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HARMONIC MAJORIZATION OF A SUBHARMONIC FUNCTION ON A CONE OR ON A CYLINDER

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To Professor N. Yanagihara on his 60th birthday

For a subharmonic function u defined on a cone or on a cylinder which is dominated on the boundary by a certain function, we generalize the classical Phragmén-Lindelöf theorem by making a harmonic majorant of u and show that if u is non-negative in addition, our harmonic majorant is the least harmonic majorant. As an application, we give a result concerning the classical Dirichlet problem on a cone or on a cylinder with an unbounded function defined on the boundary.

1. Introduction. Let $\mathbb{R}$ and $\mathbb{R}_+$ be the sets of all real numbers and all positive real numbers, respectively. The m-dimensional Euclidean space is denoted by $\mathbb{R}^m$ $(m \geq 2)$ and O denote the origin of it. By ∂S and $\overline{S}$, we denote the boundary and the closure of a set S in $\mathbb{R}^m$. Let $|P - Q|$ denote the Euclidean distance between two points P, $Q \in \mathbb{R}^m$. A point on $\mathbb{R}^m$ $(m \geq 2)$ is represented by (X, y), $X = (x_1, x_2, \ldots, x_{m-1})$. We introduce the spherical coordinates (r, Θ), $\Theta = (\theta_1, \theta_2, \ldots, \theta_{m-1})$, in $\mathbb{R}^m$ which are related to the coordinates (X, y) by

$$\begin{cases} x_1 = r\left(\prod_{j=1}^{m-1} \sin\theta_j\right), \qquad y = r\cos\theta_1, \\ x_{m+1-k} = r\left(\prod_{j=1}^{k-1} \sin\theta_j\right)\cos\theta_k \quad (m \geq 3,\ 2 \leq k \leq m-1), \\ x_1 = r\cos\theta_1, \qquad y = r\sin\theta_1 \qquad (m = 2), \end{cases}$$

where $0 \leq r < +\infty$ and $-\frac{1}{2}\pi \leq \theta_{m-1} < \frac{3}{2}\pi$ $(m \geq 2)$, $0 \leq \theta_j \leq \pi$ $(m \geq 3,\ 1 \leq j \leq m-2)$. The unit sphere and the surface area $2\pi^{m/2}\{\Gamma(m/2)\}^{-1}$ of it are denoted by $\mathbb{S}^{m-1}$ and s_m $(m \geq 2)$, respectively. The upper half unit sphere $\{(1, \Theta) \in \mathbb{S}^{m-1};\ 0 \leq \theta_1 < \frac{\pi}{2}$ (if $m = 2$, then $0 < \theta_1 < \pi$)$\}$ is also denoted by $\mathbb{S}_+^{m-1}$ $(m \geq 2)$. For simplicity, a point $(1, \Theta)$ on $\mathbb{S}^{m-1}$ and a set S, $S \subset \mathbb{S}^{m-1}$, are often identified with Θ and $\{\Theta; (1, \Theta) \in S\}$, respectively. For two sets $E_1 \subset \mathbb{R}_+$ and $E_2 \subset \mathbb{S}^{m-1}$, the set

$$\{(r, \Theta) \in \mathbb{R}^m; r \in E_1, (1, \Theta) \in E_2\}$$

in $\mathbb{R}^m$ is denoted by $E_1 \times E_2$. Given a domain Ω on $\mathbb{S}^{m-1}$ $(m \geq 2)$, the set $\mathbb{R}_+ \times \Omega$ is called a cone and denoted by $C(\Omega)$. The special cone $C(\mathbb{S}_+^{m-1})$ $(m \geq 2)$ called the half-space will be denoted by $\mathbb{T}_m$. For a positive number r, the set $\{r\} \times \mathbb{S}^{m-1}$ is denoted by $S_m(r)$ and $S_m(r) \cap \mathbb{T}_m$ by $S_m^+(r)$.

In our previous paper [**12**, Theorem 5.1], we gave a harmonic majorant of a certain subharmonic function $u(P)$ defined on a cone $C(\Omega)$ with a domain Ω having smooth boundary, such that

$$\varlimsup_{P \in C(\Omega), P \to Q} u(P) \leq 0 \tag{1.1}$$

for every $Q \in \partial C(\Omega) - \{O\}$. It can be regarded as one of the generalizations of the classical Phragmén-Lindelöf theorem. We also showed in [**12**, Corollary 5.2] that if the function $u(P)$ is non-negative in addition, our harmonic majorant is the least harmonic majorant. In this paper, we shall consider generalizations of these results, by replacing 0 of (1.1) with a general function $g(Q)$ on $\partial C(\Omega) - \{O\}$. They were motivated by the following Theorems A, B, C and D, which are special cases of our results (see Remark 5).

Nevanlinna [**10**] proved

THEOREM A. *Let $g(t)$ be a continuous function on $\mathbb{R}$ such that*

$$\int^{\infty} \frac{|g(t)| + |g(-t)|}{t^2} dt < +\infty \tag{1.2}$$

and let $f(z)$ be a regular function on $\mathbb{T}_2$ such that

$$\varlimsup_{\operatorname{Im}(z)>0, z \to t} \log|f(z)| \leq g(t)$$

for any $t \in \partial\mathbb{T}_2$. If

$$\varliminf_{r \to \infty} \frac{1}{r} \int_0^{\pi} \log^+ \left|f\left(re^{i\theta}\right)\right| \sin\theta \, d\theta = 0, \tag{1.3}$$

then

$$\log|f(z)| \leq \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{g(t)}{(t-x)^2 + y^2} dt \tag{1.4}$$

for any $z = x + iy \in \mathbb{T}_2$.

In the slightly different form from Theorem A, Boas [2, pp. 92–93] also stated

THEOREM B. *Make the same assumption as in Theorem A. If*

$$\varlimsup_{r\to\infty} \frac{1}{r} M_{\log|f|}(r) < +\infty \qquad \left(M_{\log|f|}(r) = \sup_{|z|=r,\, \mathrm{Im}(z)>0} \log|f(z)| \right),$$

then

$$\log|f(z)| \le \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{g(t)}{(t-x)^2+y^2}\, dt + a_f y \tag{1.5}$$

for any $z = x+iy \in \mathbb{T}_2$, *where*

$$a_f = \frac{2}{\pi} \lim_{r\to\infty} \frac{1}{r} \int_0^{\pi} \log|f(re^{i\theta})| \sin\theta \, d\theta.$$

Keller [7] proved an analogous result for a harmonic function on $\mathbb{T}_3$.

THEOREM C. *Let* $g(Q)$ *be a continouus function on* $\partial\mathbb{T}_3$ *such that*

$$\int^{\infty} r^{-2} \left(\int_{-\pi/2}^{3\pi/2} \left| g\left(r, \frac{\pi}{2}, \theta_2\right) \right| d\theta_2 \right) dr < +\infty$$

$$\left(Q = \left(r, \frac{\pi}{2}, \theta_2\right) \in \partial\mathbb{T}_3 \right).$$

Let $h(P)$ *be a harmonic function on* $\mathbb{T}_3$ *such that*

$$\varlimsup_{P\in\mathbb{T}_3,\, P\to Q} h(P) \le g(Q)$$

for any $Q \in \partial\mathbb{T}_3$.

(I) *There exists*

$$b_{h^+} = \lim_{r\to\infty} \frac{1}{r} \int_{S_3^+(r)} h^+(P) \cos\theta_1 \, d\sigma_{\widehat{P}}, \qquad 0 \le b_{h^+} \le +\infty,$$

where $h^+(P) = \max\{h(P), 0\}$ $(P \in S_3^+(r))$ *and* $d\sigma_{\widehat{P}} = \sin\theta_1 \, d\theta_1 \, d\theta_2$ *is the surface element on* $\mathbb{S}^2$ *at the radial projection* $\widehat{P} = (1, \theta_1, \theta_2)$ *of* $P = (r, \theta_1, \theta_2) \in S_3^+(r)$.

(II) *For any* $P \in \mathbb{T}_3$,

$$h(P) \le \frac{y}{2\pi} \int_{\partial\mathbb{T}_3} g(Q)|P-Q|^{-3} \, dQ + \frac{3}{2\pi} b_{h^+} y,$$

where dQ *is the area element on* $\partial\mathbb{T}_3$.

With respect to the least harmonic majorant of a subharmonic function on $\mathbb{T}_m$, Kuran [**8**, Theroem 3] proved

THEOREM D. *Let* $c < 0$ *and let* $u(X, y)$ *be subharmonic on*

$$\{(X, y) \in \mathbb{R}^m ; X \in \mathbb{R}^{m-1}, y > c\}$$

such that $u \geq 0$ *on* $\mathbb{T}_m$.

(I) *If*

$$\int_{\mathbb{R}^{m-1}} (1 + |X|^2)^{-1/2m} u(X, 0)\, dX < +\infty, \tag{1.6}$$

then there exists the limit

$$l_u = \lim_{r\to\infty} 2ms_m^{-1} r^{-m-1} \int_{S_m^+(r)} yu(Q)\, d\sigma_Q, \qquad 0 \leq l_u \leq +\infty,$$

where $|X| = \sqrt{x_1^2 + \cdots + x_{m-1}^2}$, dX *is the* $(m-1)$*-dimensional volume element at* $X = (x_1, \ldots, x_{m-1}) \in \mathbb{R}^{m-1}$ $(m \geq 2)$ *and* $d\sigma_Q$ *is the surface element of the sphere* $S_m(r)$ *at* $Q = (X, y) \in S_m^+(r)$. *Further if*

$$l_u < +\infty, \tag{1.7}$$

then

$$l_u y + 2s_m^{-1} y \int_{\mathbb{R}^{m-1}} |P - Q|^{-m} u(X, 0)\, dX \tag{1.8}$$

$$(P = (X, y) \in \mathbb{T}_m,\ Q = (X, 0) \in \partial\mathbb{T}_m)$$

is the least harmonic majorant of $u(P)$ *on* $\mathbb{T}_m$.

(II) *If* u *possesses a harmonic majorant on* $\mathbb{T}_m$, *then* (1.6) *and* (1.7) *hold.*

As an application, we shall give a result concerning the classical Dirichlet problem on a cone with an unbounded function defined on the boundary. Our method in this paper can be applied to a subharmonic function $u(X, y)$ defined on an infinite cylinder

$$\{(X, y) \in \mathbb{R}^m ; X \in D, y \in \mathbb{R}\},$$

where D is a bounded domain in $\mathbb{R}^{m-1}$ $(m \geq 2)$. We shall state some results in the cylindrical case.

2. Preliminaries. Let Λ_m be the spherical part of the Laplace operator

$$\Delta_m = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_{m-1}^2} + \frac{\partial^2}{\partial y^2} \qquad (m \geq 2)$$

relative to the system of spherical coordinates:

$$\Delta_m = \frac{m-1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + r^{-2}\Lambda_m.$$

Given a domain Ω on $\mathbb{S}^{m-1}$, consider the Dirichlet problem

$$\begin{aligned} (\Lambda_m + \lambda)F &= 0 \quad \text{on } \Omega, \\ F &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

We denote the least positive eigenvalue of it by $\lambda_\Omega^{(1)}$ and write $f_\Omega(\Theta)$ for the normalized positive eigenfunction corresponding to $\lambda_\Omega^{(1)}$, when they exist. Thus

$$\int_\Omega f_\Omega^2(\Theta)\, d\sigma_\Theta = 1, \tag{2.2}$$

where $d\sigma_\Theta$ is the surface element on $\mathbb{S}^{m-1}$. Two solutions of the equation

$$t^2 + (m-2)t - \lambda_\Omega^{(1)} = 0$$

are denoted by α_Ω, $-\beta_\Omega$ $(\alpha_\Omega, \beta_\Omega > 0)$.

Let $\Phi(r, \Theta)$ be a function on $C(\Omega)$. For any given r $(r \in \mathbb{R}_+)$, the integral

$$\int_\Omega \Phi(r, \Theta) f_\Omega(\Theta)\, d\sigma_\Theta$$

is denoted by $N_\Phi(r)$, when it exists. The finite or infinite limits

$$\lim_{r\to\infty} r^{-\alpha_\Omega} N_\Phi(r) \quad \text{and} \quad \lim_{r\to 0} r^{\beta_\Omega} N_\Phi(r)$$

are denoted by μ_Φ and η_Φ, respectively, when they exist. The maximum modulus $M_\Phi(r)$ $(0 < r < +\infty)$ of $\Phi(r, \Theta)$ is defined as

$$M_\Phi(r) = \sup_{\Theta\in\Omega} \Phi(r, \Theta).$$

We denote $\max\{\Phi(P), 0\}$ and $\max\{-\Phi(P), 0\}$ by $\Phi^+(P)$ and $\Phi^-(P)$, respectively.

This paper is essentially based on some results in Yoshida [**11**]. Hence, in the subsequent consideration, we make the same assumption on Ω as in it: if $m \geq 3$, then Ω is a $C^{2,\sigma}$-domain $(0 < \sigma < 1)$ on S^{m-1} surrounded by a finite number of mutually disjoint closed hypersurfaces (e.g., see Gilbarg and Trudinger [**4**, pp. 88–89] for the definition of $C^{2,\sigma}$-domain). Then there exist two positive constants L_1 and L_2 such that

$$L_1 \operatorname{dis}(\Theta, \partial\Omega) \leq f_\Omega(\Theta) \leq L_2 \operatorname{dis}(\Theta, \partial\Omega) \qquad (\Theta \in \Omega) \tag{2.3}$$

(by modifying Miranda's method [**9**, pp. 7–8], we can prove this inequality).

REMARK 1. Let $\Omega = \mathbb{S}_+^{m-1}$. Then $\alpha_\Omega = 1$, $\beta_\Omega = m-1$ and

$$f_\Omega(\Theta) = \begin{pmatrix} (2ms_m^{-1})^{1/2}\cos\theta_1 & (m \geq 3) \\ \frac{2}{\pi}\sin\theta & (m = 2) \end{pmatrix}$$
$$= (2m_m^{-1})^{1/2}\frac{y}{r} \qquad (m \geq 2).$$

Let $X = (x_1, x_2, \dots, x_{m-1})$ be a point of $\mathbb{R}^{m-1}$ $(m \geq 2)$. Given a bounded domain D in $\mathbb{R}^{m-1}$ $(m \geq 2)$, consider the Dirichlet problem

$$(\Delta_{m-1} + \lambda)F = 0 \quad \text{on } D,$$
$$F = 0 \quad \text{on } \partial D.$$

Let λ_D be the least positive eigenvalue of it and let $f_D(X)$ be the normalized eigenfunction corresponding to λ_D. As in the conical case, we assume that the boundary ∂D of $D \subset \mathbb{R}^{m-1}$ $(m \geq 3)$ is sufficiently smooth. The set

$$D \times \mathbb{R} = \{(X, y) \in \mathbb{R}^m;\ X \in D,\ y \in \mathbb{R}\}$$

in $\mathbb{R}^m$ is called a cylinder and denoted by $\Gamma(D)$ $(m \geq 2)$. Let $\Psi(X, y)$ be a function on $\Gamma(D)$. The integral

$$\int_D \Psi(X, y) f_D(X)\, dX$$

of $\Psi(X, y)$ is denoted by $N_\Psi^\Gamma(y)$ when it exists, where dX denotes the $(m-1)$-dimensional volume element. The finite or infinite limits

$$\lim_{y\to\infty} e^{-\sqrt{\lambda_D}y} N_\Psi(y) \quad \text{and} \quad \lim_{y\to-\infty} e^{\sqrt{\lambda_D}y} N_\Psi(y)$$

are denoted by μ_Ψ^Γ and η_Ψ^Γ, respectively, when they exist.

Let $G_\Omega(P, Q)$ (resp. $G_D(P, Q)$) be the Green function of a cone $C(\Omega)$ (resp. a cylinder $\Gamma(D)$) with pole at $P \in C(\Omega)$ (resp. $P \in \Gamma(D)$), and let $\partial G_\Omega(P, Q)/\partial n$ (resp. $\partial G_D(P, Q)/\partial n$) be the differentiation at $Q \in \partial C(\Omega) - \{O\}$ (resp. $Q \in \partial\Gamma(D)$) along the inward normal into $C(\Omega)$ (resp. $\Gamma(D)$). It follows from our assumption on Ω (resp. D) that $\partial G_\Omega(P, Q)/\partial n$ (resp. $\partial G_D(P, Q)/\partial n$) is continuous on $\partial C(\Omega) - \{O\}$ (resp. $\partial\Gamma(D)$) (see Gilbarg and Trudinger [**4**, Theorem 6.15]).

Let $g(Q)$ be a locally integrable function on $\partial C(\Omega) - \{O\}$ (resp. $\partial\Gamma(D)$) such that

$$\int^{+\infty} r^{-\alpha_\Omega - 1}\left(\int_{\partial\Omega} |g(r, \Theta)|\, d\sigma_\Theta\right) dr < +\infty, \tag{2.4}$$

$$\int_0 r^{\beta_\Omega - 1}\left(\int_{\partial\Omega} |g(r, \Theta)|\, d\sigma_\Theta\right) dr < +\infty,$$

(resp.

$$\int_{-\infty}^{+\infty} e^{-\sqrt{\lambda_D}|y|}\left(\int_{\partial D} |g(X, y)|\, d\sigma_X\right) dy < +\infty), \tag{2.5}$$

where $d\sigma_\Theta$ (resp. $d\sigma_X$) is the surface area element of $\partial\Omega$ (resp. ∂D) at $\Theta \in \partial\Omega$ (resp. $X \in \partial D$). If $m = 2$ and $\Omega = (\gamma, \delta)$ (resp. $D = (\gamma, \delta)$), then

$$\int_{\partial\Omega} |g(r, \Theta)|\, d\sigma_\Theta \quad \left(\text{resp. } \int_{\partial D} |g(X, y)|\, d\sigma_X\right)$$
$$= |g(r, \gamma)| + |g(r, \delta)| \quad (\text{resp. } |g(\gamma, y)| + |g(\delta, y)|).$$

The Poisson integral $\mathrm{PI}_g(P)$ (resp. $\mathrm{PI}_g^\Gamma(P)$) of g relative to $C(\Omega)$ (resp. $\Gamma(D)$) is defined as follows:

$$\mathrm{PI}_g(P) = \frac{1}{c_m}\int_{\partial C(\Omega) - \{O\}} g(Q) \frac{\partial}{\partial n} G_\Omega(P, Q)\, d\sigma_Q$$
$$\left(\text{resp. } \mathrm{PI}_g^\Gamma(P) = \frac{1}{c_m}\int_{\partial\Gamma(D)} g(Q) \frac{\partial}{\partial n} G_D(P, Q)\, d\sigma_Q\right),$$

where

$$c_m = \begin{cases} 2\pi & (m = 2), \\ (m-2)s_m & (m \geq 3) \end{cases}$$

and $d\sigma_Q$ is the surface area element on $\partial C(\Omega) - \{O\}$ (resp. $\partial\Gamma(D)$).

REMARK 2. Let $\Omega = \mathbb{S}_+^{m-1}$. Then

$$G_\Omega(P, Q) = \begin{cases} |P - Q|^{2-m} - |P - \overline{Q}|^{2-m} & (m \geq 3), \\ -\log|P - Q| + \log|P - \overline{Q}| & (m = 2), \end{cases}$$

where $\overline{Q} = (X, -y)$, that is, $\overline{Q}$ is the mirror image of $Q = (X, y)$ with respect to $\partial\mathbb{T}_m$. Hence, for two points $P = (X, y) \in \mathbb{T}_m$ and $Q \in \partial\mathbb{T}_m$,

$$\frac{\partial}{\partial n} G_\Omega(P, Q) = \begin{cases} 2(m-2)|P - Q|^{-m} y & (m \geq 3), \\ 2|P - Q|^{-2} y & (m = 2). \end{cases}$$

3. Statement of results. The following Theorem 1 is a fundamental result in this paper.

THEOREM 1. *Let* $g(Q)$ *be a locally integrable function on* $\partial C(\Omega) - \{O\}$ *satisfying* (2.4) *and let* $u(P)$ *be a subharmonic function on* $C(\Omega)$ *such that*

$$\overline{\lim}_{P\in C(\Omega),\, P\to Q}\{u(P) - \mathrm{PI}_g(P)\} \le 0 \tag{3.1}$$

and

$$\overline{\lim}_{P\in C(\Omega),\, P\to Q}\{u^+(P) - \mathrm{PI}_{|g|}(P)\} \le 0 \tag{3.2}$$

for any $Q \in \partial C(\Omega) - \{O\}$. *Then all of the limits* μ_{u^+}, η_{u^+}, μ_u *and* η_u ($0 \le \mu_{u^+}, \eta_{\mu^+} \le +\infty$, $-\infty < \mu_u, \eta_u \le +\infty$) *exist, and if*

$$\mu_{u^+} < +\infty \quad \textit{and} \quad \eta_{u^+} < +\infty, \tag{3.3}$$

then

$$u(P) \le \mathrm{PI}_g(P) + (\mu_u r^{\alpha_\Omega} + \eta_u r^{-\beta_\Omega}) f_\Omega(\Theta) \tag{3.4}$$

for any $P = (r, \Theta) \in C(\Omega)$.

REMARK 3. It is evident that (3.3) follows from

$$\underline{\lim}_{r\to\infty} r^{-\alpha_\Omega} M_u(r) < +\infty \quad \text{and} \quad \underline{\lim}_{r\to 0} r^{\beta_\Omega} M_u(r) < +\infty. \tag{3.5}$$

It is proved in Yoshida [**12**, Remark 9.1] that if

$$\overline{\lim}_{P\in C_m(\Omega),\, P\to Q} u(P) \le 0,$$

for any $Q \in \partial C(\Omega) - \{O\}$, (3.5) also follows from (3.3).

REMARK 4. If $u(P)$ is a positive harmonic function on $C(\Omega)$, then (3.3) is always satisfied. To see it, apply (I) of Lemma 2 (which will be stated in §4) to $-u(P)$. We immediately obtain that $-\infty < \mu_{-u}, \eta_{-u} \le +\infty$, so that $\mu_{u^+} = \mu_u < +\infty$ and $\eta_{u^+} = \eta_u < +\infty$.

The following Theorem 2 generalizes a result of Yoshida [**11**, Theorem 5].

THEOREM 2. *Let* $g(Q)$ *be a continuous function on* $\partial C(\Omega) - \{O\}$ *satisfying* (2.4) *and let* $u(P)$ *be a subharmonic function on* $C(\Omega)$ *such that*

$$\overline{\lim}_{P\in C(\Omega),\, P\to Q} u(P) \le g(Q) \tag{3.6}$$

for any $Q \in \partial C(\Omega) - \{O\}$. *Then all of the limits* μ_{u^+}, η_{u^+}, μ_u *and* η_u ($0 \le \mu_{u^+}$, $\eta_{u^+} \le +\infty$, $-\infty < \mu_u$, $\eta_u \le +\infty$) *exist, and if*

$$\mu_{u^+} < +\infty \quad \text{and } \eta_{u^+} < +\infty, \tag{3.7}$$

then

$$u(P) \le \mathrm{PI}_g(P) + (\mu_u r^{\alpha_\Omega} + \eta_u r^{-\beta_\Omega}) f_\Omega(\Theta) \tag{3.8}$$

for any $P = (r, \Theta) \in C(\Omega)$.

COROLLARY 1. *Let* $g(Q)$ *be a continuous function on* $\partial \mathbb{T}_m$ ($m \ge 2$) *such that*

$$\int^{+\infty} r^{-2} \left(\int_{\partial \mathbb{S}_+^{m-1}} |g(r, \Theta)| \, d\sigma_\Theta \right) dr < +\infty. \tag{3.9}$$

Let $u(P)$ *be a subharmonic function on* $\mathbb{T}_m$ *such that*

$$\overline{\lim_{P \in \mathbb{T}_m, P \to Q}} u(P) \le g(Q) \tag{3.10}$$

for any $Q \in \partial \mathbb{T}_m$. *Then both of the limits* μ_{u^+} ($0 \le \mu_{u^+} \le +\infty$) *and* μ_u ($-\infty < \mu_u \le +\infty$) *exist, and*

$$u(P) \le 2 s_m^{-1} \int_{\partial \mathbb{T}_m} g(Q) |P - Q|^{-m} \, d\sigma_Q + (2 m s_m^{-1})^{1/2} \mu_{u^+} y \tag{3.11}$$

for any $P = (X, y) \in \mathbb{T}_m$. *If*

$$\underline{\lim}_{r \to \infty} r^{-1} M_u(r) < +\infty,$$

then

$$u(P) \le 2 s_m^{-1} \int_{\partial \mathbb{T}_m} g(Q) |P - Q|^{-m} \, d\sigma_Q + (2 m s_m^{-1})^{1/2} \mu_u y \tag{3.12}$$

for any $P = (X, y) \in \mathbb{T}_m$.

REMARK 5. Let $f(z)$ be a regular function on $\mathbb{T}_2$. Put $m = 2$ and $u(P) = \log |f(z)|$ in Corollary 1. Then (3.9) is equal to (1.2). Since (1.3) gives

$$\mu_{\log^+ |f|} = 0,$$

(1.4) follows from (3.11). Since

$$\mu_{\log |f|} = \frac{2}{\pi} \lim_{r \to \infty} \frac{1}{r} \int_0^\pi \log |f(re^{i\theta})| \sin \theta \, d\theta = \frac{\pi}{2} a_f,$$

(3.12) gives (1.5). Thus we obtain Theorems A and B.

Next, to obtain Theorem C, put $m=3$ and $u=h$ in Corollary 1. From (3.11), we have

$$h(P) \leq \frac{y}{2\pi}\int_{\partial \mathbb{T}_3} g(Q)|P-Q|^{-3}\,d\sigma_\Theta + \left(\frac{3}{2\pi}\right)^{1/2}\mu_{h^+}y$$

for any $P=(X,y)\in \mathbb{T}_3$. Since

$$\mu_{h^+} = \left(\frac{3}{2\pi}\right)^{1/2} b_{h^+}$$

(Remark 1 with $m=3$), we immediately obtain Theorem C.

EXAMPLE 1. Let $\lambda_\Omega^{(2)}$ be the second least positive eigenvalue of (2.1) and let $F_\Omega(\Theta)$ be a normalized eigenfunction corresponding to $\lambda_\Omega^{(2)}$. Let A_Ω be the positive solution of the equation

$$t^2+(m-2)t-\lambda_\Omega^{(2)}=0.$$

The harmonic function

$$H(P)=r^{A_\Omega}F_\Omega(\Theta) \quad (P=(r,\Theta)\in C_m(\Omega))$$

on $\partial C(\Omega)$ has the property

$$\lim_{P\in C(\Omega),\,P\to Q} H(P)=0, \tag{3.13}$$

for any $Q\in\partial C(\Omega)-\{O\}$. Since $\lambda_\Omega^{(2)}>\lambda_\Omega^{(1)}$, it is evident that

$$\lim_{r\to\infty} r^{-\alpha_\Omega}M_H(r)=+\infty.$$

Hence it follows from Remark 3 that

$$\mu_{H^+}=+\infty. \tag{3.14}$$

This $H(P)$ shows that (3.6) with a continuous function on $\partial C(\Omega)-\{O\}$ satisfying (2.4) does not always give (3.7). Further, let $g(Q)$ be a continuous function on $\partial C(\Omega)-\{O\}$ satisfying (2.4). Put

$$I(P)=H(P)+\mathrm{PI}_g(P)$$

on $C(\Omega)$. Then we see from (3.13) that $I(P)$ is a harmonic function on $C(\Omega)$ satisfying

$$\lim_{P\in C(\Omega),\,P\to Q} I(P)=g(Q)$$

for any $Q \in \partial C(\Omega) - \{O\}$ (see Lemma 3 and Lemma 6). Hence (3.6) is valid for the function $g(Q)$ on $\partial C(\Omega) - \{O\}$. However it is easy to see that (3.8) is not true. Since $F_\Omega(\Theta)$ is orthogonal to $f_\Omega(\Theta)$ and

$$N_H(r) = 0 \qquad (0 < r < +\infty),$$

it follows from Lemma 3 that

$$\mu_I = \mu_H + \mu_{\mathrm{PI}_g} = 0, \quad \eta_I = \eta_H + \eta_{\mathrm{PI}_g} = 0.$$

Since

$$I^+(P) \geq H^+(P) - \mathrm{PI}_{|g|}(P)$$

on $C(\Omega)$, we see from (3.14) and Lemma 3 that

$$\mu_{I^+} \geq \mu_{H^+} = +\infty.$$

Hence this $I(P)$ shows that (3.8) does not always follow without (3.7).

EXAMPLE 2. There exists a subharmonic function $u(P)$ such that (3.7) is satisfied and (3.6) holds for no locally integrable function $g(Q)$ on $\partial C(\Omega) - \{O\}$ satisfying (2.4). Let ξ be a number satisfying $0 < \xi < \frac{\pi}{2}$ and let

$$\Omega = \left\{\Theta = (\theta_1, \theta_2, \ldots, \theta_{m-1}) \in \mathbb{S}^{m-1};\ |\theta_1| < \xi < \frac{\pi}{2}\right\}.$$

Consider the subharmonic function

$$v(r, \Theta) = r^{\alpha_\Omega}$$

on $C(\Omega)$ and any locally integrable function $g(Q)$ on $\partial C(\Omega) - \{O\}$ such that

$$\varlimsup_{P \in C(\Omega), P \to Q} v(r, \Theta) \leq g(Q)$$

at every $Q = (r, \Theta) \in \partial C(\Omega) - \{O\}$. Then we always have

$$\int^{+\infty} r^{-\alpha_\Omega - 1} \left(\int_{\partial\Omega} |g(r, \Theta)|\, d\sigma_\Theta \right) dr = +\infty.$$

On the other hand, we have that

$$\lim_{r \to \infty} r^{-\alpha_\Omega} M_v(r) = 1,$$

so that $\mu_{v^+} < +\infty$.

Let W be a domain in $\mathbb{R}^m$ and let $g(Q)$ be a function on ∂W. A harmonic function on W satisfying

$$\lim_{P \in W, P \to Q} h(P) = g(Q)$$

for any $Q \in \partial W$ is called the solution of the *classical Dirichlet problem* on W with g. In comparison with a result of Keller [**7**, Satz in p. 25], from Theorem 2 we obtain the following Theorem 3 which gives a kind of uniqueness of solutions of the classical Dirichlet problem on an unbounded domain $C(\Omega)$. It must be remarked that the classical Dirichlet problem on unbounded domains has no unique solution (e.g. see Helms [**6**, p. 42 and p. 158]).

THEOREM 3. *Let $g(Q)$ be a continuous function on $\partial C(\Omega) - \{O\}$ satisfying* (2.4)

(I) *The Poisson integral $\mathrm{PI}_g(P)$ is a solution of the classical Dirichlet problem on $C(\Omega)$ with g.*

(II) *Let $h(P)$ be any solution of the classical Dirichlet problem on $C(\Omega)$ with g. Then all of the limits μ_h, η_h $(-\infty < \mu_h, \eta_h \leq +\infty)$, $\mu_{|h|}$ and $\eta_{|h|}$ $(0 \leq \mu_{|h|}, \eta_{|h|} \leq +\infty)$ exist, and if*

$$\mu_{|h|} < +\infty \quad \text{and} \quad \eta_{|h|} < +\infty, \tag{3.15}$$

then

$$h(P) = \mathrm{PI}_g(P) + (\mu_h r^{\alpha_\Omega} + \eta_h r^{-\beta_\Omega}) f_\Omega(\Theta) \tag{3.16}$$

for any $P = (r, \Theta) \in C(\Omega)$.

REMARK 6. The harmonic function $I(P)$ in Example 1 is one of the solutions of the classical Dirichlet problem on $C(\Omega)$, which do not satisfy (3.15). In fact, (3.14) gives

$$\mu_{|I|} = \mu_{|\mathrm{PI}_g + H|} = +\infty,$$

because

$$\mu_{|\mathrm{PI}_g|} = 0$$

from Lemma 3 and

$$\mu_{|\mathrm{PI}_g + H|} \geq \mu_{|H|} - \mu_{|\mathrm{PI}_g|} \geq \mu_{H^+} - \mu_{|\mathrm{PI}_g|} = \mu_{H^+}.$$

COROLLARY 2. *Let $g(Q)$ be a continuous function on $\partial C(\Omega) - \{O\}$ satisfying* (2.4). *If $h(P)$ is a positive harmonic function on $C(\Omega)$ which is the solution of the classical Dirichlet problem on $C(\Omega)$ with g, then* (3.16) *holds.*

The following Theorem 4 generalizes a result of Yoshida [**12**, Corollary 5.2].

THEOREM 4. *Let u be subharmonic on a domain containing $\overline{C(\Omega)}-\{O\}$ and let*

$$u \geq 0 \quad on\ C(\Omega).$$

(I) *If $\tilde{u} = u|\partial C(\Omega) - \{O\}$ (the restriction of u to $\partial C(\Omega) - \{O\}$) satisfies* (2.4), *then both of the limits μ_n and η_u $(0 \leq \mu_n,\ \eta_u \leq +\infty)$ exist. Further, if*

$$\mu_u < +\infty \quad and \quad \eta_u < +\infty, \tag{3.17}$$

then

$$h_u(P) = \mathrm{PI}_{\tilde{u}}(P) + (\mu_u r^{\alpha_\Omega} + \eta_u r^{-\beta_\Omega}) f_\Omega(\Theta) \quad (P = (r, \Theta) \in C(\Omega))$$

is the least harmonic majorant of u on $C(\Omega)$.

(II) *If u possesses a harmonic majorant on $C(\Omega)$, then $\tilde{u}$ satisfies* (2.4) *and* (3.17) *holds.*

REMARK 7. When $u(P)$ satisfies the additional condition

$$\lim_{P \in C(\Omega),\, P \to Q} u(P) = 0$$

for any $Q \in \partial C(\Omega) - \{O\}$, we extend $u(P)$ to $\mathbb{R}^m - \{O\}$ by defining $u(P) = 0$ for any $P \in \mathbb{R}^m - C(\Omega) - \{O\}$. Then $u(P)$ is subharmonic on $\mathbb{R}^m - \{O\}$. From Remark 3 and (I) of Theorem 4, we obtain a result of Yoshida [**12**, Corollary 5.2].

COROLLARY 3. *Let u be subharmonic on a domain containing $\overline{\mathbb{T}_m}$ $(m \geq 2)$ and let*

$$u \geq 0 \quad on\ \mathbb{T}_m.$$

(I) *If $\tilde{u} = u|\partial \mathbb{T}_m$ satisfies*

$$\int^{+\infty} r^{-2} \left(\int_{\partial \mathbb{S}_+^{m-1}} \tilde{u}(r, \Theta)\, d\sigma_\Theta \right) dr < +\infty, \tag{3.18}$$

then the limit μ_u $(0 \leq \mu_u \leq +\infty)$ exists. Further, if

$$\mu_u < +\infty, \tag{3.19}$$

then

$$2s_m^{-1} y \int_{\partial \mathbb{T}_m} \tilde{u}(Q) |P - Q|^{-m}\, d\sigma_Q + (2m s_m^{-1})^{1/2} \mu_u y \tag{3.20}$$

is the least harmonic majorant of u on $\mathbb{T}_m$.

(II) *If* u *possesses a harmonic majorant on* $\mathbb{T}_m$, *then* $\tilde{u}$ *satisfies* (3.18) *and* (3.19) *holds.*

REMARK 8. Theorem D immediately follows from Corollary 3. In fact, u is bounded above on any compact subset of $\overline{T_m}$. Hence (3.19) is equivalent to (1.6). We also see from Remark 1 that

$$l_u = (2ms_m^{-1})^{1/2}\mu_u$$

and (3.20) is equal to (1.8).

Finally we shall state some results in the cylindrical case.

THEOREM 5. *Let* $g(Q)$ *be a continuous function on* $\partial\Gamma(D)$ *satisfying* (2.5) *and let* $u(P)$ *be a subharmonic function on* $\Gamma(D)$ *such that*

$$\overline{\lim_{P\in\Gamma(D),\,P\to Q}}\, u(P) \le g(Q)$$

for any $Q \in \partial\Gamma(D)$. *Then all of the limits* $\mu_{u^+}^\Gamma \eta_{u^+}^\Gamma \mu_u^\Gamma$ *and* η_u^Γ $(0 \le \mu_{u^+}^\Gamma, \eta_{u^+}^\Gamma \le +\infty, -\infty < \mu_u^\Gamma, \eta_u^\Gamma \le +\infty)$ *exist, and if*

$$\mu_{u^+}^\Gamma < +\infty \quad \text{and} \quad \eta_{u^+}^\Gamma < +\infty$$

then

$$u(P) \le \mathrm{PI}_g(P) + (\mu_u^\Gamma e^{\sqrt{\lambda_D}y} + \eta_u^\Gamma e^{-\sqrt{\lambda_D}y}) f_D(X)$$

for any $P = (X, y) \in \Gamma(D)$.

THEOREM 6. *Let* $g(Q)$ *be a continuous function on* $\partial\Gamma(D)$ *satisfying* (2.5).

(I) *The Poisson integral* $\mathrm{PI}_g^\Gamma(P)$ *is a solution of the classical Dirichlet problem on* $\Gamma(D)$ *with* g.

(II) *Let* $h(P)$ *be any solution of the classical Dirichlet problem on* $\Gamma(D)$ *with* g. *Then all of the limits* μ_h^Γ, η_h^Γ $(-\infty < \mu_h^\Gamma, \eta_h^\Gamma \le +\infty)$, $\mu_{|h|}^\Gamma$ *and* $\eta_{|h|}^\Gamma$ $(0 \le \mu_{|h|}^\Gamma, \eta_{|h|}^\Gamma \le +\infty)$ *exist, and if*

$$\mu_{|h|}^\Gamma < +\infty \quad \text{and} \quad \eta_{|h|}^\Gamma < +\infty,$$

then

$$h(P) = \mathrm{PI}_g^\Gamma(P) + (\mu_h^\Gamma e^{\sqrt{\lambda_D}y} + \eta_h^\Gamma e^{-\sqrt{\lambda_D}y}) f_D(X) \tag{3.21}$$

for any $P = (X, y) \in \Gamma(D)$.

COROLLARY 4. *Let* $g(Q)$ *be a continuous function on* $\partial\Gamma(D)$ *satisfying* (2.5). *If* $h(P)$ *is a positive harmonic function on* $\Gamma(D)$ *which* *is the solution of the classical Dirichlet problem on* $\Gamma(D)$ *with* g, *then* (3.21) *holds.*

THEOREM 7. *Let* u *be subharmonic on a domain containing* $\overline{\Gamma(D)}$ *and let*

$$u \geq 0 \quad \text{on } \Gamma(D).$$

(I) *If* $\tilde{u} = u|\partial\Gamma(D)$ (*the restriction of* u *to* $\partial\Gamma(D)$) *satisfies* (2.5), *then both of the limits* μ_u^Γ *and* η_u^Γ $(0 \leq \mu_u^\Gamma, \eta_u^\Gamma \leq +\infty)$ *exist. Further, if*

$$\mu_u^\Gamma < +\infty \quad \text{and} \quad \eta_u^\Gamma < +\infty, \tag{3.22}$$

then

$$\mathrm{PI}_{\tilde{u}}^\Gamma(P) + (\mu_u^\Gamma e^{\sqrt{\lambda_D} y} + \eta_u^\Gamma e^{-\sqrt{\lambda_D} y}) f_D(X) \qquad (P = (X, y) \in \Gamma(D))$$

is the least harmonic majorant of u *on* $\Gamma(D)$.

(II) *If* u *possesses a harmonic majorant on* $\Gamma(D)$, *then* $\tilde{u}$ *satisfies* (2.5) *and* (3.22) *holds.*

4. Proof of Theorem 1. For a domain $\Omega \subset \mathbb{S}^{m-1}$ $(m \geq 2)$ and a number t $(0 < t < +\infty)$, the sets

$$\{(r, \Theta) \in \mathbb{R}^m;\ 0 < r \leq t,\ \Theta \in \partial\Omega\} \quad \text{and}$$
$$\{(r, \Theta) \in \mathbb{R}^m;\ r \geq t,\ \Theta \in \partial\Omega\}$$

are denoted by $S_\Omega^-(t)$ and $S_\Omega^+(t)$, respectively. For two numbers t_1 and t_2 $(0 < t_1 < t_2 < +\infty)$, let $S_\Omega(t_1, t_2)$ denote the set

$$\{(r, \Theta) \in \mathbb{R}^m;\ t_1 \leq r \leq t_2,\ \Theta \in \partial\Omega\}.$$

For a point $Q \in \mathbb{R}^m$, the set $\{P \in \mathbb{R}^m;\ |P - Q| < \rho\}$ $(\rho > 0)$ is represented by $U_\rho(Q)$. We write $G_\Omega^j(P, Q)$ for the Green function of

$$C^j(\Omega) = (j^{-1}, j) \times \Omega \qquad (j \text{ is a positive integer})$$

with pole at P. For an upper semicontinuous function $\phi(Q)$ on $\partial C^j(\Omega)$, the Perron-Wiener-Brelot solution of the Dirichlet problem with respect to $C^j(\Omega)$ is denoted by $H_\phi^j(P)$ (e.g. see Helms **[6]**). Since the harmonic measure $\omega(P, E)$ of $E \subset \partial C^j(\Omega)$ with respect to $C^j(\Omega)$ is equal to

$$c_m^{-1} \int_E \frac{\partial}{\partial n} G_\Omega^j(P, Q)\, d\sigma_Q$$

(see Dahlberg [3, Theorem 3]), we know that $H^j_\phi(P)$ is equal to

$$c_m^{-1}\int_{S(j^{-1},j)\cup(\{j^{-1}\}\times\Omega)\cup(\{j\}\times\Omega)}\phi(Q)\frac{\partial}{\partial n}G^j_\Omega(P,Q)\,d\sigma_Q.$$

To prove Theorem 1, we need some lemmas.

LEMMA 1. *There exist two positive constants* k_1 *and* k_2 (*resp.* k_3 *and* k_4) *such that*

$$\begin{aligned}k_1 r^{\alpha_\Omega}t^{-\beta_\Omega-1}f_\Omega(\Theta)\qquad &(\textit{resp. } k_3 r^{-\beta_\Omega}t^{\alpha_\Omega-1}f_\Omega(\Theta))\\ \le \frac{\partial}{\partial n}G_\Omega(P,Q)&\le k_2 r^{\alpha_\Omega}t^{-\beta_\Omega-1}f_\Omega(\Theta)\\ &\qquad(\textit{resp. } k_4 r^{-\beta_\Omega}t^{\alpha_\Omega-1}f_\Omega(\Theta))\end{aligned}$$

for $P=(r,\Theta)\in C(\Omega)$ *and* $Q=(t,\Phi)\in\partial C(\Omega)-\{O\}$ *satisfying* $0<r<\frac{1}{2}t$ (*resp.* $0<t<\frac{1}{2}r$).

Proof. These immediately follow from Azarin's inequalities [**1**, Lemma 1] and (2.3).

LEMMA 2 (*Yoshida* [**12**, *Theorem* 3.31]). *Let* $u(P)$ *be a subharmonic function on* $C(\Omega)$ $(m\ge 2)$ *such that*

$$\varlimsup_{P\in C(\Omega),\,P\to Q}u(P)\le 0$$

for any $Q\in\partial C(\Omega)-\{O\}$.

(I) *Both of the limits* μ_u *and* η_u $(-\infty<\mu_u,\eta_u\le+\infty)$ *exist.*

(II) *If* $\eta_u\le 0$, *then* $r^{-\alpha_\Omega}N_u(r)$ *is non-decreasing on* $(0,+\infty)$.

(III) *If* $\mu_u\le 0$, *then* $r^{\beta_\Omega}N_u(r)$ *is non-increasing on* $(0,+\infty)$.

LEMMA 3. *Let* $g(Q)$ *be a locally integrable function on* $\partial C(\Omega)-\{O\}$ *satisfying* (2.4). *Then* $\mathrm{PI}_{|g|}(P)$ (*resp.* $\mathrm{PI}_g(P)$) *is a harmonic function on* $C(\Omega)$ *such that both of the limits* $\mu_{\mathrm{PI}_{|g|}}$ *and* $\eta_{\mathrm{PI}_{|g|}}$ (*resp.* μ_{PI_g} *and* η_{PI_g}) *exist, and*

$$\mu_{\mathrm{PI}_{|g|}}=\eta_{\mathrm{PI}_{|g|}}=0\qquad(\textit{resp. } \mu_{\mathrm{PI}_g}=\eta_{\mathrm{PI}_g}=0).$$

Proof. Take any $P=(r,\Theta)\in C(\Omega)$ and two numbers R_1, R_2 $(R_1<\frac{1}{2}r,\ R_2>2r)$. Then by Lemma 1

$$\begin{aligned}(4.1)\qquad c_m^{-1}&\int_{S^+_\Omega(R_2)}|g(Q)|\frac{\partial}{\partial n}G_\Omega(P,Q)\,d\sigma_Q\\ &\le k_5\int_{R_2}^{+\infty}t^{-\alpha_\Omega-1}\left(\int_{\partial\Omega}|g(t,\Phi)|\,d\sigma_\Phi\right)dt,\end{aligned}$$

where $k_5 = k_2 c_m^{-1} r^{\alpha_\Omega} f_\Omega(\Theta)$, and

$$(4.2) \qquad c_m^{-1} \int_{S_\Omega^-(R_1)} |g(Q)| \frac{\partial}{\partial n} G_\Omega(P, Q)\, d\sigma_Q \le k_6 \int_0^{R_1} t^{\beta_\Omega - 1} \left(\int_{\partial\Omega} |g(t, \Phi)|\, d\sigma_\Phi \right) dt$$

where $k_6 = k_4 c_m^{-1} r^{-\beta_\Omega} f_\Omega(\Theta)$. Hence we see from (2.4) that $\mathrm{PI}_{|g|}(P)$ and $\mathrm{PI}_g(P)$ are finite for any $P \in C(\Omega)$. Thus $\mathrm{PI}_g(P)$ and $\mathrm{PI}_{|g|}(P)$ are harmonic on $C(\Omega)$.

Let $\nu_{R,P}^{(1)}(E)$ and $\nu_{R,P}^{(2)}(E)$ $(0 < R < +\infty,\ P \in C(\Omega))$ be two positive measures on $\partial C(\Omega) - \{O\}$ such that

$$\nu_{R,P}^{(1)}(E) = c_m^{-1} \int_{E \cap S_\Omega^+(R)} \frac{\partial}{\partial n} G_\Omega(P, Q)\, d\sigma_Q$$

and

$$\nu_{R,P}^{(2)}(E) = c_m^{-1} \int_{E \cap S_\Omega^-(R)} \frac{\partial}{\partial n} G_\Omega(P, Q)\, d\sigma_Q$$

for every Borel subset E of $\partial C(\Omega) - \{O\}$. Then $\mathrm{PI}_{|g|}(P)$ is the sum of two positive harmonic functions:

$$(4.3) \qquad \mathrm{PI}_{|g|}(P) = h_{1,R}(P) + h_{2,R}(P),$$

where

$$h_{1,R}(P) = \int_{\partial C(\Omega) - \{O\}} |g|\, d\nu_{R,P}^{(1)}$$

and

$$h_{2,R}(P) = \int_{\partial C(\Omega) - \{O\}} |g|\, d\nu_{R,P}^{(2)}.$$

Let r_1 $(r_1 > 0)$ be a number and let ε be any positive number. From (2.4) we can choose a number r^* $(r^* > 2r_1)$ so large that

$$(4.4) \qquad \int_{S_\Omega^+(r^*)} |g(t, \Phi)| t^{-\beta_\Omega - 1}\, d\sigma_Q \le \frac{c_m}{2k_2} \varepsilon \qquad (Q = (t, \Phi)).$$

By applying Lemma 1, we see from (4.4) that

$$N_{h_{1,r^*}}(r_1) \le \frac{1}{2} \varepsilon r_1^{\alpha_\Omega}$$

and hence

$$(4.5) \qquad r_1^{-\alpha_\Omega} N_{h_{1,r^*}}(r_1) \ge -\frac{1}{2} \varepsilon.$$

Since

$$r^{-\alpha_\Omega} N_{h_{1,r^*}}(r)$$

is non-decreasing from (II) of Lemma 2, (4.5) gives that

$$(4.6) \qquad 0 \leq r^{-\alpha_\Omega} N_{h_{1,r^*}}(r) \leq \frac{1}{2}\varepsilon \qquad (r \geq r_1).$$

By using Lemma 1 again, we obtain that

$$N_{h_{2,r^*}}(r) \leq k_4 r^{-\beta_\Omega} \int_0^{r^*} t^{\beta_\Omega - 1} \left(\int_{\partial\Omega} |g(t, \Phi)|\, d\Phi \right) dt \qquad (r > 2r^*).$$

By (2.4) we can choose a number r_2 $(r_2 > 2r^*)$ so large that

$$(4.7) \qquad 0 \leq r^{-\alpha_\Omega} N_{h_{2,r^*}}(r) \leq \frac{1}{2}\varepsilon \qquad (r \geq r_2).$$

We finally conclude from (4.3), (4.6) and (4.7) that

$$0 \leq r^{-\alpha_\Omega} N_{\mathrm{PI}_{|g|}}(r) \leq \varepsilon \qquad (r \geq r_2),$$

which gives the eixstence of $\mu_{\mathrm{PI}_{|g|}}$ and

$$(4.8) \qquad \mu_{\mathrm{PI}_{|g|}} = 0.$$

In the same way we can also prove the existence of $\eta_{\mathrm{PI}_{|g|}}$ and

$$(4.9) \qquad \eta_{\mathrm{PI}_{|g|}} = 0.$$

Since

$$N_{\mathrm{PI}_{|g|}}(r) \geq N_{|\mathrm{PI}_g|}(r) \geq |N_{\mathrm{PI}_g}(r)| \qquad (0 < r < +\infty),$$

it immediately follows from (4.8) and (4.9) that both limits μ_{PI_g} and η_{PI_g} exist and are zero.

LEMMA 4 (*Yoshida* [**12**, *Theorem* 5.1] *and Remark* 3). *Let $u(P)$ be a subharmonic function on $C(\Omega)$ $(m \geq 2)$ such that*

$$\varlimsup_{P \in C(\Omega),\, P \to Q} u(P) \leq 0$$

for every $Q \in \partial C(\Omega) - \{O\}$. If (3.3) is satisfied, then

$$u(r, \Theta) \leq (\mu_u r^{\alpha_\Omega} + \eta_u r^{-\beta_\Omega}) f_\Omega(\Theta) \quad \text{on } C(\Omega).$$

Proof of Theorem 1. Consider two subharmonic functions

$$U(P) = u(P) - \mathrm{PI}_g(P) \quad \text{and} \quad U^*(P) = u^+(P) - \mathrm{PI}_{|g|}(P)$$

on $C(\Omega)$. Then we have from (3.1) and (3.2) that

$$\varlimsup_{P \in C(\Omega),\, P \to Q} U(P) \leq 0 \quad \text{and} \quad \varlimsup_{P \in C(\Omega),\, P \to Q} U^*(P) \leq 0$$

for every $Q \in \partial C(\Omega) - \{O\}$. Hence it follows from (I) of Lemma 2 that four limits μ_U, η_U, μ_{U^*} and η_{U^*} $(-\infty < \mu_U, \eta_U, \mu_{U^*}, \eta_{U^*} \leq +\infty)$ exist. Since

$$N_U(r) = N_u(r) - N_{\mathrm{PI}_g}(r) \quad \text{and} \quad N_{U^*}(r) = N_{u^+}(r) - N_{\mathrm{PI}_{|g|}}(r),$$

Lemma 3 gives the existence of four limits μ_u, η_u, μ_{u^+} and η_{u^+}, and that

$$\mu_U = \mu_u, \quad \eta_U = \eta_u, \quad \mu_{U^*} = \mu_{u^+}, \quad \eta_{U^*} = \eta_{u^+}. \tag{4.10}$$

Since

$$U^+(P) \leq u^+(P) + (\mathrm{PI}_g)^-(P) \quad \text{on } C(\Omega),$$

it also follows from Lemma 3 and (3.3) that

$$\mu_{U^+} \leq \mu_{u^+} < +\infty, \quad \eta_{U^+} \leq \eta_{u^+} < +\infty.$$

Hence by applying Lemma 4 to U, we can obtain from (4.10) that

$$U(P) \leq \mathrm{PI}_g(P) + (\mu_u r^{\alpha_\Omega} + \eta_u r^{-\beta_\Omega}) f_\Omega(\Theta) \quad \text{on } C(\Omega) \qquad (P = (r, \Theta)),$$

which is (3.4).

5. Proofs of Theorems 2 and 3, Corollaries 1 and 2. The following lemma is not obvious for unbounded functions.

LEMMA 5. *Let $g(Q)$ be an upper semicontinuous function on $\partial C(\Omega) - \{O\}$ satisfying* (2.4). *Then*

$$\overline{\lim_{P \in C(\Omega), P \to Q}} \mathrm{PI}_g(P) \leq g(Q)$$

for any $Q \in \partial C(\Omega) - \{O\}$,

Proof. Let $Q^* = (r^*, \Theta^*)$ be any point of $\partial C(\Omega) - \{O\}$ and let ε be any positive number. Take a number δ $(0 < \delta < r^*)$. From (2.4), we can choose a number R_2^*, $R_2^* > 2(r^* + \delta)$ (resp. R_1^*, $0 < R_1^* < \frac{1}{2}(r^* - \delta)$) so large (resp. small) that

$$\int_{R_2^*}^{+\infty} t^{-\alpha_\Omega - 1} \left(\int_{\partial\Omega} |g(t, \Phi)| \, d\sigma_\Phi \right) dt < \frac{c_m}{8k_2 K_\Omega} (r^* + \delta)^{-\alpha_\Omega} \varepsilon$$

$$\left(\text{resp. } \int_0^{R_1^*} t^{\beta_\Omega - 1} \left(\int_{\partial\Omega} |g(t, \Phi)| \, d\sigma_\Phi \right) | \, dt < \frac{c_m}{8k_4 K_\Omega} (r^* - \delta)^{\beta_\Omega} \varepsilon \right),$$

where

$$K_\Omega = \max_{\Theta \in \Omega} f_\Omega(\Theta).$$

From (4.1) and (4.2), we obtain that

$$(5.1)\qquad c_m^{-1}\int_{S_\Omega^+(R_2^*)}|g(Q)|\frac{\partial}{\partial n}G_\Omega(P, Q)\,d\sigma_Q<\frac{\varepsilon}{8}$$

and

$$(5.2)\qquad c_m^{-1}\int_{S_\Omega^-(R_1^*)}|g(Q)|\frac{\partial}{\partial n}G_\Omega(P, Q)\,d\sigma_Q<\frac{\varepsilon}{8}$$

for any $P=(r, \Theta)\in C(\Omega)\cap U_\delta(Q^*)$. Let φ be a continuous function on $\partial C(\Omega)-\{O\}$ such that $0\le\varphi\le 1$ on $\partial C(\Omega)-\{O\}$ and

$$\varphi=\begin{cases}1 & \text{on } S_\Omega(R_1^*, R_2^*),\\ 0 & \text{on } S_\Omega^+(2R_2^*)\cup S_\Omega^-(\frac{1}{2}R_1^*).\end{cases}$$

Since the positive harmonic function $G_\Omega(P, Q)-G_\Omega^j(P, Q)$ on $C^j(\Omega)$ converges monotonically to 0 as $j\to\infty$, we can find an integer j_0 ($j_0^{-1}<2^{-1}R_1^*$, $j_0>2R_2^*$) such that

$$(5.3)\qquad c_m^{-1}\int_{S_\Omega(2^{-1}R_1^*, 2R_2^*)}|\varphi(Q)g(Q)| \times\left|\frac{\partial}{\partial n}G_\Omega^{j_0}(P, Q)-\frac{\partial}{\partial n}G_\Omega(P, Q)\right|d\sigma_Q<\frac{\varepsilon}{4}$$

for any $P=(r, \Theta)\in C(\Omega)\cap U_\delta(Q^*)$. It follows from (5.1), (5.2) and (5.3) that

$$
\begin{aligned}
(5.4)\quad & c_m^{-1}\int_{\partial C(\Omega)-\{O\}}g(Q)\frac{\partial}{\partial n}G_\Omega(P, Q)\,d\sigma_Q\\
&\le c_m^{-1}\int_{S_\Omega(2^{-1}R_1^*, 2R_2^*)}\varphi(Q)g(Q)\frac{\partial}{\partial n}G_\Omega^{j_0}(P, Q)\,d\sigma_Q\\
&\quad+\left|c_m^{-1}\int_{S_\Omega(2^{-1}R_1^*, 2R_2^*)}\varphi(Q)g(Q)\frac{\partial}{\partial n}G_\Omega^{j_0}(P, Q)\,d\sigma_Q\right.\\
&\quad\left.-c_m^{-1}\int_{S_\Omega(2^{-1}R_1^*, 2R_2^*)}\varphi(Q)g(Q)\frac{\partial}{\partial n}G_\Omega(P, Q)\,d\sigma_Q\right|\\
&\quad+2c_m^{-1}\int_{S_\Omega^+(R_2^*)}|g(Q)|\frac{\partial}{\partial n}G_\Omega(P, Q)\,d\sigma_Q\\
&\quad+2c_m^{-1}\int_{S_\Omega^-(R_1^*)}|g(Q)|\frac{\partial}{\partial n}G_\Omega(P, Q)\,d\sigma_Q\\
&< c_m^{-1}\int_{S_\Omega(2^{-1}R_1^*, 2R_2^*)}\varphi(Q)g(Q)\frac{\partial}{\partial n}G_\Omega^{j_0}(P, Q)\,d\sigma_Q\\
&\quad+\frac{3}{4}\varepsilon
\end{aligned}
$$

for any $P = (r, \Theta) \in C(\Omega) \cap U_\delta(Q^*)$. Consider the upper semicontinuous function

$$V(Q) = \begin{cases} \varphi(Q)g(Q) & \text{on } S_\Omega(2^{-1}R_1^*, 2R_2^*), \\ 0 & \text{on } Z \end{cases}$$

$$(Z = S_\Omega(j_0^{-1}, 2^{-1}R_1^*) \cup S_\Omega(2R_2^*, j_0) \cup (\{j_0^{-1}\} \times \Omega) \cup (\{j_0\} \times \Omega))$$

on $\partial C^{j_0}(\Omega)$. Since

$$\varlimsup_{p \in C(\Omega), P \to Q^*} H_V^j(P) \leq \varlimsup_{Q \in \partial C(\Omega) - \{O\}, Q \to Q^*} V(Q) = g(Q^*)$$

(e.g. see Helms [6, Lemma 8.20]), we finally obtain from (5.4) that

$$\varlimsup_{P \in C(\Omega), P \to Q^*} c_m^{-1} \int_{\partial C(\Omega) - \{O\}} g(Q) \frac{\partial}{\partial n} G_\Omega(P, Q)\, d\sigma_Q \leq g(Q^*).$$

From Lemma 5, immediately follows

LEMMA 6. *If $g(Q)$ is a continuous function on $\partial C(\Omega) - \{O\}$ satisfying* (2.4), *then*

$$\lim_{P \in C(\Omega), P \to Q} \mathrm{PI}_g(P) = g(Q)$$

for every $Q \in \partial C(\Omega) - \{O\}$.

Proof of Theorem 2. First, we see from Lemma 6 that

$$\lim_{P \in C(\Omega), P \to Q} \mathrm{PI}_g(P) = g(Q) \quad \text{and} \quad \lim_{P \in C(\Omega), P \to Q} \mathrm{PI}_{|g|}(P) = |g(Q)|$$

for every $Q \in \partial C(\Omega) - \{O\}$. Hence we see from (3.6) that

$$\varlimsup_{P \in C(\Omega), P \to Q} \{u(P) - \mathrm{PI}_g(P)\} \leq 0$$

and

$$\varlimsup_{P \in C(\Omega), P \to Q} \{u^+(P) - \mathrm{PI}_{|g|}(P)\} \leq 0$$

for every $Q \in \partial C(\Omega) - \{O\}$. Theorem 1 immediately gives Theorem 2.

Proof of Corollary 1. Put $\Omega = \mathbb{S}_+^{m-1}$ in Theorem 2. Since $g(Q)$ is continuous at $Q = O$ of $\partial \mathbb{T}_m$, $|g(Q)|$ is bounded in the neighborhood of $Q = O$. Hence we see from Remark 1 and (3.9) that $g(Q)$ is admissible on $\partial \mathbb{T}_m$ and from (3.10) that $\eta_u \leq \eta_{u^+} = 0$. If $\mu_{u^+} = +\infty$, then (3.11) is evidently satisfied. When $\mu_{u^+} < +\infty$, (3.11) also follows from (3.8), Remark 1, Remark 2 and the inequality $\mu_u \leq \mu_{u^+}$. It is easily seen that Remark 3 and (3.8) give (3.12).

Proof of Theorem 3. It follows from Lemma 3 and Lemma 6 that $\mathrm{PI}_g(P)$ is one of the solutions. To prove (II), put $u(P) = h(P)$ and $-h(P)$ in Theorem 2. Then Theorem 2 gives the existence of all limits μ_h, η_h, μ_h^+, η_h^+,

$$(5.5) \qquad \mu_{(-h)^+} = \mu_{h^-} \quad \text{and} \quad \eta_{(-h)^+} = \eta_{h^-}.$$

Since

$$(5.6) \qquad \mu_{h^+} + \mu_{h^-} = \mu_{|h|} \quad \text{and} \quad \eta_{h^+} + \eta_{h^-} = \eta_{|h|},$$

it follows that both limits $\mu_{|h|}$ and $\eta_{|h|}$ exist. Suppose that h satisfies (3.15). Then we see from (5.5) and (5.6) that μ_{h^+}, $\mu_{(-h)^+}$, η_{h^+} and $\eta_{(-h)^+} < +\infty$. Hence, by applying Theorem 2 to $u(P) = h(P)$ and $-h(P)$ again, we obtain from (3.8) that

$$h(P) \leq \mathrm{PI}_g(P) + (\mu_h r^{\alpha_\Omega} + \eta_h r^{-\beta_\Omega}) f_\Omega(\Theta)$$

and

$$h(P) \geq \mathrm{PI}_g(P) + (\mu_h r^{\alpha_\Omega} + \eta_h r^{-\beta_\Omega}) f_\Omega(\Theta),$$

respectively, which give (3.16).

Proof of Corollary 2. It follows from Remark 4 that

$$\mu_{|h|} = \mu_{h^+} < +\infty \quad \text{and} \quad \eta_{|h|} = \eta_{h^+} < +\infty.$$

Thus Theorem 3 implies Corollary 2.

6. Proof of Theorem 4.

LEMMA 7. *Let $g(Q)$ be a non-negative lower semicontinuous function on $\partial C(\Omega) - \{O\}$ satisfying* (2.4) *and let $u(P)$ be a non-negative subharmonic function on $C(\Omega)$ such that*

$$(6.1) \qquad \overline{\lim_{P \in C(\Omega),\, P \to Q}}\, u(P) \leq g(Q)$$

for every $Q \in \partial C(\Omega) - \{O\}$. Then both of the limits μ_u and η_u $(0 \leq \mu_u,\ \eta_u \leq +\infty)$ exist, and if $\mu_u < +\infty$ and $\eta_u < +\infty$, then

$$u(P) \leq \mathrm{PI}_g(P) + (\mu_u r^{\alpha_\Omega} + \eta_u r^{-\beta_\Omega}) f_\Omega(\Theta)$$

for any $P = (r, \Theta) \in C(\Omega)$.

Proof. To apply Theorem 1, we shall show that (3.1) and (3.2) hold. Since $-g(Q)$ is upper semicontinuous on $\partial C(\Omega) - \{O\}$, it follows from Lemma 5 that

$$\underline{\lim}_{P \in C(\Omega), P \to Q} \mathrm{PI}_g(P) \geq g(Q) \tag{6.2}$$

for every $Q \in \partial C(\Omega) - \{O\}$. Hence we see from (6.1) and (6.2) that

$$\begin{aligned} &\overline{\lim}_{P \in C(\Omega), P \to Q} \{u(P) - \mathrm{PI}_g(P)\} \\ &\quad \leq \overline{\lim}_{P \in C(\Omega), P \to Q} u(P) - \underline{\lim}_{P \in C(\Omega), P \to Q} \mathrm{PI}_g(P) \leq g(Q) - g(Q) = 0 \end{aligned}$$

for every $Q \in \partial C(\Omega) - \{O\}$, which provides (3.1). Since g and u are non-negative, (3.2) also holds. Thus we obtain Lemma 7 from Theorem 1.

LEMMA 8. *Let u be subharmonic on a domain containing $\overline{C(\Omega)} - \{O\}$ such that $\tilde{u} = u|\partial C(\Omega) - \{O\}$ satisfies* (2.4) *and*

$$u \geq 0 \quad \text{on } C(\Omega).$$

Then

$$\mathrm{PI}_{\tilde{u}}(P) \leq h(P) \quad \text{on } C(\Omega)$$

for every harmonic majorant h of u on $C(\Omega)$.

Proof. Take any $P^* = (r^*, \Theta^*) \in C(\Omega)$. Let ε be any positive number. In the same way as in the proof of Lemma 5, we can choose two numbers R_1 and R_2 $(2R_1 < r < 2^{-1}R_2)$ such that

$$c_m^{-1} \int_{S_\Omega^+(R_2)} \tilde{u}(Q) \frac{\partial}{\partial n} G_\Omega(P^*, Q)\, d\sigma_Q < \frac{\varepsilon}{3} \tag{6.3}$$

and

$$c_m^{-1} \int_{S_\Omega^-(R_1)} \tilde{u}(Q) \frac{\partial}{\partial n} G_\Omega(P^*, Q)\, d\sigma_Q < \frac{\varepsilon}{3}. \tag{6.4}$$

Further, take an integer j_0 $(j_0^{-1} < R_1$ and $j_0 > R_2)$ such that

$$\begin{aligned} c_m^{-1} \int_{S_\Omega(R_1, R_2)} \tilde{u}(Q) \Big\{ &\frac{\partial}{\partial n} G_\Omega(P^*, Q) \\ &- \frac{\partial}{\partial n} G_\Omega^{j_0}(P^*, Q) \Big\}\, d\sigma_Q < \frac{\varepsilon}{3}. \end{aligned} \tag{6.5}$$

Since

$$c_m^{-1}\int_{S_\Omega(R_1,R_2)}\tilde{u}(Q)\frac{\partial}{\partial n}G_\Omega^{j_0}(P,Q)\,d\sigma_Q\le H_u^{j_0}(P)$$

for any $P\in C^{j_0}(\Omega)$, we have from (6.3), (6.4) and (6.5) that

$$\begin{aligned}(6.6)\qquad &\mathrm{PI}_{\tilde{u}}(P^*)-H_u^{j_0}(P^*)\\ &\le c_m^{-1}\int_{S_\Omega(R_1,R_2)}\tilde{u}(Q)\left\{\frac{\partial}{\partial n}G_\Omega(P^*,Q)\right.\\ &\qquad\left.-\frac{\partial}{\partial n}G_\Omega^{j_0}(P^*,Q)\right\}d\sigma_Q\\ &\quad+c_m^{-1}\int_{S_\Omega^+(R_2)}\tilde{u}(Q)\frac{\partial}{\partial n}G_\Omega(P^*,Q)\,d\sigma_Q\\ &\quad+c_m^{-1}\int_{S_\Omega^-(R_1)}\tilde{u}(Q)\frac{\partial}{\partial n}G_\Omega(P^*,Q)\,d\sigma_Q<\varepsilon.\end{aligned}$$

Here, note that $H_u^{j_0}(P)$ is the least harmonic majorant of $u(P)$ on $C^{j_0}(\Omega)$ (see Hayman [**5**, Theorem 3.15]). If h is a harmonic majorant of u on $C(\Omega)$, then

$$H_u^{j_0}(P^*)\le h(P^*).$$

Thus we obtain from (6.6) that

$$\mathrm{PI}_{\tilde{u}}(P^*)<h(P^*)+\varepsilon,$$

which gives the conclusion of Lemma 8.

Proof of Theorem 4. Let $P=(r,\Theta)$ be any point of $C(\Omega)$ and let ε be any positive number. By the Vitali-Carathéodory theorem (e.g. see [**11**, p. 56]), we can find a lower semicontinuous function $g_\varepsilon(Q)$ on $\partial C(\Omega)-\{O\}$ such that

$$(6.7)\qquad \tilde{u}(Q)\le g_\varepsilon(Q)\quad \text{on } \partial C(\Omega)-\{O\}$$

and

$$(6.8)\qquad \mathrm{PI}_{g_\varepsilon}(P)<\mathrm{PI}_{\tilde{u}}(P)+\varepsilon.$$

Since

$$\varlimsup_{P\in C(\Omega),\,P\to Q}u(P)\le\tilde{u}(Q)\le g_\varepsilon(Q)$$

for any $q\in\partial C(\Omega)-\{O\}$ from (6.7), it follows from Lemma 7 that two limits μ_u, η_u exist and if $\mu_u<+\infty$ and $\eta_u<+\infty$, then

$$(6.9)\qquad u(P)\le\mathrm{PI}_{g_\varepsilon}(P)+(\mu_u r^{\alpha_\Omega}+\eta_u r^{-\beta_\Omega})f_\Omega(\Theta).$$

Hence we have from (6.8) and (6.9) that

$$u(P) \le \mathrm{PI}_{\tilde{u}}(P) + \varepsilon + (\mu_u r^{\alpha_\Omega} + \eta_u r^{-\beta_\Omega}) f_\Omega(\Theta).$$

Since ε was arbitrary, we obtain

$$u(P) \le \mathrm{PI}_{\tilde{u}}(P) + (\mu_u r^{\alpha_\Omega} + \eta_u r^{-\beta_\Omega}) f_\Omega(\Theta)$$

for any $P = (r, \Theta) \in C(\Omega)$. This shows that $h_u(P)$ is a harmonic majorant of u on $C(\Omega)$.

To prove that h_u is the least harmonic majorant of u on $C(\Omega)$, let $h(P)$ be any harmonic function on $C(\Omega)$ such that

$$u(P) \le h(P) \quad \text{on } C(\Omega). \tag{6.10}$$

Consider the harmonic function

$$h^*(p) = h_u(P) - h(P) \quad \text{on } C(\Omega).$$

Since

$$h^*(P) \le h_u(P) \quad \text{on } C(\Omega),$$

we see from Lemma 3 that $h^*(P)$ satisfies (3.3). We also see from Lemma 8 that

$$\overline{\lim}_{P\in C(\Omega),\, P\to Q} h^*(P) = \overline{\lim}_{P\in C(\Omega),\, P\to Q} \{\mathrm{PI}_{\tilde{u}}(P) - h(P)\} \le 0$$

for any $Q \in \partial C(\Omega) - \{O\}$. We have from Lemma 3 and (6.10) that

$$\mu_{h^*} = \mu_{h_u} - \mu_h = \mu_u - \mu_h \le \mu_u - \mu_u = 0$$

and similarly $\eta_{h^*} \le 0$. Thus we obtain from Lemma 4 that

$$h^*(P) \le 0 \quad \text{on } C(\Omega),$$

which shows that $h_u(P)$ is the least harmonic majorant of $u(P)$ on $C(\Omega)$.

To prove (II), let $h_1(P)$ be a harmonic majorant of $u(P)$ on $C(\Omega)$. Since

$$\mu_u \le \mu_{h_1} < +\infty \quad \text{and} \quad \eta_u \le \eta_{h_1} < +\infty$$

from Remark 4, we immediately have (3.17). Fix $P_0 = (1, \Theta_0)$, $\Theta_0 \in \Omega$. Take any two numbers R_1, R_2 $(0 < R_1 < 2^{-1}, 2 < R_2 < +\infty)$ and choose a sufficiently large integer j^*, $j^* > \mathrm{Max}(R_1^{-1}, R_2)$, such that

$$c_m^{-1} \int_{S_\Omega(R_1, 2^{-1})} \tilde{u}(Q) \left\{ \frac{\partial}{\partial n} G_\Omega(P_0, Q) - \frac{\partial}{\partial n} G_\Omega^{j^*}(P_0, Q) \right\} d\sigma_Q \le 1$$

and

$$c_m^{-1}\int_{S_\Omega(2,R_2)} \tilde{u}(Q)\left\{\frac{\partial}{\partial n}G_\Omega(P_0, Q) - \frac{\partial}{\partial n}G_\Omega^{j^*}(P_0, Q)\right\} d\sigma_Q \le 1.$$

Since $H_u^{j^*}(P)$ is the least harmonic majorant of $u(P)$ on $C^{j^*}(\Omega)$,

$$\begin{aligned} h_1(P_0) \ge H_u^{j^*}(P) &\ge c_m^{-1}\int_{S_\Omega(j^{*-1}, j^*)} \tilde{u}(Q)\frac{\partial}{\partial n}G_\Omega^{j^*}(P_0, Q)\, d\sigma_Q \\ &\ge \begin{cases} c_m^{-1}\int_{S_\Omega(R_1, 2^{-1})} \tilde{u}(Q)\frac{\partial}{\partial n}G_\Omega^{j^*}(P_0, Q)\, d\sigma_Q \\ c_m^{-1}\int_{S_\Omega(2, R_2)} \tilde{u}(Q)\frac{\partial}{\partial n}G_\Omega^{j^*}(P_0, Q)\, d\sigma_Q. \end{cases} \end{aligned}$$

Hence it follows from Lemma 1 that

$$\begin{aligned} +\infty &> h_1(P_0) + 1 \\ &\ge \begin{cases} c_m^{-1}\int_{S_\Omega(R_1, 2^{-1})} \tilde{u}(Q)\frac{\partial}{\partial n}G_\Omega(P_0, Q)\, d\sigma_Q \\ \qquad \ge k_1 \int_{R_1}^{2^{-1}} r^{-\alpha_\Omega - 1}\left(\int_{\partial\Omega} \tilde{u}(r, \Theta)\, d\sigma_\Theta\right) dr \\ c_m^{-1}\int_{S_\Omega(2, R_2)} \tilde{u}(Q)\frac{\partial}{\partial n}G_\Omega(P_0, Q)\, d\sigma_Q \\ \qquad \ge k_3 \int_2^{R_2} r^{\beta_\Omega - 1}\left(\int_{\partial\Omega} \tilde{u}(r, \Theta)\, d\sigma_\Theta\right) dr, \end{cases} \end{aligned}$$

which shows that $\tilde{u}$ satisfies (2.4).

7. Proofs of Theorems 5, 6 and 7. These proofs proceed in the completely parallel way to the proofs of Theorems 2, 3 and 4, on the basis of two results of Yoshida [**12**, Theorems 7.2 and 7.5] and the following inequality corresponding to Lemma 1:

$$\begin{aligned} k_1' e^{-\sqrt{\lambda_D}(y^* - y)} f_D(X) \quad &(\text{resp. } k_3' e^{-\sqrt{\lambda_D}(-y^* + y)} f_D(X)) \\ \le \frac{\partial}{\partial n}G_D(P, Q) &\le k_2' e^{-\sqrt{\lambda_D}(y^* - y)} f_D(X) \\ &\qquad (\text{resp. } k_4' e^{-\sqrt{\lambda_D}(-y^* + y)} f_D(X)) \end{aligned}$$

for $P = (X, y) \in \Gamma(D)$ and $Q = (X^*, y^*) \in \partial\Gamma(D)$ satisfying $y^* > y + 1$ (resp. $y^* < y - 1$), where k_1' and k_2' (resp. k_3' and k_4') are two positive constants.

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