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QUASI-ROTATION C^*-ALGEBRAS

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The main result in this paper is to classify the isomorphism classes of certain non-commutative 3-tori obtained by taking the crossed product C^*-algebra of continuous functions on the 2-torus $\mathbf{T}^2$ by the irrational affine quasi-rotations. Each such quasi-rotation is represented by a pair (a, A), where $a \in \mathbf{T}^2$ and $A \in \mathrm{GL}(2, \mathbf{Z})$, and its associated C^*-algebra is shown to be determined (up to isomorphism) by an analogue of the rotation angle, namely its primitive eigenvalue, by its orientation $\det(A) = \pm 1$ and a certain positive integer $m(A)$ which comes from the K_1-group of the algebra and which determines the conjugacy class of A in $\mathrm{GL}(2, \mathbf{Z})$.

Introduction. In this paper we study the C^*-crossed products of the continuous functions on the 2-torus $C(\mathbf{T}^2)$ by certain transformations φ of $\mathbf{T}^2$ which we call quasi-rotations. They are like rotations in that they have an eigenvalue $\lambda = e^{2\pi i\theta}$ and a unitary eigenfunction $f \in C(\mathbf{T}^2)$, and unlike rotations in that their degree matrix $D(\varphi) \in \mathrm{GL}(2, \mathbf{Z})$ does not equal the identity matrix I_2. Clearly they contain the rotation C^*-algebra $\mathscr{A}_\theta$.

Recall that an affine transformation of a group G is a mapping $\sigma: G \to G$ of the form $\sigma(z) = aA(z)$, (for $z \in G$), where $a \in G$ and $A \in \mathrm{Aut}(G)$.

Let $\mathscr{A}(\varphi)$ denote the associated crossed product C^*-algebra $C(\mathbf{T}^2) \times_{\alpha_\varphi} \mathbf{Z}$, (cf. [**9**, 7.6]) where α_φ is the automorphism on $C(\mathbf{T}^2)$ associated with φ. We shall construct an integer-valued function m defined on the 2×2 matrices $A \in \mathrm{GL}(2, \mathbf{Z})$ which are of the form $D(\varphi)$, for some quasi-rotation φ, such that

(i) $\mathbf{Z}_{m(D(\varphi))}$ is the torsion subgroup of $K_1(\mathscr{A}(\varphi))$,

(ii) $m(A)$ and $\det(A)$ determine the conjugacy class of A in $\mathrm{GL}(2, \mathbf{Z})$.

When this is combined with the computation of the tracial range on $K_0(\mathscr{A}(\varphi))$ (see §4) a classification of the isomorphism classes of these algebras is obtained (Theorem 5.2) for the affine quasi-rotations of $\mathbf{T}^2$ associated with irrational θ. This is the main result, while for the rational case a partial answer is given. The determination of the strong Morita equivalence classes of these algebras has been studied in [**17**].

The K-groups of the crossed products of $C(\mathbf{T}^2)$ by any transformation have been computed elsewhere ([**6**]; and independently in [**15**]) using the Pimsner-Voiculescu six-term exact sequence. Here we shall merely state the results (§1).

Some results concerning the non-affine quasi-rotation algebras are given in [**16**].

1. K-groups. Every continuous function $f\colon \mathbf{T}^2 \to \mathbf{T}$ has the form $f(x, y) = x^m y^n e^{2\pi i F(x,y)}$ for some integers m, n and some continuous real-valued function F on $\mathbf{T}^2$. Call the 1×2 integral matrix $[m\ n]$ the bidegree of f and denote it by $D(f)$. Let φ be a transformation (i.e., a homeomorphism) of the 2-torus $\mathbf{T}^2$. Write φ as $\varphi = (\varphi_1, \varphi_2)$. Define the degree matrix of φ to be the 2×2 integral matrix

$$D(\varphi) = \begin{pmatrix} D(\varphi_1) \\ D(\varphi_2) \end{pmatrix}.$$

It is easy to verify that $D(\varphi \circ \psi) = D(\varphi)D(\psi)$ for any two transformations φ, ψ of $\mathbf{T}^2$. Replacing ψ by φ^{-1} we see that $D(\varphi) \in \mathrm{GL}(2, \mathbf{Z})$, i.e. $\det D(\varphi) = \pm 1$. This latter determinant determines whether φ is orientation preserving or reversing. Let I_2 denote the identity matrix in $\mathrm{GL}(2, \mathbf{Z})$.

THEOREM 1.1 ([**6**], *Chapter* 3; [**15**], *Chapter* 2). *Let φ be a transformation of $\mathbf{T}^2$.*

(1) *If* $\det D(\varphi) = 1$, *then*

$$K_0(\mathscr{A}(\varphi)) \cong \begin{cases} \mathbf{Z}^4 & \text{if } D(\varphi) = I_2, \\ \mathbf{Z}^3 & \text{if } \det(D(\varphi) - I_2) = 0 \text{ and } D(\varphi) \neq I_2, \\ \mathbf{Z}^2 & \text{if } \det(D(\varphi) - I_2) \neq 0. \end{cases}$$

(2) *If* $\det D(\varphi) = -1$, *then*

$$K_0(\mathscr{A}(\varphi)) \cong \begin{cases} \mathbf{Z}^2 \oplus \mathbf{Z}_2 & \text{if } \det(D(\varphi) - I_2) = 0, \\ \mathbf{Z} \oplus \mathbf{Z}_2 & \text{if } \det(D(\varphi) - I_2) \neq 0. \end{cases}$$

(3) *Write* $D(\varphi)^{-1} = \left(\begin{smallmatrix} m & n \\ p & q \end{smallmatrix}\right)$ *and let J denote the quotient group*

$$J = \frac{\mathbf{Z} \oplus \mathbf{Z}}{(m-1, n)\mathbf{Z} + (p, q-1)\mathbf{Z}} = \frac{\mathbf{Z}^2}{\operatorname{Im}(D(\varphi^{-1})^T - I_2)}.$$

Then

$$K_1(\mathscr{A}(\varphi)) \cong \begin{cases} \mathbf{Z}^2 \oplus J & \text{if } \det D(\varphi) = 1, \\ \mathbf{Z} \oplus J & \text{if } \det D(\varphi) = -1. \end{cases}$$

The proof of this theorem relies on the Pimsner-Voiculescu cyclic six-term exact sequence for K-theory **[11]**. A closer look at the proof yields the following corollary.

COROLLARY 1.2. *Let φ be a transformation of $\mathbf{T}^2$ such that*

$$\det(D(\varphi) - I_2) = 0,$$

and let P denote the Bott projection in $M_2(C(\mathbf{T}^2))$. In this case there is an x such that $\delta(x)$ is a generator of $\ker(\alpha_{\varphi_*} - \mathrm{id}_*)$ in $K_1(C(\mathbf{T}^2))$, where δ is the connecting homomorphism in the Pimsner-Voiculescu sequence $\delta\colon K_0(\mathscr{A}(\varphi)) \to K_1(C(\mathbf{T}^2))$.

(i) *If $\det D(\varphi) = 1$ and $D(\varphi) \neq I_2$, then $K_0(\mathscr{A}(\varphi)) \cong \mathbf{Z}^3$ is generated by $[1]$, $[P]-[1]$, and x.*

(ii) *If $\det D(\varphi) = -1$, then $K_0(\mathscr{A}(\varphi)) \cong \mathbf{Z}^2 \oplus \mathbf{Z}_2$ is generated by $[1]$, $[P]-[1]$ (which has order 2 in this case) and x.*

This corollary focuses only on transformations such that $\det(D(\varphi) - I_2) = 0$ because these include the quasi-rotations.

2. Lemmas. In this section we shall construct the integer-valued function m indicated in the introduction which classifies the conjugacy class of certain integral matrices in $\mathrm{GL}(2, \mathbf{Z})$ which arise as $D(\varphi)$ where φ is a quasi-rotation. As it turns out these are the matrices A which have eigenvalue 1, i.e. $\det(A - I_2) = 0$ (cf. §3).

Two matrices $A, B \in \mathrm{GL}(2, \mathbf{Z})$ are conjugate if there exists $S \in \mathrm{GL}(2, \mathbf{Z})$ such that $SAS^{-1} = B$. Let us express this by $A \sim B$. It will be shown later that for quasi-rotations φ and ψ of $\mathbf{T}^2$, if $\mathscr{A}(\varphi) \cong \mathscr{A}(\psi)$, then $D(\varphi) \sim D(\psi)$ (cf. Proposition 2.8). If, in addition, φ and ψ are affine, it will follow that they are topologically conjugate (i.e., there exists a transformation h of $\mathbf{T}^2$ such that $h \circ \psi = \varphi \circ h$).

The construction of m is divided up into two cases.

LEMMA 2.1. *Let $A \in \mathrm{GL}(2, \mathbf{Z})$ be such that $\det(A - I_2) = 0$ and $\det A = 1$, say*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let $e = \gcd(a - 1, b)$, when $b \neq 0$, and define

$$m(A) = \begin{cases} \dfrac{e^2}{|b|} & \text{if } b \neq 0, \\ |c| & \text{if } b = 0. \end{cases}$$

Then

$$A \sim \begin{pmatrix} 1 & 0 \\ m(A) & 1 \end{pmatrix}.$$

Hence, $A \sim B$ *if and only if* $m(A) = m(B)$, *for all matrices* A, B *satisfying the hypotheses of this lemma.*

Proof. From $(a-1)(d-1) - bc = 0$ and $ad - bc = 1$ one obtains $a + d = 2$ and $-(a-1)^2 = bc$. If $b = 0$, the lemma is clear. Suppose that $b \neq 0$. Since $e = \gcd(a-1, b)$, there exist integers s, t such that

$$\left(\frac{a-1}{e}\right)t - \left(\frac{b}{e}\right)s = 1,$$

so that

$$S = \begin{pmatrix} \frac{a-1}{e} & \frac{b}{e} \\ s & t \end{pmatrix} \in \mathrm{GL}(2, \mathbf{Z}).$$

One then checks that

$$SA = \begin{pmatrix} 1 & 0 \\ \frac{-e^2}{b} & 1 \end{pmatrix} S:$$

$$\begin{aligned} SA &= \begin{pmatrix} \frac{a-1}{e}\frac{b}{e} & \\ s & t \end{pmatrix} \begin{pmatrix} a & b \\ c & 2-a \end{pmatrix} \\ &= \begin{pmatrix} \left(\frac{a-1}{e}\right)a + \frac{bc}{e} & \left(\frac{a-1}{e}\right)b + \frac{b(2-a)}{e} \\ sa + tc & sb + t(2-a) \end{pmatrix} \end{aligned}$$

and

$$\begin{pmatrix} 1 & 0 \\ \frac{-e^2}{b} & 1 \end{pmatrix} \begin{pmatrix} \frac{a-1}{e} & \frac{b}{e} \\ s & t \end{pmatrix} = \begin{pmatrix} \frac{a-1}{e} & \frac{b}{e} \\ \frac{-e^2}{b}\left(\frac{a-1}{e}\right) + s & t - e \end{pmatrix}.$$

These can be seen to be equal using the relations $-(a-1)^2 = bc$ and $(a-1)t - bs = e$. Thus

$$SAS^{-1} = \begin{pmatrix} 1 & 0 \\ \pm m(A) & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ m(A) & 1 \end{pmatrix}. \qquad \sqcup$$

Henceforth we shall write $m(\varphi) = m(D(\varphi))$.

COROLLARY 2.2. *Let φ be a transformation of $\mathbf{T}^2$ with* $\det D(\varphi) = 1$ *and* $\det(D(\varphi) - I_2) = 0$. *Then φ is topologically conjugate to a transformation ψ with*

$$D(\psi) = \begin{pmatrix} 1 & 0 \\ m(\varphi) & 1 \end{pmatrix}.$$

Proof. Since $D(\varphi)$ satisfies the hypotheses of the previous lemma, we have

$$SD(\varphi)S^{-1} = \begin{pmatrix} 1 & 0 \\ m(\varphi) & 1 \end{pmatrix}$$

for some $S \in \mathrm{GL}(2, \mathbf{Z})$. We can choose an automorphism σ of $\mathbf{T}^2$ (as a group) with $D(\sigma) = S$. For example, if

$$S = \begin{pmatrix} m & n \\ p & q \end{pmatrix},$$

let $\sigma(x, y) = (x^m y^n, x^p y^q)$. Letting $\psi = \sigma \circ \varphi \circ \sigma^{-1}$, we obtain

$$D(\psi) = D(\sigma)D(\varphi)D(\sigma)^{-1} = \begin{pmatrix} 1 & 0 \\ m(\varphi) & 1 \end{pmatrix}. \qquad \square$$

COROLLARY 2.3. *Let φ be a transformation of $\mathbf{T}^2$ with* $\det D(\varphi) = 1$ *and* $\det(D(\varphi) - I_2) = 0$. *Then*

$$K_1(\mathscr{A}(\varphi)) \cong \mathbf{Z}^3 \oplus \mathbf{Z}_{m(\varphi)}.$$

Proof. Since by the preceding corollary ψ is topologically conjugate to φ, we can use Theorem 1.1 to obtain

$$\begin{aligned} K_1(\mathscr{A}(\varphi)) \cong K_1(\mathscr{A}(\psi)) &\cong \mathbf{Z}^2 \oplus \left(\frac{\mathbf{Z}^2}{(0, 0)\mathbf{Z} + (m(\varphi), 0)\mathbf{Z}} \right) \\ &\cong \mathbf{Z}^3 \oplus \mathbf{Z}_{m(\varphi)}. \end{aligned} \qquad \square$$

Consequently, if φ and ψ are transformations of $\mathbf{T}^2$ satisfying the hypotheses of the above corollary and if $\mathscr{A}(\varphi)$ and $\mathscr{A}(\psi)$ are isomorphic, strongly Morita equivalent, or, more generally, have isomorphic K_1-groups, then $m(\varphi) = m(\psi)$ so that $D(\varphi) \sim D(\psi)$.

LEMMA 2.4. *Let $A \in \mathrm{GL}(2, \mathbf{Z})$ be such that* $\det A = -1$ *and* $\det(A - I_2) = 0$, *so that A has the form*

$$A = \begin{pmatrix} k & x \\ y & -k \end{pmatrix},$$

where $k^2+xy=1$. *Let* $e=\gcd(k-1, x)$, *when* $x\neq 0$, *and consider the integer-valued function*

$$m(A)=\begin{cases}\gcd\left(e, \dfrac{e(k+1)}{x}\right) & \text{if } x\neq 0,\\ \gcd(2, y) & \text{if } x=0.\end{cases}$$

Then $m(A)\in\{1, 2\}$, *and*

(i) $m(A)=1\Leftrightarrow A\sim\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$,

(ii) $m(A)=2\Leftrightarrow A\sim\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$.

Consequently, for such matrices A *and* B *one has* $A\sim B\Leftrightarrow m(A)=m(B)$. (*Hence there are two conjugacy classes in this case.*)

Proof. Since $(k-1)/e$ and x/e are relatively prime integers and $xy=(1-k)(1+k)$ or $(x/e)y=((1-k)/e)(1+k)$, it follows that x/e divides $k+1$; hence $e(k+1)/x$ is an integer (when $x\neq 0$), so that $m(A)$ makes sense.

To see that $m(A)\in\{1, 2\}$, note that

$$m(A)|e|(k-1) \quad\text{and}\quad m(A)|(e(k+1)/x)|(k+1).$$

Hence $m(A)|(k+1)-(k-1)$ or $m(A)|2$, as desired.

Now assume that $m(A)=1$ and suppose that $k\neq\pm 1$, so that $x\neq 0$. We shall seek an integral matrix

$$S=\begin{pmatrix}a&b\\c&d\end{pmatrix}$$

such that

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}k&x\\y&-k\end{pmatrix}=\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix},$$

and $ad-bc=1$. This implies that

$$ka+yb=c, \quad xa-kb=d, \quad kc+yd=a, \quad xc-kd=b,$$

and one easily checks that the last two of these equations follow from the first two. Substituting the first two equations into $ad-bc=1$ we get $a(xa-kb)-b(ka+yb)=1$, or $xa^2-2kab-yb^2=1$, which may be factored as

$$\left[\frac{x}{e}a-\frac{k-1}{e}b\right]\left[ea+\frac{ey}{k-1}b\right]=1,$$

where $ey/(k-1)=-e(k+1)/x$ is an integer (since $k\neq 1$). Therefore, the existence of S is guaranteed provided the equations

$$\frac{x}{e}a-\frac{k-1}{e}b=1, \quad ea+\frac{ey}{k-1}b=1,$$

have integer solutions a, b.

Multiplying the first of these equations by e and the second by x/e and subtracting the two gives $2b = e - (x/e)$. Similarly, if we multiply these equations by $k+1$ and $k-1$, respectively, we obtain $2a = (e(k+1)/x) - ((k-1)/e)$. To show that b exists we must show that e and x/e have the same parity, i.e., either both are even or odd. This may be shown as follows.

Assume that x/e is odd and e is even. Then x is even and $k-1$ is even (since $2|e|(k-1))$. So $k+1$ is even. But then $2|(e(k+1)/x)$ since x/e is odd, and hence, $2|m(A) = 1$, a contradiction. A similar contradiction argument follows if x/e is even and e is odd.

To show that a exists one shows that $e(k+1)/x$ and $(k-1)/e$ have the same parity. If $(k-1)/e$ is even, then x/e is odd. Since $k-1$ is even, $k+1$ is even and so $e(k+1)/x$ is even since x/e is odd. Conversely, if $e(k+1)/x$ is even then since $1 = m(A)$, e must be odd. Now as $k+1$ is even, so is $k-1$, and so $(k-1)/e$ is even since e is odd.

Now we assume that $m(A) = 2$ and $k \neq \pm 1$, so that $x \neq 0$. Then e and $e(k+1)/x$ are even so that the matrix

$$S = \begin{pmatrix} \dfrac{e(k+1)}{2x} & \dfrac{e}{2} \\ \dfrac{k-1}{e} & \dfrac{x}{e} \end{pmatrix}$$

has integer entries and determinant 1. Using the relation $xy = (1-k)(1+k)$ one can easily check that

$$SA = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} S.$$

Now the cases when $k = \pm 1$ are easily handled by similar arguments as above. □

The matrices satisfying the hypotheses of Lemma 2.4 are the "orientation reversing" square roots of the identity matrix. Using this lemma we can show that there is a quick way to find the conjugacy class of A when its entries have known parity.

COROLLARY 2.5. *Let A satisfy the hypotheses of Lemma* 2.4.

(1) k *even* $\Rightarrow m(A) = 1$.

(2) *Suppose* k *is odd. Then*

  (i) x *or* y *is odd* $\Rightarrow m(A) = 1$,

  (ii) x *and* y *are even* $\Rightarrow m(A) = 2$.

Proof. If $m(A) \neq 1$, then $m(A) = 2$ so that $2|e|(k-1)$ and hence k is odd. This proves (1). We now prove (2).

(i) Without loss of generality suppose x is odd. Since $m(A)|e|x$, it follows that $m(A) = 1$.

(ii) Suppose that x and y are even. Since $k-1$ is even, e is even. We assert that $e(k+1)/x$ is even, so that $m(A) = 2$. To see this, write $y = (e(k+1)/x)((1-k)/e)$ where we may assume $x \neq 0$ (if $x = 0$ then $k = \pm 1$ so $m(A) = 2$). If x/e is even, then $(1-k)/e$ is odd (being relatively prime), so y is even implies that $e(k+1)/x$ is even. Now if x/e is odd, then $k+1$ being even it follows that $e(k+1)/x$ is even, and hence $m(A) = 2$. □

Setting $m(I_2) = 0$, we may now summarize the contents of Lemmas 2.1 and 2.4 as follows:

COROLLARY 2.6. *Let* $A, B \in \mathrm{GL}(2, \mathbf{Z})$ *be such that* $\det(A - I_2) = \det(B - I_2) = 0$. *Then* $A \sim B$ *if and only if* $\det A = \det B$ *and* $m(A) = m(B)$.

COROLLARY 2.7. *Let* φ *be a transformation of* $\mathbf{T}^2$ *such that* $\det D(\varphi) = -1$ *and* $\det(D(\varphi) - I_2) = 0$. *Then*

$$K_1(\mathscr{A}(\varphi)) \cong \mathbf{Z}^2 \oplus \mathbf{Z}_{m(\varphi)}.$$

Proof. Arguing as in the proof of Corollary 2.2 φ is topologically conjugate to a transformation ψ of $\mathbf{T}^2$ such that

$$D(\psi) = \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } m(\varphi) = 1, \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \text{if } m(\varphi) = 2. \end{cases}$$

On applying Theorem 1.1(3) to ψ we obtain

$$\begin{aligned} K_1(\mathscr{A}(\varphi)) &\cong K_1(\mathscr{A}(\psi)) \\ &\cong \begin{cases} \mathbf{Z} \oplus \left(\dfrac{\mathbf{Z}^2}{(-1, 1)\mathbf{Z} + (1, -1)\mathbf{Z}} \right) & \text{if } m(\varphi) = 1, \\ \mathbf{Z} \oplus \left(\dfrac{\mathbf{Z}^2}{(0, 0)\mathbf{Z} + (0, -2)\mathbf{Z}} \right) & \text{if } m(\varphi) = 2, \end{cases} \\ &\cong \mathbf{Z}^2 \oplus \mathbf{Z}_{m(\varphi)}. \end{aligned}$$

Combining the results of this section together with those of the previous one we arrive at the following result.

Proposition 2.8. *Let φ_1 and φ_2 be transformations of $\mathbf{T}^2$ such that* $\det(D(\varphi_i) - I_2) = 0$, $i = 1, 2$. *If $\mathscr{A}(\varphi_1)$ and $\mathscr{A}(\varphi_2)$ have isomorphic K_i-groups ($i = 0, 1$), then* $\det D(\varphi_1) = \det D(\varphi_2)$ *and* $m(\varphi_1) = m(\varphi_2)$, *so that* $D(\varphi_1) \sim D(\varphi_2)$.

Proof. Since they have isomorphic K_0-groups, Theorem 1.1 implies that $\det D(\varphi_1) = \det D(\varphi_2)$. Since they have isomorphic K_1-groups, we may combine Corollaries 2.3 and 2.7 to get $m(\varphi_1) = m(\varphi_2)$. By Corollary 2.6 we deduce that $D(\varphi_1) \sim D(\varphi_2)$. □

Remark. The quantity $\det(D(\varphi) - I_2)$ turns out to be the so-called Lefschetz number of φ, which is defined in algebraic topology as the alternating sum of the traces of the induced maps of φ on the cohomology groups of the underlying space (in our case $\mathbf{T}^2$). The Lefschetz fixed point theorem states that if φ is a diffeomorphism on a smooth manifold which has no fixed points, then its Lefschetz number is zero. In our case, for the 2-torus, the Lefschetz number is

$$\det(D(\varphi) - I_2) = 1 - \operatorname{trace}(D(\varphi)) + \det(D(\varphi)).$$

(see Bott and Tu [**1**, Theorem 11.25].)

3. Quasi-rotations.

Definition. A transformation φ of $\mathbf{T}^2$ is said to be a quasi-rotation if $D(\varphi) \neq I_2$ and if φ has a "non-singular" eigenvalue $\lambda \neq 1$. That is, $\exists f \in C(\mathbf{T}^2)$ invertible such that $f \circ \varphi = \lambda f$.

Taking the supremum on both sides of $f \circ \varphi = \lambda f$ yields $|\lambda| = 1$. Thus $f/|f|$ is a unitary eigenfunction with eigenvalue λ. Hence we will always assume, without loss of generality, that f is unitary. It is easy to show that the *affine* quasi-rotations have eigenvalues which are automatically non-singular.

Crossed products of $C(\mathbf{T}^n)$ by affine rotations of $\mathbf{T}^n$, i.e. $D(\varphi) = I_2$, have been classified by Riedel [**13**, Corollary 3.7].

Lemma 3.1. *Let φ be a quasi-rotation with non-singular eigenvalue $\lambda \neq 1$ so that $f \circ \varphi = \lambda f$, where $f \in C(\mathbf{T}^2)$ is unitary. Then*

(i) $D(f) \neq [0\ 0]$,
(ii) $\det(D(\varphi) - I_2) = 0$.

Proof. Assume that $D(f) = [0\ 0]$ so that one can write $f(x, y) = e^{2\pi i F(x,y)}$, for some continuous real-valued function F on $\mathbf{T}^2$. The relation $f \circ \varphi = \lambda f$ then becomes

$$e^{2\pi i(F(\varphi(x,y)) - F(x,y))} = \lambda.$$

Thus $F(\varphi(x, y)) - F(x, y) = c$, for all $(x, y) \in \mathbf{T}^2$, where c is a real constant. By induction this becomes

$$F(\varphi^{(k)}(x, y)) - F(x, y) = kc,$$

for every positive integer k. But since the left-hand side is bounded, it follows that $c = 0$ and so $\lambda = 1$, a contradiction.

Upon taking degrees on both sides of $f \circ \varphi = \lambda f$ we obtain $D(f)D(\varphi) = D(f)$, or $D(f)(D(\varphi) - I_2) = 0$, where $D(f) \neq [0\ 0]$. Therefore, $\det(D(\varphi) - I_2) = 0$. □

Definition. Let φ be a quasi-rotation of $\mathbf{T}^2$ and λ a non-singular eigenvalue of φ. We call λ a "primitive" eigenvalue if it has an associated unitary eigenfunction $f \in C(\mathbf{T}^2)$ such that $D(f)$ has relatively prime entries.

Lemma 3.2. *Every quasi-rotation φ of $\mathbf{T}^2$ has a primitive non-singular eigenvalue ($\neq 1$), which is unique up to complex conjugation.*

Proof. Suppose that $f \circ \varphi = \lambda f$, $\lambda \neq 1$, and $f \in C(\mathbf{T}^2)$ is a unitary with $D(f) = [m\ n] \neq [0\ 0]$ (by Lemma 3.1). Let $d = \gcd(m, n)$. Choose a unitary $g \in C(\mathbf{T}^2)$ such that $g^d = f$, where g^d is the d-fold pointwise product of g. Thus $g^d \circ \varphi = \lambda g^d$, or $[(g \circ \varphi)\overline{g}]^d = \lambda$. By continuity, $(g \circ \varphi)\overline{g} = \lambda_0$ for some dth-root λ_0 of λ. Hence $g \circ \varphi = \lambda_0 g$ and $\lambda_0 \neq 1$ is primitive since the entries of $D(g) = [(m/d)\ (n/d)]$ are relatively prime.

To prove the uniqueness part suppose that in addition to $g \circ \varphi = \lambda_0 g$ (λ_0 primitive) we have $h \circ \varphi = \mu h$, where μ is primitive and $D(h)$ has relatively prime entries. Taking degrees on both sides of these two equations we get $D(g)(D(\varphi) - I_2) = 0$, and $D(h)(D(\varphi) - I_2) = 0$. Since $D(\varphi) - I_2 \neq 0$, it follows that $D(g)$ and $D(h)$ are rationally dependent, that is, there are non-zero integers a and b such that

$$aD(g) + bD(h) = [0\ 0].$$

But since $D(g)$ and $D(h)$ have relatively prime entries it follows that $D(g) = \pm D(h)$, and so $D(gh^{\pm 1}) = [0\ 0]$. From the above two eigenvalue equations we have

$$(gh^{\pm 1}) \circ \varphi = (\lambda_0 \mu^{\pm 1})(gh^{\pm 1}).$$

Since $gh^{\pm 1}$ has zero bidegree, Lemma 3.1(i) implies that $\lambda_0 \mu^{\pm 1} = 1$ Hence, $\mu = \lambda_0^{\mp 1}$, as desired. □

Examples. 1. Let $\lambda = e^{2\pi i \theta}$, $0 < \theta < 1$, and consider the Anzai transformation $\varphi_\theta(x, y) = (\lambda x, xy)$. Since $D(\varphi_\theta) \neq I_2$ and $u \circ \varphi_\theta = \lambda u$ where $u(x, y) = x$ and $\lambda \neq 1$, φ_θ is a quasi-rotation. In fact, it is clear that φ_θ is affine. If θ is irrational, then φ_θ is minimal on the 2-torus (using the minimality criterion in [**8**, p. 84], or [**15**, Prop. 1.1.4]). Hence the associated crossed product C^*-algebra $\mathscr{A}(\varphi_\theta)$ is simple (cf. Power [**12**]) and has a unique faithful tracial state since φ_θ is uniquely ergodic, i.e. has a unique invariant Borel probability measure (cf. [**5**, Prop. 1.12] or [**15**, Lemma 1.3.4]). The isomorphism classes of these algebras (for θ irrational) were studied by Packer [**7**], and also by Ji [**5**] in his more general setting of Furstenberg transformations of n-tori. Here we shall classify these crossed products within the slightly broader family of those associated with affine quasi-rotations.

2. Furstenberg [**4**, p. 597] proved that a minimal transformation φ of $\mathbf{T}^2$ which is not homotopic to the identity, i.e. such that $D(\varphi) \neq I_2$, has an irrational eigenvalue λ, so that any (non-zero) eigenfunction will automatically be invertible. Hence φ is a quasi-rotation.

3. There are only two orientation reversing affine quasi-rotations of $\mathbf{T}^2$ up to topological conjugation (by Lemma 2.4 above). The first one is of the form $(x, y) \mapsto (ay, bx)$, with degree matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

having primitive eigenvalue $\lambda = ab$ (say $\lambda \neq 1$) and eigenfunction $f(x, y) = xy$. The second one has the form $(x, y) \mapsto (\lambda x, \bar{y})$, with degree matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and has primitive eigenvalue λ (say $\lambda \neq 1$) and eigenfunction $u(x, y) = x$.

4. In [**16**] certain techniques of Furstenberg have been used to construct a (non-affine) quasi-rotation ψ which does not have topologically quasi-discrete spectrum. This settled a question of Ji [**5**, pp. 75–76] in the negative; namely, whether in general a transformation of the form $(x, y) \mapsto (e^{2\pi i\theta}x, f(x)y)$, where $f\colon \mathbf{T} \to \mathbf{T}$ is continuous with degree ± 1, is topologically conjugate to the Anzai transformation φ_θ or to its inverse. The latter has topologically quasi-discrete spectrum and so cannot be topologically conjugate to ψ. An interesting question in this regard is whether the associated crossed product C^*-algebras are isomorphic. They have the same K-groups, same tracial range, have unique tracial states, and are both simple.

4. The range of the trace. In this section we wish to compute the range of the trace for the algebras $\mathscr{A}(\varphi)$ for any quasi-rotation φ.

This computation follows closely that of the irrational rotation algebras studied by Rieffel [**14**] and Pimsner and Voiculescu [**11**].

Let us note that almost every C^*-crossed product of a commutative unital C^*-algebra by $\mathbf{Z}$ has a tracial state. If X is a compact metric space and φ is a transformation of X, then a theorem of Krylov and Bogolioubov (cf. [**18**, p. 132]) ensures that there is a Borel probability measure μ on X which is φ-invariant, that is, $\mu(\varphi^{-1}(E)) = \mu(E)$ for every Borel subset E of X. The map

$$\tau(f) = \int_X f\,d\mu$$

is a tracial state on $C(X)$ which is α-invariant, where α is the automorphism of $C(X)$ associated with φ, i.e. $\alpha(f) = f \circ \varphi^{-1}$. This means that τ induces a tracial state $\hat{\tau}$ on $C(X) \times_\alpha \mathbf{Z}$.

THEOREM 4.1. *Let φ be a quasi-rotation of $\mathbf{T}^2$ with primitive eigenvalue $\lambda = e^{2\pi i\theta}$. Then for any tracial state τ on $\mathscr{A}(\varphi)$ we have*

$$\tau_* K_0(\mathscr{A}(\varphi)) = \mathbf{Z} + \theta\mathbf{Z}.$$

Note that we did not assume that θ is irrational, only that it is not an integer.

Proof. Let $f \in C(\mathbf{T}^2)$ be a unitary such that $f \circ \varphi = \lambda f$ and $D(f)$ has relatively prime entries. This f induces a C^*-homomorphism $\rho\colon C(\mathbf{T}) \to C(\mathbf{T}^2)$ given by $\rho(g) = g \circ f$.

If we let β denote the automorphism on $C(\mathbf{T})$ associated with rotation by λ, namely, $\beta(g)(x) = g(\bar{\lambda}x)$, for $g \in C(\mathbf{T})$ and $x \in \mathbf{T}$, then ρ is an equivariant homomorphism between the C^*-dynamical systems $(C(\mathbf{T}), \beta, \mathbf{Z})$ and $(C(\mathbf{T}^2), \alpha_\varphi, \mathbf{Z})$. To see this we verify that $\rho \circ \beta = \alpha_\varphi \circ \rho$ as follows:

$$\begin{aligned}\alpha_\varphi(\rho(g))(z) &= \rho(g)(\varphi^{-1}(z)) = g \circ f \circ \varphi^{-1}(z) = g(\bar{\lambda} f(z)) \\ &= \beta(g)(f(z)) = \rho(\beta(g))(z),\end{aligned}$$

for all $z \in \mathbf{T}^2$ and $g \in C(\mathbf{T})$.

Using the naturality of the Pimsner-Voiculescu sequence, this ρ induces a morphism between their associated Pimsner-Voiculescu sequences yielding the commutative diagram

$$\begin{array}{ccccccccc}\cdots & \to & K_0(C(\mathbf{T})) & \xrightarrow{i_*} & K_0(C(\mathbf{T}) \times_\beta \mathbf{Z}) & \xrightarrow{\delta_0} & K_1(C(\mathbf{T})) & \to & \cdots \\ & & \downarrow \rho_* & & \downarrow \tilde{\rho}_* & & \downarrow \rho_* & & \\ \cdots & \to & K_0(C(\mathbf{T}^2)) & \xrightarrow{i_*} & K_0(\mathscr{A}(\varphi)) & \xrightarrow{\delta_0'} & K_1(C(\mathbf{T}^2)) & \to & \cdots\end{array}$$

where $\tilde{\rho}$: $C(\mathbf{T}) \times_\beta \mathbf{Z} \to \mathscr{A}(\varphi)$ is the induced homomorphism from ρ.

If θ is irrational one can construct the Rieffel projection e in $C(\mathbf{T}) \times_\beta \mathbf{Z} = \mathscr{A}_\theta$ having trace θ (cf. [**14**, pp. 418f]). If θ is rational one can still construct the Rieffel projection in the same way and it can be shown that $\tau'(e) = \theta$, for any tracial state τ' on $\mathscr{A}_\theta$ (cf. Elliott [**3**, Lemma 2.3, pp. 170-171]). In both cases one has $\delta_0[e] = [f_0]$, which is the generator of $K_1(C(\mathbf{T}))$, where $f_0(z) = z$, $z \in \mathbf{T}$. Since the diagram commutes, one has

$$\delta_0'[\tilde{\rho}(e)] = \delta_0'\tilde{\rho}_*[e] = \rho_*\delta_0[e] = \rho_*[f_0] = [f],$$

and since $D(f)$ has relatively prime entries, $[f]$ is generator of $\ker((\alpha_\varphi)_* - \mathrm{id}_*)$ in $K_1(C(\mathbf{T}))$. Hence the projection $\tilde{\rho}(e)$ yields a generator in $K_0(\mathscr{A}(\varphi))$ which, along with the two generators as in Corollary 1.2 (having traces 0 and 1), gives the range of the trace as

$$\begin{aligned}\tau_*K_0(\mathscr{A}(\varphi)) &= \mathbf{Z} + \tau(\tilde{\rho}(e))\mathbf{Z} \\ &= \mathbf{Z} + \tau'(e)\mathbf{Z} \\ &= \mathbf{Z} + \theta\mathbf{Z},\end{aligned}$$

where $\tau' = \tau \circ \tilde{\rho}$ is a tracial state on $\mathscr{A}_\theta$. □

REMARK. One could use Pimsner's computation of the tracial range [**10**] to prove the above theorem using the concept of the determinant associated with a trace. But for our purposes the above short proof suffices.

Now let us look at some of the consequences of this theorem and the results of the preceding section.

COROLLARY 4.2. *Let φ_j be a quasi-rotation of $\mathbf{T}^2$ with primitive eigenvalue $\lambda_j = e^{2\pi i\theta_j}$, $j = 1, 2$. If $\mathscr{A}(\varphi_1) \cong \mathscr{A}(\varphi_2)$, then*

(1) $\mathbf{Z} + \theta_1\mathbf{Z} = \mathbf{Z} + \theta_2\mathbf{Z}$,

(2) $\det D(\varphi_1) = \det D(\varphi_2)$,

(3) $m(\varphi_1) = m(\varphi_2)$.

Consequently, $D(\varphi_1) \sim D(\varphi_2)$.

Proof. The preceding theorem yields (1), and Proposition 2.8 yields (2) and (3). □

COROLLARY 4.3 (*Packer* [**7**, p. 49]; Ji [**5**, p. 39]). *For each irrational number* $0 < \theta < 1$ *and each non-zero integer* k, *let* $H_{\theta,k}$ *denote* *the crossed product C^*-algebra of $C(\mathbf{T}^2)$ by the Anzai transformation $\varphi(x, y) = (e^{2\pi i\theta}x, x^k y)$. Then*

$$H_{\theta, k} \cong H_{\theta', k'} \Leftrightarrow |k| = |k'| \quad \text{and} \quad \theta' \in \{\theta, 1-\theta\}.$$

Proof. (Note that if $k = 0$, then $H_{\theta, k} \cong \mathscr{A}_\theta \otimes C(\mathbf{T})$ and the conclusion easily holds.)

($\Rightarrow$) In this case the tracial ranges being equal (by the preceding corollary) implies that $\theta' \in \{\theta, 1-\theta\}$, as the latter are irrational. Since these algebras have isomorphic K_1-groups, Corollary 2.3 shows that $|k| = |k'|$. The converse easily follows. □

Let us recall that the natural action of the group $\mathrm{GL}(2, \mathbf{Z})$ on the irrational numbers is given by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \theta = \frac{a\theta + b}{c\theta + d}.$$

COROLLARY 4.4. *Let φ_j be an irrational quasi-rotation of $\mathbf{T}^2$ with primitive eigenvalue $\lambda_j = e^{2\pi i\theta_j}$, $j = 1, 2$ (i.e. θ_j is irrational). If $\mathscr{A}(\varphi_1)$ and $\mathscr{A}(\varphi_2)$ are strongly Morita equivalent, then*

(1) $\theta_2 = A\theta_1$, *for some* $A \in \mathrm{GL}(2, \mathbf{Z})$,

(2) $\det D(\varphi_1) = \det D(\varphi_2)$,

(3) $m(\varphi_1) = m(\varphi_2)$.

Consequently, $D(\varphi_1) \sim D(\varphi_2)$.

Proof. Conclusions (2) and (3) follow from Proposition 2.8 since strongly Morita equivalent C^*-algebras have isomorphic K-groups. Theorem 4.1 allows one to apply Rieffel's argument [**14**, Proposition 2.5] to derive (1). □

COROLLARY 4.5. *No $\mathscr{A}_\theta$ is isomorphic to any $C(\mathbf{T}^2) \times_\alpha \mathbf{Z}$. For θ irrational, no $\mathscr{A}_\theta$ is strongly Morita equivalent to any $C(\mathbf{T}^2) \times_\alpha \mathbf{Z}$.*

Proof. Assume that $\mathscr{A}_\theta \cong C(\mathbf{T}^2) \times_\alpha \mathbf{Z}$. Then $K_0(C(\mathbf{T}^2) \times_\alpha \mathbf{Z}) \cong K_0(\mathscr{A}_\theta) \cong \mathbf{Z}^2$, and the proof of Theorem 1.1(1) shows that $K_0(C(\mathbf{T}^2) \times_\alpha \mathbf{Z})$ is generated by the classes $[1]$ and $[P]$, where P is the Bott projection. These, however, have traces equal to 1, and so looking at their tracial ranges yields $\mathbf{Z} = \mathbf{Z} + \theta\mathbf{Z}$. Thus $\theta \in \mathbf{Z}$ and hence $\mathscr{A}_\theta \cong C(\mathbf{T}^2)$ which is isomorphic to $C(\mathbf{T}^2) \times_\alpha \mathbf{Z}$, and being therefore commutative implies that $\alpha = \mathrm{id}$. Thus, $C(\mathbf{T}^2) \cong C(\mathbf{T}^2) \times_{\mathrm{id}} \mathbf{Z} \cong C(\mathbf{T}^2) \otimes C(\mathbf{T}) \cong C(\mathbf{T}^3)$, a contradiction. A similar argument shows the second assertion of the corollary. □

The second assertion of this corollary is still true for θ rational, but it requires a little more work which we defer to a future paper **[17]**.

Let us now extend Theorem 4.1 to matrix algebras over $\mathscr{A}(\varphi)$.

If A is a unital C^*-algebra, then any tracial state τ on $M_n \otimes A$ has the form $(1/n)\mathrm{tr} \otimes \tau'$ for some tracial state τ' on A, where tr is the usual trace on matrices (for instance see **[5**, Lemma 3.3**]**). Furthermore, if all tracial states on A induce the same map on $K_0(A)$, then all tracial states on $M_n \otimes A$ induce the same map on $K_0(M_n \otimes A)$ (cf. **[5**, Lemma 3.5**]**). In fact one has in this case

$$\tau_* K_0(M_n \otimes A) = \frac{1}{n}\tau'_* K_0(A),$$

for all tracial states τ, τ' on $M_n \otimes A$ and A, respectively. This yields the following.

COROLLARY 4.6. *Let φ be a quasi-rotation of $\mathbf{T}^2$ with primitive eigenvalue $\lambda = e^{2\pi i\theta}$. Then*

$$\tau_* K_0(M_n \otimes \mathscr{A}(\varphi)) = \frac{1}{n}(\mathbf{Z} + \theta\mathbf{Z}),$$

for any tracial state τ on $M_n \otimes \mathscr{A}(\varphi)$.

COROLLARY 4.7. *Let φ_j be a quasi-rotation of $\mathbf{T}^2$ with primitive eigenvalue $\lambda_j = e^{2\pi i\theta_j}$, $j = 1, 2$. If $M_n \otimes \mathscr{A}(\varphi_1) \cong M_k \otimes \mathscr{A}(\varphi_2)$, then*

(1) $n = k$,

(2) $\mathbf{Z} + \theta_1\mathbf{Z} = \mathbf{Z} + \theta_2\mathbf{Z}$,

(3) $\det D(\varphi_1) = \det D(\varphi_2)$,

(4) $m(\varphi_1) = m(\varphi_2)$.

Proof. It will suffice to prove (1) since the other conclusions will then follow from Corollaries 4.2 and 4.6. For brevity denote $B_j = \mathscr{A}(\varphi_j)$, $j = 1, 2$. The proof of (1) is easy if θ_j is irrational, but requires a little more work otherwise. To do so it suffices (by symmetry) to prove that if M_k can be unitally embedded in $M_n \otimes B_1$, then $k|n$.

Recall that $K_0(B_1)$ is generated by a projection $e \in B_1$ of trace θ_1, and two other classes $[1]$ and $x = [P] - [1]$.

Let $\{e_{ij}^{(n)}\}_{i,j=1,\dots,n}$ be the standard matrix units for M_n, so that $K_0(M_n \otimes B_1)$ has independent generators $[e_{11}^{(n)} \otimes e]$, $[e_{11}^{(n)} \otimes 1]$, and $e_{11}^{(n)} \otimes x = [e_{11}^{(n)} \otimes P] - [e_{11}^{(n)} \otimes 1]$.

Suppose that $\sigma: M_k \to M_n \otimes B_1$ is a unital embedding and σ_*: $K_0(M_k) \to K_0(M_n \otimes B_1)$, where $K_0(M_k) = \mathbf{Z}[e_{11}^{(k)}]$. Then

$$\sigma_*[e_{11}^{(k)}] = a[e_{11}^{(n)} \otimes e] + b[e_{11}^{(n)} \otimes 1] + c(e_{11}^{(n)} \otimes x),$$

for some integers a, b, c. Now since $I_k = \sum_i e_{ii}^{(k)}$ is the sum of equivalent projections, we get from

$$I_n \otimes 1 = \sigma(I_k) = \sum_i \sigma(e_{ii}^{(k)})$$

that $[I_n \otimes 1] = k[\sigma(e_{ii}^{(k)})] \in K_0(M_n \otimes B_1)$. Thus

$$\begin{aligned} n[e_{11}^{(n)} \otimes 1] &= [I_n \otimes 1] = k\sigma_*[e_{11}^{(k)}] \\ &= ka[e_{11}^{(n)} \otimes e] + kb[e_{11}^{(n)} \otimes 1] + kc(e_{11}^{(n)} \otimes x), \end{aligned}$$

and therefore $n = kb$. □

REMARK. The argument in the above elementary proof can also be used to show a similar result for the rotation C^*-algebras $\mathscr{A}_\theta$. Recall that Rieffel **[14]** showed this for θ irrational, while in **[3]** and **[19]** it was shown for rational θ and $n = k = 1$.

5. Main Theorem. Before embarking on the main result let us introduce some notation and characterize the affine quasi-rotations. Later a partial result is given for the rational affine quasi-rotation algebras.

If $A \in \mathrm{GL}(2, \mathbf{Z})$, say

$$A = \begin{pmatrix} m & n \\ p & q \end{pmatrix}$$

then its action on $\mathbf{T}^2$ is defined by $A(x, y) = (x^m y^n, x^p y^q)$. This actually gives the group isomorphism $\mathrm{Aut}(\mathbf{T}^2) \cong \mathrm{GL}(2, \mathbf{Z})$. It is easy to check that

$$A_1(A_2 z) = (A_1 A_2)(z),$$

for all $A_1, A_2 \in \mathrm{GL}(2, \mathbf{Z})$ and $z \in \mathbf{T}^2$.

Now if $X = [m\ n]$ is a 1×2 integral matrix, it induces a continuous function (actually a character) X: $\mathbf{T}^2 \to \mathbf{T}$ given by $X(x, y) = x^m y^n$. Clearly, $X(Az) = (XA)(z)$ for any X, and $A \in \mathrm{GL}(2, \mathbf{Z})$. Also, since X is a homomorphism, $X(zw) = X(z)X(w)$.

Let us suppose that $A \in \mathrm{GL}(2, \mathbf{Z})$ is such that $A \neq I_2$ and $\det(A - I_2) = 0$. Then the proof of Lemma 3.2 (uniqueness part) shows that there exists an integral matrix $X_A = [m\ n]$ having relatively prime entries such that

$$X_A(A - I_2) = [0\ 0],$$

and that X_A is unique up to sign. So $X_A A = X_A$.

Now let us determine the affine quasi-rotations of $\mathbf{T}^2$.

LEMMA 5.1. *Let $\varphi(z) = aA(z)$ be an affine transformation of $\mathbf{T}^2$. Then φ is a quasi-rotation if and only if the following conditions hold*:

(i) $A \neq I_2$,
(ii) $\det(A - I_2) = 0$,
(iii) $X_A(a) \neq 1$.

Proof. Suppose these three conditions hold. Then for $z \in \mathbf{T}^2$ one has

$$X_A \circ \varphi(z) = X_A(aA(z)) = X_A(a)X_A(A(z)) = X_A(a)X_A(z),$$

so that $X_A \circ \varphi = X_A(a)X_A$, where $X_A(a) \neq 1$ is a non-singular eigenvalue which is primitive as X_A has relatively prime entries. Since also $D(\varphi) = A$, φ is a quasi-rotation.

Conversely, suppose that φ is a quasi-rotation. By definition (i) holds, and by Lemma 3.1 condition (ii) holds. It remains to check (iii). By Lemma 3.2 φ has a primitive eigenvalue $\lambda \neq 1$ so that $f \circ \varphi = \lambda f$, where f is unitary with $D(f)$ having relatively prime entries. Taking D on both sides gives $D(f)(A - I_2) = 0$. By the uniqueness of X_A, we get that $D(f) = \pm X_A$. Replacing f by $\overline{f}$, if necessary, we may assume that $D(f) = X_A = [m\ n]$. So let us then write f as $f(x, y) = x^m y^n e^{2\pi i F(x,y)}$, where F is real-valued. This we may re-write as $f(z) = X_A(z)e^{2\pi i F(z)}$, where $z \in \mathbf{T}^2$. Thus the equation $f \circ \varphi = \lambda f$ becomes

$$X_A(\varphi(z))e^{2\pi i F(\varphi(z))} = \lambda X_A(z)e^{2\pi i F(z)}.$$

Now since $X_A \circ \varphi = X_A(a)X_A$, as we computed above, this equation reduces to

$$e^{2\pi i\{F(\varphi(z))-F(z)\}} = \lambda\overline{X_A(a)},$$

which, by arguing as in the proof of Lemma 3.1, implies that $\lambda\overline{X_A(a)} = 1$. Hence $X_A(a) = \lambda \neq 1$. □

Let $\mathscr{B}(a, A)$ denote the crossed product C^*-algebra associated with the affine quasi-rotation corresponding to the pair (a, A) satisfying the conditions of the preceding lemma. The inverse of such a quasi-rotation can easily be checked to correspond to the pair $(A^{-1}(\overline{a}), A^{-1})$.

THEOREM 5.2 [**15**, *Theorem* 4.3.2]. *Let (a_j, A_j) be a pair corresponding to the irrational affine quasi-rotation φ_j of $\mathbf{T}^2$, $j = 1, 2$. Then the following are equivalent*:

(1) $\mathscr{B}(a_1, A_1) \cong \mathscr{B}(a_2, A_2)$,

(2) *φ_1 and φ_2 are topologically conjugate via an affine transformation,*

(3) *The following conditions hold:*

(i) $X_{A_2}(a_2) = X_{A_1}(a_1)^{\pm 1}$,

(ii) $\det(A_1) = \det(A_2)$,

(iii) $m(A_1) = m(A_2)$.

Proof. By Lemma 5.1, $A_j \neq I_2$ and $\det(A_j - I_2) = 0$, so that $X_j = X_{A_j}$, with relatively prime entries, exists such that $X_j A_j = X_j$, $j = 1, 2$.

In view of Corollary 4.2 condition (1) implies (3), as $X_j(a_j)$ is irrational. Clearly (2) implies (1). So we need to check that (3) implies (2).

Assuming that (i), (ii), (iii) hold we shall construct an affine transformation $\psi(z) = kK(z)$ which intertwines φ_1 and φ_2. By Corollary 2.6, $A_1 \sim A_2$ so choose $K \in \mathrm{GL}(2, \mathbf{Z})$ such that $KA_1K^{-1} = A_2$. The equation $X_2A_2 = X_2$ becomes $(X_2K)A_1 = X_2K$. Since X_2 has relatively prime entries then so does $X_2K = \pm X_1$. Replacing K by $-K$, if necessary, we may choose the $\pm$ in $X_2K = \pm X_1$ according to whether $X_2(a_2) = X_1(a_1)^{\pm 1}$, respectively.

We need to choose k so that $\psi \circ \varphi_1 = \varphi_2 \circ \psi$. The left-hand side of this is

$$\psi \circ \varphi_1(z) = kK(a_1A_1(z)) = kK(a_1)KA_1(z),$$

and the right side is

$$\varphi_2 \circ \psi(z) = a_2A_2(kK(z)) = a_2A_2(k)A_2K(z).$$

These expressions are equal if and only if

$$(*) \qquad kK(a_1) = a_2A_2(k),$$

and it suffices to show that this equation has a solution $k \in \mathbf{T}^2$.

To do this, first extend the equation $X_2K = \pm X_1$ to

$$\begin{pmatrix} X_2 \\ R_2 \end{pmatrix} K = \begin{pmatrix} \pm X_1 \\ R_1 \end{pmatrix},$$

for some 1×2 integral matrices R_1 and R_2 such that

$$T_j = \begin{pmatrix} X_j \\ R_j \end{pmatrix}$$

has determinant ± 1 (which is possible since X_2 has relatively prime entries). Now apply T_2 to both sides of $(*)$ to get

$$\binom{X_2}{R_2}(k)\binom{X_2}{R_2}K(a_1) = \binom{X_2}{R_2}(a_2)\binom{X_2}{R_2}A_2(k),$$

or

$$\binom{X_2}{R_2}(k)\binom{\pm X_1}{R_1}(a_1) = \binom{X_2}{R_2}(a_2)\binom{X_2}{R_2'}(k),$$

where $R_2' = R_2A_2$. Note that $R_2' \neq R_2$; for otherwise $R_2(A_2 - I_2) = 0$ which implies that $T_2(A_2 - I_2) = 0$ hence $A_2 - I_2 = 0$ as T_2 is invertible. Thus the above equation becomes

$$(X_2(k), R_2(k))(X_1(a_1)^{\pm 1}, R_1(a_1)) = (X_2(a_2), R_2(a_2))(X_2(k), R_2'(k)).$$

By condition (i) the first coordinates of both sides of this equation are equal for all k. The second coordinates become

$$R_2(k)R_1(a_1) = R_2(a_2)R_2'(k),$$

or

$$(R_2 - R_2')(k) = R_2(a_2)\overline{R_1(a_1)},$$

and this clearly has a solution k since $R_2 - R_2' \neq [0\ 0]$. □

Therefore, the irrational affine quasi-rotation algebras $\mathscr{B}(a, A)$ are completely determined up to isomorphism by the triple $(X_A(a), \det(A), m(A))$, up to conjugacy of $X_A(a)$, where $X_A(a)$ is the primitive eigenvalue coming from the tracial range, $\det(A) = \pm 1$ is known from the K_0-group and $m(A)$ is known from the K_1-group.

COROLLARY 5.3. *For irrational affine quasi-rotations of* $\mathbf{T}^2$, *we have*: $M_n \otimes \mathscr{B}(a_1, A_1) \cong M_k \otimes \mathscr{B}(a_2, A_2)$ *if and only if* $k = n$, $X_{A_1}(a_1) = X_{A_2}(a_2)^{\pm 1}$, $\det(A_1) = \det(A_2)$, *and* $m(A_1) = m(A_2)$.

As a final remark let us note that an argument due to Yin **[19]** for the rational rotation algebras can be used to show Theorem 5.2 (and hence Corollary 5.3) for the rational case for the orientation reversing quasi-rotations. Condition (3) in Theorem 5.2 implies (2) in exactly the same way as in the proof. It only remains to check $(1) \Rightarrow (3)$. Let $X_{A_j}(a_j) = e^{2\pi i\theta_j}$, $j = 1, 2$. Clearly, (ii) and (iii) follow as before, so we need to check (i). An isomorphism $\sigma: \mathscr{B}(a_1, A_1) \to \mathscr{B}(a_2, A_2)$ induces one on their K_0-groups which on their generators (cf. Corollary 1.2(ii)) is of the form

$$\begin{aligned}\sigma_*[1] &= [1],\\ \sigma_*([P_1]-[1]) &= [P_2]-[1], \quad \text{being elements of order two},\\ \sigma_*[e_{\theta_1}] &= r[1]+s([P_2]-[1])+t[e_{\theta_2}],\end{aligned}$$

for some integers r, s, t. Taking traces of the last of these equations gives $\theta_1 = r + t\theta_2$. Since the matrix of σ_* is

$$\begin{pmatrix} 1 & 0 & r \\ 0 & 1 & s \\ 0 & 0 & t \end{pmatrix},$$

and is invertible over $\mathbf{Z}$ one has that $t = \pm 1$; hence $\theta_1 = r \pm \theta_2$ which yields (1).

This argument however fails for the orientation preserving case since the above gives us conditions on certain integers that do not necessarily imply that $\theta_1 = \pm\theta_2 \bmod \mathbf{Z}$. This we do not know how to prove since the role of the Bott projection here is not so clear.

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