < \infty$. But then $\sum_{I_1} r_j \leq p^{-1} \sum_{I_1} t_j < \infty$.

(ii): Similar to (i).

(iii):
$$\begin{aligned}&\sum_{I_3} (t_j - r_j)^2 r_j^{-1} (1-r_j)^{-1} \\ &\quad = \sum_{I_3} (t_j - r_j)^2 t_j^{-1} (1-t_j)^{-1} \left(t_j r_j^{-1}\right) (1-t_j) (1-r_j)^{-1} \\ &\quad < p^2 \sum_{I_3} (t_j - r_j)^2 t_j^{-1} (1-t_j)^{-1} < \infty.\end{aligned}$$
□

LEMMA 2.2. *Let $\{t_j\}$ and $\{r_j\}$ be weight sequences in $(0, 1)$ with $\{t_j\} \sim \{r_j\}$. Then there is a weight sequence $\{s_j\}$ in $(0, 1)$ such that $\{t_j\}\mathscr{R}\{s_j\}$ and $\{r_j\}\mathscr{R}\{s_j\}$.*

Proof. Let
$$s_j = \begin{cases} \min(r_j, t_j), & j \in I_1, \\ \max(r_j, t_j), & j \in I_2, \\ t_j, & j \in I_3. \end{cases}$$

Then
$$\begin{aligned}&\sum (s_j - t_j)^2 t_j^{-1} (1-t_j)^{-1} \\ &\quad = \sum_{I_1} (s_j - t_j)^2 t_j^{-1} (1-t_j)^{-1} + \sum_{I_2} (s_j - t_j)^2 t_j^{-1} (1-t_j)^{-1} \\ &\quad \leq \sum_{I_1} t_j (1-t_j)^{-1} + \sum_{I_2} t_j^{-1} (1-t_j) < \infty, \quad \text{and}\end{aligned}$$

$$\begin{aligned}\sum (s_j - r_j)^2 r_j^{-1} (1-r_j)^{-1} &\le \sum_{I_1} r_j (1-r_j)^{-1} + \sum_{I_2} r_j^{-1}(1-r_j) \\ &\quad + \sum_{I_3} (t_j - r_j)^2 r_j^{-1} (1-r_j)^{-1} < \infty. \qquad \square\end{aligned}$$

THEOREM 2.3. *Let $\{t_j\}_{j\ge j_0}$ and $\{r_j\}_{j\ge j_1}$ be weight sequences in $(0, 1)$, and let f and g be the corresponding weights on A. Let $T\colon \eta_f(x) \in \eta_f(B) \mapsto \eta_g(x) \in H_g$. The following are equivalent:*

(1) $\{t_j\}\mathscr{R}\{r_j\}$,

(2) $\eta_g(B) \subset D(T^*)$ (and hence T is preclosed),

(3) $\eta_g(B) \subset D(T^*)$, T^*T has dense range, and polar$(\overline{T})$ *is a unitary operator (intertwining π_f and π_g).*

Proof. (1) $\Rightarrow$ (3): This implication is contained in the proof of Lemma 3.8 of **[11]**.

(3) $\Rightarrow$ (2): Immediate.

(2) $\Rightarrow$ (1): The computation in the proof of Lemma 3.8(i) of **[11]** shows that if x is in B, then $\eta_g(x) \in D(T^*)$ implies that

$$\lim_{n\to\infty} \langle \eta_f(xh_n), \xi\rangle$$

exists for $\xi \in \eta_f(B)$, and defines a bounded linear function of ξ. (The elements $h_n \in B$ are given by

$$h_n = \bigotimes_{j=-n}^{n} (\Omega_j \Lambda_j^{-1})^{(j)},$$

where $\{\Omega_j\}$ are obtained from $\{r_j\}$ in the same way that $\{\Lambda_j\}$ are obtained from $\{t_j\}$.) It follows by the Banach-Steinhaus theorem that $\{\eta_f(xh_n)\colon n \in \mathbb{N}\}$ is a bounded set. A computation similar to the first computation in the proof of Lemma 3.3 of **[11]** shows that if x is in B_k,

$$\|\eta_f(xh_n)\|^2 = f(h_k x^* x h_k) \prod_{j=k+1}^{n} \left[1 + (t_j - r_j)^2 t_j^{-1} (1-t_j)^{-1}\right].$$

Hence the infinite product $\prod[1 + (t_j - r_j)^2 t_j^{-1}(1-t_j)^{-1}]$ converges. Equivalently, $\{t_j\}\mathscr{R}\{r_j\}$. $\square$

THEOREM 2.4. *Let* $\{t_j\}_{j\geq j_0}$ *and* $\{r_j\}_{j\geq j_1}$ *be weight sequences in* $(0, 1)$, *and let* f *and* g *be the corresponding weights on* A. *Let* $T: \eta_f(x) \in \eta_f(B) \mapsto \eta_g(x) \in H_g$. *The following are equivalent*:

(1) π_f *and* π_g *are unitarily equivalent,*

(2) $\{t_j\} \sim \{r_j\}$,

(3) T *is preclosed,*

(4) T *is preclosed, and* $\operatorname{polar}(\overline{T})$ *is a unitary operator* (*intertwining* π_f *and* π_g).

Proof. (2) $\Rightarrow$ (4) : By Lemma 2.2 there is a weight sequence in $(0, 1)$ with corresponding weight γ such that $f\mathscr{R}\gamma$ and $g\mathscr{R}\gamma$. Let

$$\begin{aligned} &S_f: \eta_f(x) \in \eta_f(B) \mapsto \eta_\gamma(x), \quad \text{and} \\ &S_g: \eta_g(x) \in \eta_g(B) \mapsto \eta_\gamma(x). \end{aligned}$$

Note that $T = S_g^{-1}S_f$. By Theorem 2.3, $\eta_\gamma(B) \subset D(S_f^*) \cap D(S_g^*)$. Let $x, y \in B$. Then

$$\begin{aligned} \langle S_g^*\eta_\gamma(x), T\eta_f(y)\rangle &= \langle S_g^*\eta_\gamma(x), \eta_g(y)\rangle \\ &= \langle \eta_\gamma(x), \eta_\gamma(y)\rangle = \langle \eta_\gamma(x), S_f\eta_f(y)\rangle \\ &= \langle S_f^*\eta_\gamma(x), \eta_f(y)\rangle. \end{aligned}$$

This implies that $S_g^*\eta_\gamma(B) \subset D(T^*)$. By Theorem 2.3, again, $S_g^*\eta_\gamma(B) = R(S_g^*S_g)$ is dense in H_g. Thus T is preclosed. Since $\sim$ is symmetric, T^{-1} is also preclosed. This implies that $\overline{T}$ is one-to-one. Since $\overline{T}$ has dense range, it follows that $\operatorname{polar}(\overline{T})$ is unitary.

(4) $\Rightarrow$ (3) : Immediate.

(3) $\Rightarrow$ (1) : Since $\overline{T}$ has dense range, $W = \operatorname{polar}(\overline{T})$ is a co-isometry. From $T\pi_f(x)|\eta_f(B) = \pi_g(x)T$, it follows routinely that $\pi_f|W^*WH_f$ is unitarily equivalent to π_g. Since π_f and π_g are factor representations, π_f and π_g are quasi-equivalent. Since f and g are faithful and of product type, π_f and π_g are unitarily equivalent.

(1) $\Rightarrow$ (2) : It is shown in **[3]** that π_f is unitarily equivalent to π_g if and only if

$$\sum \left(1 - (r_jt_j)^{1/2} - [(1 - r_j)(1 - t_j)]^{1/2}\right) < \infty.$$

(In **[3]** it is actually shown that this condition is equivalent to quasi-equivalence. Since we are working here with faithful product weights, this is the same as unitary equivalence.)

By rationalizing the numerator one obtains

$$\begin{aligned}1 &- (ab)^{1/2} - [(1-a)(1-b)]^{1/2}\\ &= C^{-1}\left(a+b-2ab-2[ab(1-a)(1-b)]^{1/2}\right),\end{aligned}$$

where $1 \le C \le 2$ for $0 \le a,\ b \le 1$. Rationalizing again, one obtains

$$\begin{aligned}1 &- (ab)^{1/2} - [(1-a)(1-b)]^{1/2}\\ &= C^{-1}(a-b)^2\left((a-b)^2 + \left[(a-a^2)^{1/2} + (b-b^2)^{1/2}\right]^2\right)^{-1}.\end{aligned}$$

Let

$$\begin{aligned}K_1 &= \left\{j : r_j - r_j^2 < t_j - t_j^2\right\},\\ K_2 &= \left\{j : r_j - r_j^2 > t_j - t_j^2\right\},\\ K_3 &= \{j : r_j = t_j\},\\ K_4 &= \{j : r_j = 1 - t_j\}.\end{aligned}$$

We have

$$\begin{aligned}\infty &> \sum_{K_1} \frac{(r_j - t_j)^2}{(r_j - t_j)^2 + 4t_j(1-t_j)}\\ &= \sum_{K_1} \frac{1}{1 + 4\left((r_j - t_j)^2 t_j^{-1}(1-t_j)^{-1}\right)^{-1}}.\end{aligned}$$

Equivalently,

$$\sum_{K_1} (r_j - t_j)^2 t_j^{-1} (1-t_j)^{-1} < \infty.$$

Similarly,

$$\sum_{K_2} (r_j - t_j)^2 r_j^{-1} (1-r_j)^{-1} < \infty.$$

Clearly,

$$\sum_{K_3} (r_j - t_j)^2 r_j^{-1} (1-r_j)^{-1} < \infty.$$

Finally,

$$\begin{aligned}\infty &> \sum_{K_4} \frac{(1-2t_j)^2}{(1-2t_j)^2 + 2t_j(1-t_j)}\\ &> \sum_{K_4} (1-2t_j)^2.\end{aligned}$$

Hence, $\sum_{K_4}(t_j - 1/2)^2 < \infty$, $\sum_{K_4}(r_j - 1/2)^2 < \infty$, and so

$$\sum_{K_4}(r_j - t_j)^2 r_j^{-1}(1 - r_j)^{-1} < \infty.$$

It is now clear from Lemma 2.1 that $\{t_j\} \sim \{r_j\}$. □

REMARKS. The relation $\sim$ makes sense for arbitrary weight sequences, where we impose the following additional restriction on $\sim$ in Definition 2: $t_j = 0$ or $r_j = 0 \Rightarrow j \notin I_3$. In this generality, the condition of [3] used in (1) $\Rightarrow$ (2) above is equivalent to quasi-equivalence of the weights. It is easy to see that our proof of the equivalence of $\sim$ and the condition of [3] holds for arbitrary weight sequences.

COROLLARY 2.5. *Let $\{t_j\}$ and $\{r_j\}$ be arbitrary weight sequences, and let f and g be the corresponding weights. The following are equivalent*:

(1) *π_f and π_g are quasi-equivalent,*
(2) $\{t_j\} \sim \{r_j\}$. □

3. Equivalence of states on $\mathcal{O}_2$.

DEFINITION. Let $\{t_j\}_{j \geq j_0}$ and $\{r_j\}_{j \geq j_1}$ be weight sequences. We say that $\{t_j\}$ and $\{r_j\}$ have *equivalent tails* if there is an integer n such that $\{t_j\}_{j \geq j_0} \sim \{r_{j-n}\}_{j \geq j_1 + n}$.

LEMMA 3.1. *Let $\{r_j\}_{j \geq j_1}$ be a weight sequence. Let g and h be the weights corresponding to $\{r_j\}_{j \geq j_1}$ and $\{r_{j-n}\}_{j \geq j_1+n}$, respectively. Then $h = g \circ \alpha^n$.*

Proof. Let $l \geq |j_1| + |n|$. For $x \in (B_l)_+$, and $k > l + |n|$, a straightforward calculation shows that

$$h_k(e_k x e_k) = g_k(e_k \alpha^n(x) e_k).$$

Since h_k and g_k are bounded for each k, it follows that for $x \in (A_l)_+$, and $k > l + |n|$,

$$h_k(e_k x e_k) = g_k(e_k \alpha^n(x) e_k).$$

Now let $x \in A_+$. We have for $l \geq |j_1| + |n|$ and $k > l + |n|$:

$$\begin{aligned} h(e_l x e_l) &= \sup_k h_k(e_k e_l x e_l e_k) = \sup_k g_k(e_k \alpha^n(e_l x e_l) e_k) \\ &= \sup_k g_k(e_k e_{l-n} \alpha^n(x) e_{l-n} e_k) \\ &= g(e_{l-n} \alpha^n(x) e_{l-n}) \leq g \circ \alpha^n(x). \end{aligned}$$

It follows that $h(x) \leq g \circ \alpha^n(x)$. An analogous calculation shows that $g \circ \alpha^n(x) \leq h(x)$. □

REMARKS. Let $J = C_c(\mathbb{Z}, B) \subset A \times_\alpha \mathbb{Z}$. Let f be a weight on A arising from a weight sequence. Since $\eta_f(B)$ is dense in H_f, and $\eta_f(B) \subset L_f$, it follows from the proof of Lemma 1.2 that $\eta_{\tilde{f}}(J)$ is dense in $H_{\tilde{f}}$. The map $W_f \colon H_{\tilde{f}} \to l^2(\mathbb{Z}, H_f)$ defined by

$$W_f \eta_{\tilde{f}}(x)(k) = \eta_f \circ \alpha^{-k}(x(k)), \qquad x \in J,$$

extends to a unitary operator, and

$$W_f \pi_{\tilde{f}}(x) W_f^* = \sum_k \pi(x(k)) U^k$$

(see the proof of Lemma 1.2). We will let α also denote the automorphism of $A \times_\alpha \mathbb{Z}$ defined by $(\alpha(x))(k) = \alpha(x(k))$, for $x \in J$.

LEMMA 3.2. *Let $\{t_j\}$ and $\{r_j\}$ be weight sequences in $(0, 1)$, and let f and g be the corresponding weights. Let $T \colon \eta_{\tilde{f}}(y) \in \eta_{\tilde{f}}(J) \mapsto \eta_{\tilde{g}}(\alpha^n(y))$. Suppose that $\{t_j\}$ and $\{r_j\}$ have equivalent tails. Then T is preclosed and* $\operatorname{polar}(\overline{T})$ *is a unitary operator (intertwining $\pi_{\tilde{f}}$ and $\pi_{\tilde{g}} \circ \alpha^n$).*

Proof. Let $T_0 \colon \eta_f(b) \in \eta_f(B) \mapsto \eta_g(\alpha^n(b))$. By Lemma 3.1 and Theorem 2.4, T_0 is preclosed and $\operatorname{polar}(\overline{T_0})$ is unitary. Let $\xi \in C_c(\mathbb{Z}, D(T_0^*))$ and let $y \in J$. Then

$$\begin{aligned}
\left\langle W_g^* \xi, T\eta_{\tilde{f}}(y) \right\rangle &= \langle \xi, W_g \eta_{\tilde{g}}(\alpha^n(y)) \rangle \\
&= \sum_k \left\langle \xi(k), \eta_g \circ \alpha^{n-k}(y(k)) \right\rangle \\
&= \sum_k \left\langle \xi(k), T_0 \eta_f \circ \alpha^{-k}(y(k)) \right\rangle \\
&= \left\langle (T_0^* \otimes 1)\xi, W_f \eta_{\tilde{f}}(y) \right\rangle \\
&= \left\langle W_f^* (T_0^* \otimes 1)\xi, \eta_{\tilde{f}}(y) \right\rangle,
\end{aligned}$$

where we let $T_0^* \otimes 1$ have domain

$$C_c(\mathbb{Z}, D(T_0^*)) \subset l^2(\mathbb{Z}, H_g) \cong H_g \otimes l^2(\mathbb{Z}).$$

Hence $W_g^* C_c(\mathbb{Z}, D(T_0^*)) \subset D(T^*)$, and

$$T^* W_g^* | C_c(\mathbb{Z}, D(T_0^*)) = W_f^* (T_0^* \otimes 1).$$

Since $\operatorname{polar}(\overline{T_0})$ is unitary, we know that T_0^* has dense domain and dense range. Thus T is preclosed and $\overline{T}$ is one-to-one. Since T already has dense range, it follows that $\operatorname{polar}(\overline{T})$ is unitary. □

THEOREM 3.3. *Let $\{t_j\}_{j\geq j_0}$ and $\{r_j\}_{j\geq j_1}$ be weight sequences in $(0, 1)$, let f and g be the corresponding weights on A, and let $\tilde{f}_0$ and $\tilde{g}_0$ be the restrictions of $\tilde{f}$ and $\tilde{g}$, respectively, to $e_0(A \times_\alpha \mathbb{Z})e_0$.*

(1) *If $\{t_j\}$ and $\{r_j\}$ have equivalent tails, then $\tilde{f}_0$ and $\tilde{g}_0$ are quasi-equivalent.*

(2) *If $\{t_j\}$ and $\{r_j\}$ do not have equivalent tails, then $\tilde{f}_0$ and $\tilde{g}_0$ are disjoint.*

Proof. (1): Let $\{t_j\}_{j\geq j_0} \sim \{r_{j-n}\}_{j\geq j_1+n}$. By symmetry we may assume that $n \geq 0$. By altering $\{r_j\}$ up to $\sim$, we may assume that $j_1 = 0$. By Lemma 3.2 we have that $T\colon \eta_{\tilde{f}}(y) \in \eta_{\tilde{f}}(J) \mapsto \eta_{\tilde{g}}(\alpha^n(y))$ is preclosed, and that $\operatorname{polar}(\overline{T})$ is unitary. Let $\overline{T} = V|\overline{T}|$ be the polar decomposition of $\overline{T}$, and let $D = e_0(A \times_\alpha \mathbb{Z})e_0$. Note that $\alpha(D) \subset D$, and that $H_{\tilde{f}_0} = \overline{\eta_{\tilde{f}}(D)}$, and $H_{\tilde{g}_0} = \overline{\eta_{\tilde{g}}(D)}$. It now follows that $V\pi_{\tilde{f}_0}(x) = \pi_{\tilde{g}_0}(\alpha^n(x))V|H_{\tilde{f}_0}$, $x \in D$. Let $V_0 = V|H_{\tilde{f}_0}\colon H_{\tilde{f}_0} \to H_{\tilde{g}_0}$. Then V_0 is an isometry. We then have

$$\begin{aligned} \pi_{\tilde{f}_0}(x) &= V_0^* \pi_{\tilde{g}_0}(\alpha^n(x)) V_0 \\ &= \left(\pi_{\tilde{g}_0}(S_1^*)^n V_0\right)^* \pi_{\tilde{g}_0}(x) \left(\pi_{\tilde{g}_0}(S_1^*)^n V_0\right). \end{aligned}$$

Note that

$$\begin{aligned} R(V_0) = \overline{\eta_{\tilde{g}_0}\left(S_1^n D (S_1^*)^n\right)} &\subset \overline{\eta_{\tilde{g}_0}(S_1^n D)} \\ &= R\left(\pi_{\tilde{g}_0}(S_1^n)\right). \end{aligned}$$

Hence $\pi_{\tilde{g}_0}(S_1^*)^n V_0$ is an isometry. It follows that $\pi_{\tilde{f}_0}$ is unitarily equivalent to the subrepresentation of $\pi_{\tilde{g}_0}$ obtained by restricting to

$$\operatorname{range}\left(\pi_{\tilde{g}_0}\left(\left(S_1^*\right)^n\right) V_0\right) = \overline{\eta_{\tilde{g}_0}\left(D\left(S_1^*\right)^n\right)} = \overline{\pi_{\tilde{g}_0}(D)\, \eta_{\tilde{g}_0}\left(\left(S_1^*\right)^n\right)}.$$

We claim that the central support of this subrepresentation is 1. To see this we will show that

$$\overline{\pi_{\tilde{g}_0}(D)\,\pi_{\tilde{g}_0}(D)'\,\eta_{\tilde{g}_0}\left((S_1^*)^n\right)} = H_{\tilde{g}_0}.$$

In order to demonstrate this we will need certain elements of $\pi_{\tilde{g}_0}(D)'$. Let G_n be the set of words of length n in the elements S_1 and S_2. Note that

$$(*) \qquad \sum_{w\in G_n} ww^* = 1.$$

For w, $z \in G_n$ let

$$T_{w,z}\colon \eta_{\tilde{g}_0}(x) \in \eta_{\tilde{g}_0}(D) \mapsto \eta_{\tilde{g}_0}(xwz^*).$$

Let $x = S_{p_0}\cdots S_{p_k}S_{q_l}^*\cdots S_{q_0}^*$. Then $x^*x = (q_0, q_0)\otimes\cdots\otimes(q_l, q_l)$. Then

$$zw^*x^*xwz^* = \begin{cases} zz^*\otimes(q_{n+1}, q_{n+1})\otimes\cdots\otimes(q_l, q_l)\,, \\ \qquad\qquad \text{if } l \geq n \text{ and } w = S_{q_0}\cdots S_{q_n}\,, \\ zz^*\,, \quad \text{if } l < n \text{ and } w = S_{q_0}\cdots S_{q_l}S_{i_{l+1}}\cdots S_{i_n}\,, \\ 0\,, \qquad \text{otherwise.} \end{cases}$$

Hence

$$\tilde{g}_0(zw^*x^*xwz^*) = \begin{cases} \dfrac{g_0(zz^*)}{g_0((q_0, q_0)\otimes\cdots\otimes(q_n, q_n))}\cdot g_0(x^*x)\,, \\ \dfrac{g_0(zz^*)}{g_0((q_0, q_0)\otimes\cdots\otimes(q_l, q_l))}\cdot g_0(x^*x)\,, \\ 0\,, \end{cases}$$

where the three lines in the bracket correspond to the three cases defined in the previous bracket. Note that for $l \leq n$,

$$g_0((q_0, q_0)\otimes\cdots\otimes(q_l, q_l)) \geq \prod_{m=0}^{n}\min(r_m, 1-r_m) \stackrel{\text{def}}{=} C_n.$$

Thus

$$\tilde{g}_0(zw^*x^*xwz^*) \leq g_0(zz^*)\,(C_n)^{-1}\,\tilde{g}_0(x^*x)\,.$$

Let y also be a product in the elements S_1, S_2, S_1^*, S_2^*. If $x \neq y$, a simple calculation shows that $\tilde{g}_0(y^*x) = 0$. (This relies heavily on the fact that g factors through the diagonal of A_0.) It now follows easily that

$$\tilde{g}_0(zw^*x^*xwz^*) \leq g_0(zz^*)\,(C_n)^{-1}\,\tilde{g}_0(x^*x)\,,$$

for x in the *-algebra generated by S_1 and S_2, and hence by continuity, for all x in D. This implies that $T_{w,z}$ extends to a bounded linear operator (also denoted $T_{w,z}$) in $\pi_{\tilde{g}_0}(D)'$.

By $(*)$, we have that

$$\sum_{w \in G_n} \pi_{\tilde{g}_0}(w)\, T_{S_1^n, w}\, \eta_{\tilde{g}_0}\left((S_1^*)^n\right) = \eta_{\tilde{g}_0}(1).$$

Since $\eta_{\tilde{g}_0}(1)$ is cyclic for $\pi_{\tilde{g}_0}(D)$, the claim follows.

This implies that $\pi_{\tilde{f}_0}$ and $\pi_{\tilde{g}_0}$ are quasi-equivalent.

(2): Suppose that $\{t_j\}$ and $\{r_j\}$ do not have equivalent tails. Then $f \not\sim g \circ \alpha^n$ for all integers n. It follows from Theorem 2.3, and the fact that these are faithful factor weights on A, that f is disjoint from $g \circ \alpha^n$ for all n. We now need two lemmas.

LEMMA 3.4. *Let A be a C^*-algebra, let $\alpha \in \operatorname{Aut}(A)$, let f, g be lower semi-continuous weights on $A \times_\alpha \mathbb{Z}$, and let $\tilde{f}$, $\tilde{g}$ be the canonical extensions of f, g to weights on $A \times_\alpha \mathbb{Z}$. The following are equivalent*:

(a) *$\tilde{f}$ is disjoint from $\tilde{g}$,*

(b) *f is disjoint from $g \circ \alpha^n$ for all integers n.*

Proof. (a) $\Rightarrow$ (b): If (b) is false then there is an integer n, and a nonzero operator T in $L(H_f, H_g)$, such that $T\pi_f(x) = \pi_g(\alpha^n(x))T$ for $x \in A$. We will use the notations and results of the proof of Lemma 1.2. Let $\widetilde{T} = T \otimes U_0^{-n}$ on $H_f \otimes l^2(\mathbb{Z})$. Then for $x \in A$,

$$\begin{aligned}
\widetilde{T}\tilde{\pi}_f(x) &= \widetilde{T}\left(\sum_{k \in \mathbb{Z}} \pi_f \circ \alpha^{-k}(x) \otimes \delta_k\right) \\
&= \sum_{k \in \mathbb{Z}} T\pi_f \circ \alpha^{-k}(x) \otimes U_0^{-n}\delta_k \\
&= \sum_{k \in \mathbb{Z}} \pi_g \circ \alpha^{n-k}(x)\, T \otimes \delta_{k-n} U_0^{-n} = \tilde{\pi}_g(x)\, \widetilde{T}.
\end{aligned}$$

It is clear that $\widetilde{T}$ commutes with $U = 1 \otimes U_0$. Therefore $\widetilde{T}$ intertwines $\pi_{\tilde{f}}$ and $\pi_{\tilde{g}}$.

(b) $\Rightarrow$ (a): If (b) holds, then $\tilde{\pi}_f = \bigoplus_n \pi_f \circ \alpha^{-n}$ and $\tilde{\pi}_g = \bigoplus_n \pi_g \circ \alpha^{-n}$ are disjoint. Then there is a bounded sequence $\{x_j\}$ in A such that $\tilde{\pi}_f(x_j)$ tends strongly to 1, and $\tilde{\pi}_g(x_j)$ tends strongly to 0. Since

$(\tilde{\pi}_f, U)$ is unitarily equivalent to $\pi_{\tilde{f}}$ (by Lemma 1.2), and similarly for g, it follows that $\pi_{\tilde{f}}(x_j)$ tends strongly to 1, and $\pi_{\tilde{g}}(x_j)$ tends strongly to 0. Hence $\tilde{f}$ and $\tilde{g}$ are disjoint. □

LEMMA 3.5. *Let C be a C^*-algebra, let $e \in C$ be a projection, and let $D = eCe$. Let π and ρ be representations of C, and let π_0 and ρ_0 be the restrictions of π and ρ to D, acting on eH_π and eH_ρ, respectively. If π and ρ are disjoint, then π_0 and ρ_0 are disjoint.*

Proof. There is a bounded sequence $\{x_j\}$ in C such that $\pi(x_j)$ tends strongly to 1, and $\rho(x_j)$ tends strongly to 0. Then $\{ex_je\}$ is a bounded sequence in D, and $\pi_0(ex_je)$ tends strongly to 1, and $\rho_0(ex_je)$ tends strongly to 0. □

End of proof of Theorem 3.3. By Lemma 3.4, $\tilde{f}$ and $\tilde{g}$ are disjoint. Then by Lemma 3.5, $\tilde{f}_0$ and $\tilde{g}_0$ are disjoint. □

THEOREM 3.6. *Let $\{t_j\}_{j\geq 0}$ and $\{r_j\}_{j\geq 0}$ be weight sequences in $[0, 1]$. Let f and g be the corresponding weights on A, and let $\tilde{f}_0$ and $\tilde{g}_0$ be the restrictions of $\tilde{f}$ and $\tilde{g}$, respectively, to $e_0(A \times_\alpha \mathbb{Z})e_0$.*

(1) If $\{t_j\}$ and $\{r_j\}$ have equivalent tails, then $\tilde{f}_0$ and $\tilde{g}_0$ are quasi-equivalent.

(2) If $\{t_j\}$ and $\{r_j\}$ do not have equivalent tails, then $\tilde{f}_0$ and $\tilde{g}_0$ are disjoint.

Proof. (1) By Corollary 2.5, we may modify $\{t_j\}$ and $\{r_j\}$ up to equivalence so as to obtain weight sequences in $(0, 1)$. In doing so we modify f and g up to quasi-equivalence. It is easily seen that $\tilde{f}$, $\tilde{g}$ and $\tilde{f}_0$, $\tilde{g}_0$ are all modified up to quasi-equivalence. Part (1) now follows from Theorem 3.3(1).

(2) By Corollary 2.5, f and $g \circ \alpha^n$ are disjoint for all n. Part (2) now follows from Lemmas 3.4 and 3.5. □

4. Pure states on $\mathcal{O}_2$: a theorem of Archbold, Lazar, Tsui, and Wright. We now apply the results of the previous sections to give an alternative proof of a result of Archbold, Lazar, Tsui, and Wright ([**2**], Proposition 2.10). We will consider weight sequences $\{t_j\}_{j\geq j_0}$ with $t_j = 0$ or 1 for all j. It is clear that two such weight sequences are equivalent under $\sim$ if and only if they are eventually equal, and have equivalent tails if and only if they have equal tails. (We say that $\{t_j\}$ and $\{r_j\}$ have *equal tails* if $t_j = r_{j-n}$ for some n and all sufficiently large j.) Given such a weight sequence $\{t_j\}$, with corresponding weight f, we construct a pure state on A, denoted f', as follows. Let

$$\Lambda'_j = \begin{cases} \Lambda_j, & j \geq j_0, \\ e_{11}, & j < j_0. \end{cases}$$

We construct a positive functional on $\bigcup A_n$ by means of $\{\Lambda'_j\}$ instead of $\{\Lambda_j\}$. Then the resulting functional is continuous. We let f' denote its extension to A, which is always a pure state. We remark that since

$$f'|A_0 = f|A_0 = f_0,$$

and

$$\tilde{f}'|e_0(A \times_\alpha \mathbb{Z})e_0 = f' \circ E|e_0(A \times_\alpha \mathbb{Z})e_0 = f_0 \circ F_0,$$

the functionals on O_2 obtained from f and f' are equal. We remark also that $\sim$ is equivalent to quasi-equivalence for the primed functionals, as in §2.

LEMMA 4.1. *Let $\{t_j\}_{j\geq 0}$ and $\{r_j\}_{j\geq 0}$ be weight sequences with t_j, $r_j = 0$ or 1 for all j. Let f, g be the corresponding weights on A, and let $\tilde{f}_0$, $\tilde{g}_0$ be the resulting states on O_2. Then*

(1) *$\tilde{f}_0$ is pure if and only if $\{t_j\}$ is not eventually repeating.*

(2) *$\tilde{f}_0$ and $\tilde{g}_0$ are unitarily equivalent if $\{t_j\}$ and $\{r_j\}$ have equal tails, and are disjoint otherwise.*

Proof. It is clear that $\{t_j\} \sim \{t_{j-n}\}$ for some non-zero n if and only if $\{t_j\}$ is eventually repeating. Suppose that $\{t_j\}$ is not eventually repeating. Then Corollary 2.5 implies that f' and $f' \circ \alpha^n$ are disjoint for $n \neq 0$. Now by Lemma 1.2 we conclude that $\tilde{f}'$ is a pure state of $A \times_\alpha \mathbb{Z}$, and hence that $\tilde{f}_0 = \tilde{f}'|e_0(A \times_\alpha \mathbb{Z})e_0$ is a pure state of O_2. The proof of the other implication in (1) will follow from the proof of (2).

We know by Theorem 3.6 that $\tilde{f}_0$ and $\tilde{g}_0$ are quasi-equivalent or disjoint, depending on whether $\{t_j\}$ and $\{r_j\}$ have or do not have equal tails. From what we have already shown it follows that if $\{t_j\}$ and $\{r_j\}$ are not eventually repeating then $\tilde{f}_0$ and $\tilde{g}_0$ are pure, and hence quasi-equivalent if and only if unitarily equivalent. (We mention that what we have done so far already proves Theorem 4.2 below.)

We claim now that if $\{t_j\}$ is eventually repeating, then $\pi_{\tilde{f}'}(A \times_\alpha \mathbb{Z})'$ is non-trivial and abelian. Since $\pi_{\tilde{f}'}(e_0)$ has central cover 1, it follows that $\pi_{\tilde{f}'}(O_2)'$ is non-trivial and abelian. It then follows that repeating weight sequences can never yield pure states on O_2, proving the reverse implication in (1). Moreover two repeating weight sequences with equal tails yield quasi-equivalent states on O_2 with multiplicity-free GNS representations, and hence are unitarily equivalent. Thus the lemma will follow from the claim.

So assume that $\{t_j\}$ eventually repeats with period n. Then the weight sequences $\{t_j\}$ and $\{t_{j-n}\}$ differ in only finitely many places. It follows that there is a unitary element W_0 in $\widetilde{A}$ so that

$$f' \circ \alpha^n = f' \circ \mathrm{Ad}\,(W_0)\,.$$

An easy calculation shows that the map

$$\eta_{f'}(x) \in \eta_{f'}(A) \mapsto \eta_{f'}\left(\alpha^{-n}(x)\,W_0\right)$$

defines a unitary operator W on $H_{f'}$, and that

$$\pi_{f'} \circ \alpha^{-n} = \mathrm{Ad}\,(W) \circ \pi_{f'}.$$

Letting $\pi = \sum_{k\in\mathbb{Z}} \pi_{f'} \circ \alpha^{-k} \otimes \delta_k$ acting on $H_{f'} \otimes l^2(\mathbb{Z})$, we have that

$$\pi = \sum_{j=0}^{n-1} \sum_{k\in\mathbb{Z}} \mathrm{Ad}(W^k) \circ \pi_{f'} \circ \alpha^{-j} \otimes \delta_{kn+j}.$$

Letting $\widetilde{W} = W \otimes 1$, $U = 1 \otimes U_0$, and $M = \pi_{\tilde{f}'}(A \times_\alpha \mathbb{Z})''$, and using Lemma 1.2, it is clear that $\widetilde{W}U^n \in M'$. We will finish the proof by showing that M' is generated as a von Neumann algebra by $\widetilde{W}U^n$. By Lemma 1.2 we may work interchangeably with M or with $(\pi(A) \cup \{U\})''$.

Since n is the period of repetition in $\{t_j\}$, it is easily seen that

$$\pi\,(A)'' = \left\{ \sum_{k\in\mathbb{Z}} T_k \otimes \delta_k \colon T_k \in L\left(H_f\right),\ T_{k+n} = W T_k W^* \right\}.$$

Let operators on $H_{f'} \otimes l^2(\mathbb{Z})$ have matrix decompositions along the basis $\{\delta_k\}$ of $l^2(\mathbb{Z})$. Then $T = (T_{ij})$ is in $\pi(A)''$ if and only if

$$T_{ij} = \begin{cases} 0, & i \neq j, \\ W^a T_{bb} W^{-a}, & i = j = an + b, \text{ with } 0 \leq b < n. \end{cases}$$

A routine calculation now shows that $S = (S_{ij})$ is in $\pi(A)'$ if and only if

$$S_{ij} = \begin{cases} 0, & i \not\equiv j \pmod{n}, \\ \lambda_{ij} W^{(i-j)/n}, & i \equiv j \pmod{n}, \end{cases}$$

where the λ_{ij} are constants. Additionally, S commutes with U if and only if

$$S_{ij} = \begin{cases} 0, & i \not\equiv j \pmod{n}, \\ \lambda(i-j) W^{(i-j)/n}, & i \equiv j \pmod{n}, \end{cases}$$

where $\lambda(p)$ is the (necessarily unique) value of $\{\lambda_{k,pn+k} \colon k \in \mathbb{Z}\}$. Thus $S \in M'$ if and only if there is a function $\lambda \colon \mathbb{Z} \to \mathbb{C}$ such that

$$S = \sum_p \lambda(p)(\widetilde{W} U^n)^p. \qquad \square$$

THEOREM 4.2 ([2, Proposition 2.10]). *There are uncountably many inequivalent pure states of $\mathcal{O}_2$ which extend the trace on Choi's algebra.* $\square$

5. Invariants for diagonal product states on $\mathcal{O}_2$. We now indicate how the techniques of [11] can be extended to give information about the modular spectrum and period of the factors obtained from weight sequences.

DEFINITION. Let $\{t_j\}$ be a weight sequence. If there is a non-zero k for which $\{t_j\} \sim \{t_{j-k}\}$, we define the *period* of $\{t_j\}$ to be

$$\operatorname{per}\{t_j\} = \inf\{k > 0 \colon \{t_j\} \sim \{t_{j-k}\}\}.$$

If there is no such non-zero k, we say that the period of $\{t_j\}$ is infinite.

THEOREM 5.1. *Let $\{t_j\}$ be a weight sequence such that $\sum t_j(1-t_j) = \infty$, and $\operatorname{per}\{t_j\} < \infty$. Let f be the weight on A corresponding to $\{t_j\}$, and $\tilde{f}_0$ the state on $\mathcal{O}_2$ constructed from f. Let $M_f = \pi_f(A)''$, and $M_0 = \pi_{\tilde{f}_0}(\mathcal{O}_2)''$. Then M_0 is a type* III *factor, and*

$$S(M_0) \supset S(M_f),$$
$$T(M_0) \subset T(M_f).$$

(S and T are modular spectrum and modular period, respectively. These invariants were defined in [5]. *See also* [11].)

Proof. By the results of §§2 and 3 we may assume that $0 < t_j < 1$ for all j. Let $n = \operatorname{per}\{t_j\}$. Then there is a unitary operator W on H_f such that

$$\pi_f \circ \alpha^n = \operatorname{Ad}(W) \circ \pi_f.$$

Let $\pi\colon A \to L(H_f \otimes \mathbb{C}^n)$ be given by

$$\pi(x) = \sum_{j=0}^{n-1} \pi_f \circ \alpha^j \otimes \delta_j,$$

where $\{\delta_j : 0 \leq j < n\}$ is an orthonormal basis for $\mathbb{C}^n$, and let $\widetilde{W}$ be the unitary operator on $H_f \otimes \mathbb{C}^n$ whose matrix relative to $\{\delta_j\}$ is

$$\begin{pmatrix} 0 & 1 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ & & & & 1 \\ W & & & & 0 \end{pmatrix}.$$

Then $\pi \circ \alpha = \operatorname{Ad}(\widetilde{W}) \circ \pi$. Therefore α extends to an automorphism of $\pi(A)''$. Let φ be the faithful normal semi-finite weight on M_f defined in **[11]**, ($\varphi(x) = \sup_n \langle x\eta_f(e_n), \eta_f(e_n)\rangle$, $x \in M_{f^+}$), and let $\Phi = \varphi \otimes 1_n$. Then Φ is a faithful normal semi-finite weight on $\pi(A)''$. Now the proofs in **[11]** can be adapted in a fairly straightforward manner to prove the theorem. □

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