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ON ρ-MIXING EXCEPT ON SMALL SETS

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For stochastic processes, some conditions of "ρ-mixing except on small sets" are shown to be equivalent to the (Rosenblatt) strong mixing condition.

I. Introduction. Suppose $(\Omega, \mathscr{F})$ is a measurable space. For any probability measure μ on $(\Omega, \mathscr{F})$, and any two σ-fields $\mathscr{A}$ and $\mathscr{B} \subset \mathscr{F}$, define the following measures of dependence:

$$\alpha(\mathscr{A}, \mathscr{B}; \mu) := \sup|\mu(A \cap B) - \mu(A)\mu(B)|, \qquad A \in \mathscr{A},\ B \in \mathscr{B},$$

$$\phi(\mathscr{A}, \mathscr{B}; \mu) := \sup|\mu(B|A) - \mu(B)|, \qquad A \in \mathscr{A},\ B \in \mathscr{B},\ \mu(A) > 0,$$

$$\lambda(\mathscr{A}, \mathscr{B}; \mu) := \sup\frac{|\mu(A \cap B) - \mu(A)\mu(B)|}{[\mu(A)\mu(B)]^{1/2}}, \qquad A \in \mathscr{A},\ B \in \mathscr{B},$$

$$\rho(\mathscr{A}, \mathscr{B}; \mu) := \sup|\mathrm{Corr}_\mu(f, g)|, \qquad f \in L^2(\Omega, \mathscr{A}, \mu),\ g \in L^2(\Omega, \mathscr{B}, \mu),$$

$$\beta(\mathscr{A}, \mathscr{B}; \mu) := \sup(1/2)\sum_{i=1}^{I}\sum_{j=1}^{J}|\mu(A_i \cap B_j) - \mu(A_i)\mu(B_j)|$$

where the last sup is taken over all pairs of partitions $\{A_1, \dots, A_I\}$ and $\{B_1, \dots, B_J\}$ of Ω such that $A_i \in \mathscr{A}$ for all i and $B_j \in \mathscr{B}$ for all j. Here of course $\mu(B|A) := \mu(A \cap B)/\mu(A)$, $0/0$ is interpreted to be 0, and

$$\mathrm{Corr}_\mu(f, g) := \frac{E_\mu(f - E_\mu f)(g - E_\mu g)}{E_\mu^{1/2}(f - E_\mu f)^2 E_\mu^{1/2}(g - E_\mu g)^2}$$

where $E_\mu h := \int_\Omega h\, d\mu$. In what follows, we shall be working with a given probability measure P, and these definitions will be used with $\mu = P$ and with $\mu = P(\cdot|D)$ for various events D.

Suppose $X := (X_k,\ k \in \mathbb{Z})$ is a strictly stationary sequence of random variables on a probability space $(\Omega, \mathscr{F}, P)$. For $-\infty \le J \le L \le \infty$ let $\mathscr{F}_J^L$ denote the σ-field of events generated by the r.v.'s

$(X_k,\ J \le k \le L)$. This sequence X is said to be

strongly mixing [8] if $\alpha(n) := \alpha(\mathscr{F}_{-\infty}^0, \mathscr{F}_n^\infty; P) \to 0$ as $n \to \infty$,
ϕ-mixing [6] if $\phi(n) := \phi(\mathscr{F}_{-\infty}^0, \mathscr{F}_n^\infty; P) \to 0$ as $n \to \infty$,
ρ-mixing [7] if $\rho(n) := \rho(\mathscr{F}_{-\infty}^0, \mathscr{F}_n^\infty; P) \to 0$ as $n \to \infty$,
absolutely regular [10] if $\beta(n) := \beta(\mathscr{F}_{-\infty}^0, \mathscr{F}_n^\infty; P) \to 0$ as $n \to \infty$.

It is well known (see e.g. [2]) that (i) ρ-mixing implies strong mixing, (ii) absolute regularity implies strong mixing, (iii) ϕ-mixing implies ρ-mixing and absolute regularity, and (iv) aside from transitivity, there are no other implications between these four mixing conditions. Also (see [1], [3], [5]), ρ-mixing is equivalent to the condition

$$\lambda(n) := \lambda(\mathscr{F}_{-\infty}^0, \mathscr{F}_n^\infty; P) \to 0 \quad \text{as } n \to \infty.$$

The following proposition is already well known, at least in principle:

PROPOSITION 1.1. *Suppose $X := (X_k, k \in \mathbb{Z})$ is a strictly stationary sequence of random variables on a probability space $(\Omega, \mathscr{F}, P)$. Then the following two statements are equivalent:*

(1) *X is absolutely regular.*
(2) *There exists a sequence of events $D_1, D_2, D_3, \ldots \in \mathscr{F}_{-\infty}^0$ such that $\mathrm{Lim}_{n\to\infty} P(D_n) = 1$ and $\mathrm{Lim}_{n\to\infty} \phi(\mathscr{F}_{-\infty}^0, \mathscr{F}_n^\infty; P(\cdot|D_n)) = 0$.*

Thus the absolute regularity condition is in essence a condition of "ϕ-mixing except on small sets" (the "small" exceptional sets being D_1^c, $D_2^c, D_3^c, \ldots$).

The argument for Proposition 1.1 is elementary; its main features are in e.g. [9, Chapters 6 and 12], where the absolute regularity condition is formulated under the name "weak Bernoulli." Let us quickly review it here: One can show that for each $n \ge 1$,

$$\beta(n) = E_P[\sup|P(B|\mathscr{F}_{-\infty}^0) - P(B)|,\ B \in \mathscr{F}_n^\infty]$$

where $P(\cdot|\mathscr{F}_{-\infty}^0)$ is a regular conditional probability (whose existence may be assumed without loss of generality). From this and elementary calculations, Proposition 1.1 holds with the very last equation in Proposition 1.1 replaced by

$$\mathrm{Lim}_{n\to\infty}[\sup|P(B|A) - P(B)|,\ A \in \mathscr{F}_{-\infty}^0,\ A \subset D_n,\ P(A) > 0,\ B \in \mathscr{F}_n^\infty] = 0.$$

From this and an elementary calculation (using e.g. Lemma 2.2 in §2 of this note), one has Proposition 1.1 itself.

M. Peligrad and the author **[4]** studied conditions which might be described as " ρ-mixing except on small sets." Our purpose here is to show that some of these conditions are equivalent to strong mixing. Our results are formulated in Theorems 1.2 and 1.3 below; statement (2) in Theorem 1.3 gives the main version of " ρ-mixing except on small sets."

THEOREM 1.2. *For each $\varepsilon > 0$, there exists $\delta > 0$ such that the following statement holds*:

If $(\Omega, \mathscr{F}, P)$ is a probability space and $\mathscr{A}$ and $\mathscr{B}$ are σ-fields $\subset \mathscr{F}$ such that $\alpha(\mathscr{A}, \mathscr{B}; P) \leq \delta$, then there exists an event $D \in \mathscr{A}$ such that $P(D) \geq 1 - \varepsilon$ and $\rho(\mathscr{A}, \mathscr{B}; P(\cdot|D)) \leq \varepsilon$.

Theorem 1.2 will be proved in §2. By the simple inequality

$$\alpha(\mathscr{A}, \mathscr{B}; \mu) \leq \rho(\mathscr{A}, \mathscr{B}; \mu)$$

and Lemma 2.2 below, one trivially obtains the following "converse" of Theorem 1.2: If $\varepsilon > 0$, $\mathscr{A}$ and $\mathscr{B}$ are σ-fields, D is an event (not necessarily in $\mathscr{A} \vee \mathscr{B}$), $P(D) \geq 1 - \varepsilon$ and $\rho(\mathscr{A}, \mathscr{B}; P(\cdot|D)) \leq \varepsilon$, then $\alpha(\mathscr{A}, \mathscr{B}; P) \leq 4\varepsilon$. As an application of Theorem 1.2 together with this "converse", we have the following analog of Proposition 1.1:

THEOREM 1.3. *Suppose $X := (X_k, \ k \in \mathbb{Z})$ is a strictly stationary sequence of random variables on a probability space $(\Omega, \mathscr{F}, P)$. Then the following two statements are equivalent*:

(1) *X is strongly mixing.*
(2) *There exists a sequence of events $D_1, D_2, D_3, \ldots \in \mathscr{F}_{-\infty}^0$ such that*

$$\operatorname{Lim}_{n\to\infty} P(D_n) = 1 \quad \textit{and} \quad \operatorname{Lim}_{n\to\infty} \rho(\mathscr{F}_{-\infty}^0, \mathscr{F}_n^\infty; P(\cdot|D_n)) = 0.$$

REMARK 1.4. Let us define (for the moment) statements (2′) resp. (2″) to be statement (2) in Theorem 1.3 with the very last equation replaced by

$$\operatorname{Lim}_{n\to\infty} \lambda(\mathscr{F}_{-\infty}^0, \mathscr{F}_n^\infty; P(\cdot|D_n)) = 0 \quad \text{resp.} \quad \operatorname{Lim}_{n\to\infty} L(n) = 0$$

where

$$L(n) := \sup \frac{|P(A \cap B \cap D_n) - P(A \cap D_n)P(B \cap D_n)|}{[P(A \cap D_n)P(B \cap D_n)]^{1/2}},$$

$A \in \mathscr{F}_{-\infty}^{0}$, $B \in \mathscr{F}_{n}^{\infty}$. By elementary arguments, (2) (in Theorem 1.3) $\Rightarrow$ (2′) $\Rightarrow$ (2″) $\Rightarrow$ (1) (strong mixing). Hence Theorem 1.3 implies that (2′) and (2″) are also equivalent to strong mixing. Thus a negative answer is provided to a conjecture in [**4**] that conditions such as (2) or (2′) or (2″) might describe a proper subclass of the stationary strongly mixing sequences.

REMARK 1.5. Theorem 1.3 and Remark 1.4 hold with the restriction $D_n \in \mathscr{F}_{-\infty}^{0}$ replaced by, say, the weaker restriction $D_n \in \mathscr{F}_{-\infty}^{\infty}$. This holds very simply by Theorem 1.3 itself and elementary calculations.

REMARK 1.6. Under a hypothesis similar to statement (2) in Theorem 1.3, with $P(D_n)$ converging to 1 sufficiently fast, weak and strong invariance principles were proved by M. Peligrad and the author [**4**, Theorems 1 and 2]. In [**4**, Theorem 3], stationary random sequences were constructed which were covered by [**4**, Theorems 1 and 2] but were *not* covered by previous limit theorems under mixing conditions (because they were not ρ-mixing and were strongly mixing with too slow a mixing rate). Thus conditions such as (2) in Theorem 1.3 are useful in extending the class of stationary sequences which are known to satisfy certain limit theorems. For further discussion of this, see the introduction of [**4**].

II. Proof of Theorem 1.2. To prove Theorem 1.2, it will suffice to prove the following proposition:

PROPOSITION 2.1. *Suppose* $(\Omega, \mathscr{F}, P)$ *is a probability space,* $\mathscr{A}$ *and* $\mathscr{B}$ *are* σ*-fields* $\in \mathscr{F}$, *and*

$$q := \alpha(\mathscr{A}, \mathscr{B}; P) \le (1/24)^{11}. \tag{2.1}$$

Then $\exists D \in \mathscr{A}$ *such that* $P(D) \ge 1 - q^{8/11}$ *and* $\lambda(\mathscr{A}, \mathscr{B}; P(\cdot|D)) \le q^{1/11}$.

By [**1**] or [**3**, Theorem 1.1] or [**5**], one has Theorem 1.2 as an immediate consequence of Proposition 2.1.

Proof of Proposition 2.1. Suppose that the hypothesis of Proposition 2.1 is true but the conclusion is false. We shall aim for a contradiction.

We assume $q > 0$. (Otherwise the conclusion of Proposition 2.1 would hold trivially with $D = \Omega$.)

We are supposing that for each $D \in \mathscr{A}$ with $P(D) \ge 1 - q^{8/11}$, there exist events $A \in \mathscr{A}$ and $B \in \mathscr{B}$ such that

$$|P(A \cap B|D) - P(A|D)P(B|D)| > q^{1/11} \cdot [P(A|D)P(B|D)]^{1/2}. \tag{2.2}$$

When making use of this, we can and will always impose the additional restriction that $A \subset D$. (Otherwise we could simply replace A by $A \cap D$ without changing any numbers in (2.2).)

The main idea of the proof of Proposition 2.1 is roughly as follows: Equation (2.2) and some simple arithmetic will imply that $P(A \cap B)$ will be "somewhat comparable" to $P(A)$, and $P(B)$ "won't be too much larger" than $P(A)$ (which may be "very small"). One can apply (2.2) repeatedly, obtaining *disjoint* events A, and corresponding but not necessarily disjoint events B; the events $A \cap B$ will be disjoint. Taking the respective unions, one obtains "overall" events A and B with $P(A)$ "moderately small" but not "very small", $P(A \cap B)$ "somewhat comparable" to $P(A)$ (and hence "moderately small"), and $P(B)$ "not too much larger" than $P(A)$. Then $P(A) \cdot P(B)$, the product of two "moderately small" numbers, will be considerably less than the "moderately small" number $P(A \cap B)$. Thus $P(A \cap B) - P(A)P(B)$ will be "moderately small" but will fail to be "very small", and thereby (2.1) will be violated.

The following elementary lemma will be useful. We leave its proof to the reader.

LEMMA 2.2. *If D and F are events and $P(D) > 0$, then $|P(F|D) - P(F)| \le P(D^c)$.*

We need to use the ordinals associated with well-orderings of finite or countable sets. We shall refer to all of these ordinals as simply "countable ordinals." Recall that the set of countable ordinals is itself well-ordered in a natural way; it is uncountable, but each element is greater than at most countably many others. The least (or "initial") ordinal will be denoted 0, and the successor of any ordinal η will be denoted $\eta + 1$.

Construction 2.3. For each countable ordinal η, we shall define events D_η, A_η and B_η such that

$$D_\eta \in \mathscr{A}, \quad A_\eta \in \mathscr{A}, \quad A_\eta \subset D_\eta, \quad \text{and} \quad B_\eta \in \mathscr{B}. \tag{2.3}$$

The definition will be "transfinite inductive" and will use (2.2).

To start off with the least ordinal 0, define $D_0 = \Omega$, and let $A_0 \in \mathscr{A}$ and $B_0 \in \mathscr{B}$ be such that

$$|P(A_0 \cap B_0) - P(A_0)P(B_0)| > q^{1/11}[P(A_0)P(B_0)]^{1/2}. \tag{2.4}$$

Trivially (2.3) holds for $\eta = 0$.

Now suppose η is any countable ordinal > 0, and suppose that for all ordinals $\gamma < \eta$ the events D_γ, A_γ, and B_γ have already been defined and satisfy (2.3) with η replaced by γ. Define the event $D_\eta = (\bigcup_{\gamma<\eta} A_\gamma)^c$. Thus $D_\eta \in \mathscr{A}$. If $P(D_\eta) < 1 - q^{8/11}$, then define $A_\eta = B_\eta = \varnothing$ (the empty set). If instead $P(D_\eta) \geq 1 - q^{8/11}$, then define the events $A_\eta \in \mathscr{A}$ and $B_\eta \in \mathscr{B}$, with $A_\eta \subset D_\eta$, such that

$$\begin{aligned} |P(A_\eta \cap B_\eta | D_\eta) - P(A_\eta|D_\eta)P(B_\eta|D_\eta)| \\ > q^{1/11}[P(A_\eta|D_\eta)P(B_\eta|D_\eta)]^{1/2}. \end{aligned} \tag{2.5}$$

In either case (2.3) is satisfied. This completes Construction 2.3.

REMARK 2.4. For what follows, it should be kept in mind that (i) the events A_η are (pairwise) disjoint; and (ii) eqn. (2.5) (and consequently $P(A_\eta|D_\eta) > 0$ and $P(B_\eta|D_\eta) > 0$) holds for all η such that $P(D_\eta) \geq 1 - q^{8/11}$, including $\eta = 0$.

LEMMA 2.5. *Suppose η is a countable ordinal such that $P(D_\eta) \geq 1 - q^{8/11}$. Then the following six statements hold:*

(i) $P(D_\eta) > 1/2$.
(ii) $P(A_\eta|D_\eta) \geq q^{2/11}P(B_\eta|D_\eta)$.
(iii) $P(B_\eta|D_\eta) \geq q^{2/11}P(A_\eta|D_\eta)$.
(iv) $P(A_\eta) \leq P(A_\eta|D_\eta) \leq q^{5/11}$.
(v) $P(B_\eta|D_\eta) \leq q^{5/11}$.
(vi) $P(A_\eta \cap B_\eta) > (1/2)q^{2/11}P(A_\eta)$.

Proof. (i) holds by (2.1).

Proof of (ii). Suppose $P(A_\eta|D_\eta) < q^{2/11}P(B_\eta|D_\eta)$. Then

$$\begin{aligned} |P(A_\eta \cap B_\eta|D_\eta) - P(A_\eta|D_\eta)P(B_\eta|D_\eta)| \leq P(A_\eta|D_\eta) \\ < q^{1/11}[P(A_\eta|D_\eta)P(B_\eta|D_\eta)]^{1/2}, \end{aligned}$$

contradicting (2.5). Thus (ii) must hold after all.

The proof of (iii) is like that of (ii).

Proof of (iv). $P(A_\eta) \leq P(A_\eta|D_\eta)$ since $A_\eta \subset D_\eta$.
Suppose $P(A_\eta|D_\eta) > q^{5/11}$. Then by (iii), $P(B_\eta|D_\eta) > q^{7/11}$. Hence by (2.5),

$$|P(A_\eta \cap B_\eta|D_\eta) - P(A_\eta|D_\eta)P(B_\eta|D_\eta)| > q^{7/11}. \tag{2.6}$$

By Lemma 2.2 and our hypothesis $P(D_\eta) \geq 1 - q^{8/11}$, we have $|P(A_\eta \cap B_\eta|D_\eta) - P(A_\eta \cap B_\eta)| \leq q^{8/11}$ and

$$\begin{aligned}
&|P(A_\eta|D_\eta)P(B_\eta|D_\eta) - P(A_\eta)P(B_\eta)| \\
&\quad \leq P(A_\eta|D_\eta)|P(B_\eta|D_\eta) - P(B_\eta)| + P(B_\eta)|P(A_\eta|D_\eta) - P(A_\eta)| \\
&\quad \leq 2q^{8/11}.
\end{aligned}$$

Hence by (2.3), (2.6), and (2.1),

$$\begin{aligned}
\alpha(\mathscr{A}, \mathscr{B}; P) &\geq |P(A_\eta \cap B_\eta) - P(A_\eta)P(B_\eta)| \\
&\geq q^{7/11} - 3q^{8/11} > q^{8/11},
\end{aligned}$$

contradicting (2.1). Thus $P(A_\eta|D_\eta) \leq q^{5/11}$ must hold after all. This completes the proof of (iv).

The proof of (v) is like that of (iv).

Proof of (vi). By (iv) and (v),

$$P(A_\eta|D_\eta)P(B_\eta|D_\eta) \leq q^{1/11}[P(A_\eta|D_\eta)P(B_\eta|D_\eta)]^{1/2}.$$

Since (2.5) holds, we therefore must have

$$P(A_\eta \cap B_\eta|D_\eta) > q^{1/11}[P(A_\eta|D_\eta)P(B_\eta|D_\eta)]^{1/2}.$$

Hence by (i), (iii), and (iv),

$$\begin{aligned}
P(A_\eta \cap B_\eta) &\geq (1/2)P(A_\eta \cap B_\eta|D_\eta) \\
&> (1/2)q^{1/11}[P(A_\eta|D_\eta)P(B_\eta|D_\eta)]^{1/2} \\
&\geq (1/2)q^{2/11}P(A_\eta|D_\eta) \\
&\geq (1/2)q^{2/11}P(A_\eta).
\end{aligned}$$

Thus (vi) holds. This completes the proof of Lemma 2.5.

LEMMA 2.6. *There exists a countable ordinal* η *such that* $P(D_\eta^c) > q^{8/11}$.

Proof. Suppose Lemma 2.6 is false. Define the number

$$s := \sup P(D_\eta^c) \tag{2.7}$$

where this sup is taken over all countable ordinals η. Then

$$s \leq q^{8/11}. \tag{2.8}$$

For each $n = 1, 2, 3, \ldots$, let $\gamma(n)$ be a countable ordinal such that $P(D^c_{\gamma(n)}) \geq s - 1/n$. Let γ be the least ordinal such that $\forall n \geq 1$, $\gamma \geq \gamma(n)$. As is well known, γ is a countable ordinal. By Construction 2.3, one has that $\forall n \geq 1$, $D^c_\gamma \supset D^c_{\gamma(n)}$ and hence $P(D^c_\gamma) \geq s - 1/n$. Hence by (2.7) and (2.8), $P(D^c_\gamma) = s \leq q^{8/11}$. Thus (2.5) holds for $\eta = \gamma$, and this forces $P(A_\gamma) > 0$. Since $A_\gamma \subset D_\gamma$ and $D^c_{\gamma+1} = D^c_\gamma \cup A_\gamma$, we have $P(D^c_{\gamma+1}) = P(D^c_\gamma) + P(A_\gamma) > s$, contradicting (2.7). Thus Lemma 2.6 holds after all.

Using Lemma 2.6, henceforth let τ denote the least countable ordinal such that $P(D^c_\tau) > q^{8/11}$.

LEMMA 2.7. $P(D^c_\tau) \leq 2q^{5/11}$.

Proof. By Remark 2.4(i),

$$q^{8/11} < P(D^c_\tau) = \sum_{\eta<\tau} P(A_\eta).$$

Let T be a finite set of ordinals $< \tau$ such that $\sum_{\eta \in T} P(A_\eta) > q^{8/11}$. Let γ denote the greatest element of T. Then

$$P(D^c_{\gamma+1}) = \sum_{\eta \leq \gamma} P(A_\eta) > q^{8/11}.$$

Since $\gamma < \tau$, this forces $\gamma + 1 = \tau$ to hold, by the definition of τ; and it also follows that $P(D^c_\gamma) \leq q^{8/11}$. Hence by Lemma 2.5(iv), $P(A_\gamma) \leq q^{5/11}$. Hence $P(D^c_\tau) = P(D^c_\gamma) + P(A_\gamma) < 2q^{5/11}$. This completes the proof of Lemma 2.7.

Now we are ready to complete the proof of Proposition 2.1. Define the events

$$A := \bigcup_{\eta<\tau} A_\eta = D^c_\tau \quad \text{and} \quad B := \bigcup_{\eta<\tau} B_\eta. \tag{2.9}$$

By Remark 2.4(i), the events $A_\eta \cap B_\eta$ are (pairwise) disjoint. Hence

$$\begin{aligned} P(A \cap B) &\geq \sum_{\eta<\tau} P(A_\eta \cap B_\eta) \geq \sum_{\eta<\tau} (1/2) q^{2/11} P(A_\eta) \\ &= (1/2) q^{2/11} P(A) \end{aligned} \tag{2.10}$$

by Lemma 2.5(vi) and (2.9).

Next (since $D_\eta^c \subset D_\tau^c\ \forall \eta < \tau$), $B \subset D_\tau^c \cup [\bigcup_{\eta<\tau}(B_\eta \cap D_\eta)]$ and hence

$$\begin{aligned}
P(B) &\leq P(D_\tau^c) + \sum_{\eta<\tau} P(B_\eta \cap D_\eta) \\
&\leq P(A) + \sum_{\eta<\tau} P(B_\eta | D_\eta) \\
&\leq P(A) + q^{-2/11} \sum_{\eta<\tau} P(A_\eta | D_\eta) \\
&\leq P(A) + q^{-2/11} \sum_{\eta<\tau} 2P(A_\eta) \\
&= (1 + 2q^{-2/11})P(A) \leq 6q^{3/11}
\end{aligned} \tag{2.11}$$

by (2.9), Lemma 2.5(ii) and (i), (2.1), and Lemma 2.7.

Hence by (2.3), (2.9), (2.10), (2.11), (2.1), and the definition of τ,

$$\begin{aligned}
\alpha(\mathscr{A}, \mathscr{B}; P) &\geq P(A \cap B) - P(A)P(B) \\
&\geq [(1/2)q^{2/11} - 6q^{3/11}]P(A) \\
&\geq (1/4)q^{2/11}P(A) \geq (1/4)q^{10/11} > q,
\end{aligned}$$

contradicting (2.1).

Thus Proposition 2.1 must hold after all. This completes the proof of Theorem 1.2.

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