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REMOVABLE SINGULARITIES FOR SUBHARMONIC FUNCTIONS

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Let Ω be an open set in $\mathbb{R}^n$ $(n \geq 3)$ and S be a C^2 $(n-1)$-dimensional manifold in Ω. Let $\alpha \in (0, n-2)$ and E be a compact subset of S of zero α-dimensional Hausdorff measure. We show that, if s is subharmonic in $\Omega \backslash E$ and satisfies $s(X) \leq c[\operatorname{dist}(X, S)]^{\alpha+2-n}$ for $X \in \Omega \backslash S$, then s has a subharmonic extension to the whole of Ω. The sharpness of this and other similar results is also established.

1. Introduction and results. Let Ω denote an open set in Euclidean space $\mathbb{R}^n$ $(n \geq 3)$, and let E be a compact subset of Ω. This paper is concerned with results of the following type: if s is a subharmonic function in $\Omega \backslash E$, where E is "small" and s is "not too badly behaved" (near E), then s has a subharmonic extension to the whole of Ω. We say in this case that E is a *removable singularity* of s. There is an obvious analogue of this notion for harmonic functions.

It is a consequence of a classical result [7, Theorem 5.18] that, if E is polar and s is a subharmonic function on $\Omega \backslash E$ which is bounded above near E, then E is a removable singularity of s. The idea behind our results is that, by imposing constraints on the geometry and size of the set E, the boundedness requirement can be considerably relaxed. The size of E is measured in terms of its α-dimensional Hausdorff measure $m_\alpha(E)$. A discussion of Hausdorff measures in relation to subharmonic functions can be found in Hayman and Kennedy [7, §5.4].

Let O_n denote the origin of $\mathbb{R}^n$, let $|X|$ denote the Euclidean norm of a point $X \in \mathbb{R}^n$, and $B(X, r)$ be the open ball of centre X and radius r. Also, let Φ: $\Omega \to \mathbb{R}$ be a C^2 function with nonvanishing gradient throughout Ω. We put $S = \{Y \in \Omega: \Phi(Y) = 0\}$.

THEOREM 1. *Let $\alpha \in (0, n-2)$ and E be a compact subset of S such that $m_\alpha(E) = 0$. If s is subharmonic in $\Omega \backslash E$ and satisfies*

$$s(X) \leq c[\operatorname{dist}(X, S)]^{\alpha+2-n} \qquad (X \in \Omega \backslash S) \tag{1}$$

for some positive constant c, then E is a removable singularity of s.

COROLLARY. *Let α and E be as above. If h is harmonic in $\Omega\backslash E$ and satisfies*

$$|h(X)| \leq c[\operatorname{dist}(X, S)]^{\alpha+2-n} \qquad (X \in \Omega\backslash S),$$

then E is a removable singularity of h.

Regarding the bound on α, Theorem 1 is true, but not interesting, when $\alpha = n-2$: for then we are requiring s to be bounded above near the set E, which is polar by [7, Theorem 5.14]. For higher values of α, the set E need not be polar [7, Theorem 5.13].

Theorem 1 is related to work by Dahlberg [4] on subharmonic functions in Lipschitz domains. In the case of a domain with a C^2 boundary, his theorem simplifies to the following boundary analogue of Theorem 1.

THEOREM A. *Let Ω be a bounded domain in $\mathbb{R}^n$ with C^2 boundary, let $\alpha \in (0, n-1)$, and let E be a closed subset of $\partial\Omega$ such that $m_\alpha(E) = 0$. If s is subharmonic in Ω, satisfies $\limsup s(X) \leq 0$ as $X \to Y \in \partial\Omega\backslash E$, and*

$$s(X) \leq c[\operatorname{dist}(X, \partial\Omega)]^{\alpha+1-n} \qquad (X \in \Omega),$$

then $s \leq 0$ in Ω.

The sharpness of Theorem 1 and its corollary is shown by the following result, which does not require E to be a subset of S. (Clearly (2) is stronger than (1) when $E \subset S$.)

THEOREM 2. *Let $\alpha \in (0, n-2)$ and E be a compact set such that $m_\alpha(E) > 0$. Then there is a positive harmonic function h on $\mathbb{R}^n\backslash E$ such that*

$$h(X) \leq [\operatorname{dist}(X, E)]^{\alpha+2-n} \qquad (X \in \mathbb{R}^n\backslash E), \tag{2}$$

but for which E is not a removable singularity.

It is natural to ask whether Theorem 1 remains true if we drop the requirement $E \subset S$ and replace (1) by (2). The following example shows that this is far from the case.

EXAMPLE. For each $k \in \mathbb{N}$ let $S_k = \partial B(O_n, [\log(k+1)]^{-1})$ and let E_k be a finite subset of S_k such that $B(X, 1/k) \cap E_k$ is non-empty for any $X \in S_k$. Then the compact set $E = [\bigcup_k E_k] \cup \{O_n\}$ has the property that $m_\alpha(E) = 0$ for every $\alpha > 0$. On the other hand, the function $s(X) = |X|^{2-n}$ is harmonic on $\mathbb{R}^n \backslash E$ and is easily seen to satisfy

$$s(X) \leq C(n, \alpha)[\operatorname{dist}(X, E)]^{\alpha+2-n} \qquad (X \in \mathbb{R}^n \backslash E)$$

for any $\alpha \in [0, n-2)$.

Among previous work on removable singularities of subharmonic functions we mention papers by Shapiro **[14]**, Kuran **[11]**, Kaufman and Wu **[9]**, and Armitage **[1]**. Our results are new in that, by introducing the restriction $E \subset S$, we are able to permit very bad behaviour of s near E. Thus, for example, the Lebesgue integrability requirement in **[14]** is not applicable in Theorem 1 if $\alpha \leq n-3$. We mention also Cima and Graham **[3]**, who showed that an analytic subvariety E in the unit ball of $\mathbb{C}^n$ is a removable singularity for holomorphic functions which satisfy appropriate growth conditions near E.

A slight modification of the proof of Theorem 1 yields the following.

THEOREM 3. *Let $\alpha \in (0, n-2)$ and E be a compact subset of S such that $m_\alpha(E) < +\infty$. If s is subharmonic in $\Omega \backslash E$ and satisfies*

$$s(X) \leq u(\operatorname{dist}(X, S)) \qquad (X \in \Omega \backslash S)$$

where $t^{n-\alpha-2}u(t) \to 0$ $(t \to 0+)$, then E is a removable singularity of s.

Theorem 3 can be regarded as a generalization of the following simple, well-known fact, which corresponds to the case $\alpha = 0$: if s is subharmonic in $\Omega \backslash \{Y\}$ and $\limsup |X - Y|^{n-2} s(X) \leq 0$ as $X \to Y$, then $\{Y\}$ is a removable singularity of s. The following shows that Theorem 3 is sharp.

THEOREM 4. *Let $\alpha \in (0, n-2)$ and E be a compact set which is not σ-finite with respect to m_α. Then there is a positive harmonic function h on $\mathbb{R}^n \backslash E$ such that*

$$\sup\{h(X) \colon \operatorname{dist}(X, E) = \rho\} = o(\rho^{\alpha+2-n}) \qquad (\rho \to 0+),$$

but for which E is not a removable singularity.

Using ideas from **[12]**, we can apply Theorem A to give removability results based on the behaviour of the means, $\mathscr{A}(s^+; X, r)$ and $\mathscr{M}(s^+; X, r)$, of s^+ over the ball $B(X, r)$ and the sphere $\partial B(X, r)$ respectively. (Given $X \in \Omega$, the function s^+ will be defined at least almost everywhere on $\partial B(X, r)$ for all small $r > 0$.) The following theorem, which is close to a result of Shapiro **[14]**, complements removable singularity results of Armitage **[1]** based on the behaviour of spherical means.

THEOREM 5. *Let* $\alpha \in (0, n-2)$ *and* E *be a compact subset of* Ω *such that* $m_\alpha(E) = 0$. *If* s *is subharmonic in* $\Omega \backslash E$ *and satisfies*

$$\mathscr{A}(s^+; X, r) \le cr^{\alpha+2-n} \qquad (\overline{B(X, r)} \subset \Omega), \tag{3}$$

then E *is a removable singularity of* s.

THEOREM 6. *Let* $\alpha \in (0, n-2)$ *and* E *be a compact subset of* $\mathbb{R}^n$ *such that* $m_\alpha(E) > 0$. *Then there is a positive harmonic function* h *on* $\mathbb{R}^n \backslash E$ *such that*

$$\mathscr{M}(h; X, r) \le r^{\alpha+2-n} \qquad (X \in \mathbb{R}^n),$$

but for which E *is not a removable singularity.*

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2. Proofs of Theorems 1 and 3.

2.1. Let $\varphi \in (0, \pi/2)$, let $X = (x, X') \in \mathbb{R} \times \mathbb{R}^{n-1}$, and define

$$D(Q, r) = \{X \in \overline{B(Q, 2r)}: |X' - Q'| \tan\varphi \le |x - q|\}$$

for $Q \in \mathbb{R}^n$ and $r > 0$.

LEMMA A. *On the set* $R = \{X: |x| < |X'| \tan\varphi\}$ *there is a positive harmonic function of the form*

$$|X|^{k(\varphi)} F_\varphi(\tan^{-1}(|x|/|X'|)), \tag{4}$$

where $k(\varphi) > 0$, $F_\varphi(\varphi) = 0$ *and* F'_φ *is continuous on* $(0, \pi/2)$. *Further,* $k(\varphi) \to \infty$ *as* $\varphi \to 0+$.

It is well known that there is a positive harmonic function h on R of the form $|X|^{k(\varphi)} F(X/|X|)$, where $k(\varphi) > 0$ and $F = 0$ on $\partial R \cap \partial B(O_n, 1)$. Further, F is unique up to a multiplicative constant. This uniqueness and the symmetry properties of R imply that h has the form (4), where $F_\varphi(\varphi) = 0$. Consideration of the second order ordinary differential equation satisfied by F_φ shows that $F_\varphi \in C^\infty(0, \pi/2)$. For the final assertion, see Friedland and Hayman **[6]**.

We now fix φ small enough to ensure that $k(\varphi) \geq n-1$. The set $R \cap B(O_n, 1)$ is a NTA domain in the sense of Jerison and Kenig [**8**], so the boundary Harnack principle [**8**, (5.1)] can be applied to show that there is a positive constant $C(n, \varphi)$ such that the Green kernel G for the set $\mathbb{R}^n \backslash D(O_n, 1)$ satisfies

$$G(X, Y) \leq C(n, \varphi)|X|^{k(\varphi)} F_\varphi(\tan^{-1}|x|/|X'|)$$

for

$$X \in B(O_n, 1) \backslash D(O_n, 1) \quad \text{and} \quad Y \in \mathbb{R}^n \backslash B(O_n, 3).$$

2.2. Let $L = \{X' \in \mathbb{R}^{n-1}\colon |X'| < 4\}$ and $f\colon L \to \mathbb{R}$ be a C^2 function such that $f(O_{n-1}) = 0$ and $|\nabla f(O_{n-1})| = 0$. We write

$$|\nabla_2 f| = \left\{ \sum_{j,k=1}^{n-1} \left[\frac{\partial^2 f}{\partial x_j \partial x_k} \right]^2 \right\}^{1/2}.$$

Further, let $T = \{(f(X'), X')\colon X' \in L\}$ and put

$$A = \bigcup_{\{Q \in T : |Q'| \leq 1\}} D(Q, 1).$$

A simple sequence argument shows that A is a closed set.

If $J \subset \mathbb{R}^n$ is compact, we use $H[J, F]$ to denote the Perron–Wiener–Brelot solution to the generalized Dirichlet problem for the unbounded component of $\mathbb{R}^n \backslash J$, with data F on the finite boundary and 0 at infinity.

LEMMA 1. *Let $\beta \in (0, n-2)$ and define g on ∂A by*

$$g(X) = \begin{cases} [\operatorname{dist}(X, T)]^{-\beta} & (X \notin T), \\ 0 & (X \in T). \end{cases}$$

There are positive constants η, K such that, if f is as above and satisfies $|\nabla_2 f| < \eta$ on L, then g is integrable with respect to harmonic measure for $\mathbb{R}^n \backslash A$ and

$$H[A, g](Y) \leq K|Y|^{2-n} \qquad (|Y| > 5).$$

To prove this, let η be sufficiently small to ensure that $|f(Q')| \leq \sqrt{3}$ whenever $|Q'| \leq 1$. Thus $|X| \leq 4$ for $X \in A$. Let G_* be the Green kernel for the set $\mathbb{R}^n \backslash A$, and let G_Q be the Green kernel for the set $\mathbb{R}^n \backslash D(Q, 1)$, where Q is some point in T satisfying $|Q'| = 1$. Then $G_* \leq G_Q$ in $\mathbb{R}^n \backslash A$, and so from §2.1,

$$G_*(X, Y) \leq C(n, \varphi)|X - Q|^{k(\varphi)} F_\varphi(\tan^{-1}|x - q||X' - Q'|)$$

for

$$X \in B(Q, 1)\backslash A \quad \text{and} \quad Y \in \mathbb{R}^n\backslash B(O_n, 5).$$

Dividing by $\operatorname{dist}(X, A)$ and taking limits along the normal common to ∂A and $\partial D(Q, 1)$, we see that (for small η) the normal derivative $\partial G_*/\partial n_X$ at X satisfies

$$\frac{\partial G_*}{\partial n_X}(X, Y) \le C(n, \varphi, \eta)[\operatorname{dist}(X, T)]^{k(\varphi)-1},$$

where

$$X \in \{Z \in \partial A\colon\ 0 < \operatorname{dist}(X, T) < (\sin\varphi)/2\} \quad \text{and} \quad Y \in \mathbb{R}^n\backslash B(O_n, 5).$$

Since $k(\varphi) - 1 > n - 2 > \beta$, it now follows that g is integrable with respect to harmonic measure for $\mathbb{R}^n\backslash A$. Further, the surface area of $\{X \in \partial A\colon\ \operatorname{dist}(X, T) < (\sin\varphi)/2\}$ is bounded above by a constant depending on n, φ, η, but not f, so we can write

$$H[A, g](Y) \le C(n, \varphi, \eta) \qquad (|Y| = 5),$$

and hence

$$H[A, g](Y) \le C(n, \varphi, \eta)(5/|Y|)^{n-2} \qquad (|Y| \ge 5).$$

(Note: Dahlberg **[5]** has shown that, for bounded Lipschitz domains, harmonic measure is absolutely continuous with respect to surface area measure, and that the density function is given by the normal derivative of the Green function. In the above argument we have used this fact and the observation that the image of $\mathbb{R}^n\backslash A$ under inversion in $\partial B(O_n, 1)$ is a bounded Lipschitz domain, punctured at the origin.)

2.3. We now complete the proof of Theorem 1. From the implicit function theorem there exists $\gamma > 0$ such that, if $Z \in E$, then (choosing a suitable coordinate system $(\tilde{x}_1, \dots, \tilde{x}_n)$ centred at Z) there is a C^2 function f_Z such that $f_Z(\widetilde{O}_{n-1}) = 0$, $|\nabla f_Z(\widetilde{O}_{n-1})| = 0$ and

$$\{X \in S\colon |\tilde{x}| < \gamma \text{ and } |\widetilde{X}'| < \gamma\} = \{X\colon\ |\widetilde{X}'| < \gamma \text{ and } \tilde{x} = f_Z(\widetilde{X}')\}.$$

Let $\varepsilon > 0$ and $0 < \delta < \gamma/4$. Since $m_\alpha(E) = 0$, there exists a finite collection of open balls B_i of radii $r_i/2 < \delta/2$ such that

$$E \subset \bigcup_i B_i \quad \text{and} \quad \sum_i r_i^\alpha < \varepsilon. \tag{5}$$

For each i, choose $Z_i \in B_i \cap E$ (any B_i for which $B_i \cap E$ is empty is discarded) and, using the above coordinate system centred at Z_i, put

$$A_i = \bigcup_{\{Q \in S\colon\ |\widetilde{Q}'| \le r_i\}} \widetilde{D}(Q, r_i).$$

Clearly $E \subset \bigcup_i A_i$ if δ is small. We define g_i on ∂A_i by

$$g_i(X) = \begin{cases} [\operatorname{dist}(X, S)]^{\alpha+2-n} & (X \notin S), \\ 0 & (X \in S). \end{cases}$$

Applying a dilation of centre Z_i and magnification factor r_i, it follows from Lemma 1 that

$$(6) \qquad H[A_i, g_i](Y) \le K r_i^\alpha |Y - Z_i|^{2-n} \qquad (Y \in \mathbb{R}^n \backslash B(Z_i, 5r_i)).$$

(Note that, provided $\gamma > 0$ is chosen small enough, the hypothesis $|\nabla_2 f| < \eta$ is satisfied.)

Now let V be a bounded open set such that $E \subset V \subset \overline{V} \subset \Omega$ and let a be an upper bound for s on ∂V. If $X \in V \backslash \bigcup_i B(Z_i, 5\delta)$, then by (1), (5) and (6),

$$\begin{aligned}(s-a)^+(X) &\le c \sum_i H[A_i, g_i](X) \\ &\le cK \sum_i r_i^\alpha |X - Z_i|^{2-n} \\ &\le cK(5\delta)^{2-n}\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ can be arbitrarily small, we have $s(X) \le a$ for $X \in V$ satisfying $\operatorname{dist}(X, E) \ge 5\delta$. Further, since $\delta > 0$ can be arbitrarily small, we have $s \le a$ in $V \backslash E$. Thus s is bounded above near the polar set E, and so E is a removable singularity of s.

2.4. The proof of Theorem 3 requires only self-evident modification to (5) and the last paragraph of §2.3.

3. Proofs of Theorems 2 and 4.

3.1. Theorem 2 relies on a lemma due to Frostman [7, Lemma 5.4]. This says that, if $m_\alpha(E) > 0$, then there is a finite, positive measure μ on E such that $\mu(B(X, r)) \le r^\alpha$ for any ball $B(X, r)$. Clearly we can assume that $\mu(E) \le 1$. The Newtonian potential v, due to μ, is harmonic on $\mathbb{R}^n \backslash E$ but not on $\mathbb{R}^n$.

Now let $X \in \mathbb{R}^n \backslash E$ and $\rho = \operatorname{dist}(X, E)$. If $\rho < 1$ and $Y \in B(X, \rho)$, then integration by parts yields

$$\begin{aligned} v(Y) &= \int_E |Y - Z|^{2-n}\, d\mu(Z) \\ &\le (n-2) \int_{\rho - |Y-X|}^\infty \mu(B(Y, t)) t^{1-n}\, dt \\ &\le (n-2) \int_{\rho - |Y-X|}^\infty \min\{t^\alpha, 1\} t^{1-n}\, dt \\ &\le (n-2)(n-2-\alpha)^{-1}(\rho - |Y - X|)^{\alpha+2-n}. \end{aligned}$$

Putting $p = [2(n-2-\alpha)]^{-1}$, we now have

$$\left\{\rho^{-n}\int_{B(X,\rho)}[v(Y)]^p\,dY\right\}^{1/p}$$
$$\leq C(n,\alpha)\left\{\rho^{-n}\int_0^\rho t^{n-1}[\rho-t]^{(\alpha+2-n)p}\,dt\right\}^{1/p}$$
$$\leq C(n,\alpha)\{2\rho^{-1/2}\}^{1/p}$$
$$\leq C(n,\alpha)\rho^{\alpha+2-n}.$$

Applying an inequality originally due to Hardy and Littlewood in the case $n = 2$, and extended by Kuran to higher dimensions [**10**, Theorem 1], it follows that

$$v(X) \leq C(n,\alpha)\rho^{\alpha+2-n},$$

and so, letting $h = v/C(n,\alpha)$, we obtain (2).

3.2. To prove Theorem 4, we note (see [**13**, pp. 83, 84]) that, if E is not σ-finite with respect to m_α, then there is a positive, nondecreasing, continuous function w on $[0,+\infty)$ such that $t^{-\alpha}w(t) \to 0$ $(t \to 0+)$ and E is not σ-finite with respect to m_w. (Here m_w refers to the Hausdorff measure generated by w.) As in §3.1 there exists a finite, positive measure μ on E such that $\mu(B(X,r)) \leq w(r)$, for any ball $B(X,r)$. We write h for the Newtonian potential due to μ and assume that $\mu(E) \leq 1$.

Now let $\varepsilon > 0$ and choose $\delta > 0$ such that $t^{-\alpha}w(t) \leq \varepsilon$ for $t \in (0,\delta)$. Also, let $X \in \mathbb{R}^n\backslash E$ and $\rho = \operatorname{dist}(X,E)$. If $\rho < \delta$ and $Y \in B(X,\rho)$, the reasoning of §3.1 yields

$$h(Y) \leq \delta^{2-n} + \varepsilon(n-2)(n-2-\alpha)^{-1}(\rho-|Y-X|)^{\alpha+2-n}$$

and

$$h(X) \leq C(n,\alpha)(\delta^{2-n} + \varepsilon\rho^{\alpha+2-n}),$$

whence

$$\limsup_{\rho\to 0+}(\rho^{n-2-\alpha}\sup\{h(X)\colon \operatorname{dist}(X,E)=\rho\}) \leq \varepsilon C(n,\alpha).$$

Since $\varepsilon > 0$ can be arbitrarily small, the result follows.

4. Proofs of Theorems 5 and 6.

4.1. To prove Theorem 5, let V be a bounded open set such that $E \subset V \subset \overline{V} \subset \Omega$ and let a be an upper bound for s on ∂V. Let $v = (s-a)^+$ on V and let $v = 0$ on $\mathbb{R}^n\backslash V$. Clearly v is subharmonic on $\mathbb{R}^n \backslash E$. Now let I_v denote the Poisson integral of v in $\mathbb{R}^n \times (0, +\infty)$. Using integration by parts and then (3), it follows that

$$\begin{aligned} I_v(X, y) &= C(n)y \int_0^\infty t^{n+1}(y^2+t^2)^{-(n+3)/2} \mathscr{A}(v; X, t)\, dt \\ &\le C(n, \alpha, c) y^{\alpha+2-n} \qquad (X \in \mathbb{R}^n; y > 0). \end{aligned}$$

Applying Harnack's inequalities [**2**, p. 200] twice in the ball of centre (X, y) and radius $y/2$, we have

$$I_{(-\Delta v)}(X, y) = \partial^2 I_v(X, y)/\partial y^2 \le C(n, \alpha, c) y^{\alpha-n}.$$

But the distributional Laplacian Δv is non-negative on $\mathbb{R}^n \backslash E$ by the subharmonicity of v. Applying a half-space version of Theorem A (obtained by inversion from the corresponding result for the ball), it follows that $I_{(-\Delta v)} \le 0$ on $\mathbb{R}^n \times (0, +\infty)$. Hence $\Delta v \ge 0$ on $\mathbb{R}^n$ and so v has a subharmonic extension to $\mathbb{R}^n$. Since s is now bounded above on $V \backslash E$, it follows that the polar set E is a removable singularity of s.

4.2. Let μ and v be as in the proof of Theorem 2. Then (cf. [**7**, (3.9.6)])

$$\begin{aligned} \mathscr{M}(v; X, r) &= (n-2) \int_r^\infty t^{1-n} \mu(B(X, t))\, dt \\ &\le (n-2) \int_r^\infty t^{\alpha+1-n}\, dt = C(n) r^{\alpha+2-n}, \end{aligned}$$

and Theorem 6 follows.

References

[1] D. H. Armitage, *Mean values and associated measures of superharmonic functions*, Hiroshima Math. J., **13** (1983), 53–63.

[2] M. Brelot, *Élements de la théorie classique du potentiel*, Centre de documentation universitaire, Paris, 1965.

[3] J. A. Cima and I. R. Graham, *On the extension of holomorphic functions with growth conditions across analytic subvarieties*, Michigan Math. J., **28** (1981), 241–256.

[4] B. E. J. Dahlberg, *On exceptional sets at the boundary for subharmonic functions*, Ark. Mat., **15** (1977), 305–312.

[5] ——, *Estimates of harmonic measure*, Arch. Rational Mech. Anal., **65** (1977), 275–288.

[6] S. Friedland and W. K. Hayman, *Eigenvalue inequalities for the Dirichlet problem on spheres and the growth of subharmonic functions*, Comment. Math. Helvetici, **51** (1976), 133–161.

[7] W. K. Hayman and P. B. Kennedy, *Subharmonic functions*, Volume 1, Academic Press, London, 1976.
[8] D. S. Jerison and C. E. Kenig, *Boundary behaviour of harmonic functions in non-tangentially accessible domains*, Advances in Math., **46** (1982), 80–147.
[9] R. Kaufman and J.-M. Wu, *Removable singularities for analytic or subharmonic functions*, Ark. Mat., **18** (1980), 107–16. Correction, ibid., **21** (1983), 1.
[10] Ü. Kuran, *Subharmonic behaviour of* $|h|^p$ $(p > 0, h$ *harmonic*), J. London Math. Soc. (2), **8** (1974), 529–538.
[11] ——, *Some extension theorems for harmonic, superharmonic and holomorphic functions*, J. London Math. Soc. (2), **22** (1980), 269–284.
[12] A. G. O'Farrell, *The 1-reduction for removable singularities, and the negative Hölder spaces*, Proc. Roy. Irish Acad., **88A** (1988), 133–151.
[13] C. A. Rogers, *Hausdorff Measures*, Cambridge University Press, London, 1970.
[14] V. L. Shapiro, *Subharmonic functions and Hausdorff measure*, J. Differential Equations, **27** (1978), 28–45.

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