PACIFIC JOURNAL OF MATHEMATICS
Vol. 150, No. 2, 1991

CHARACTER VALUE ESTIMATES FOR GROUPS OF LIE TYPE

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Let G be a group of Lie type over the field of q elements. Let χ be a nonlinear irreducible character of G and x a noncentral element of G. Examination of character tables suggests that $|\chi(x)/\chi(1)| \leq C/q$, where C is a universal constant independent of χ, x, and G. This order of magnitude is attained when, for example, χ is the doubly transitive permutation character of $\mathrm{GL}(n, q)$ and x centralizes a hyperplane of $\mathrm{PG}(n-1, q)$; $|\chi(x)/\chi(1)|$ then approaches $1/q$ as $n \to \infty$. In this paper, we establish a bound of the above type when x is a semisimple element which has prime order modulo $Z(G)$. However, we must exclude certain groups G in characteristic 2 and 3. The most serious exclusions are the groups of type C_n in characteristic 2. Our proof, which is summarized below, does not use Deligne-Lusztig theory.

We first consider the case that x is contained in no proper parabolic subgroup of G. By character orthogonality, $|\chi(x)| \leq |C_G(x)|^{1/2}$. Since $C_G(x)$ is essentially a torus, the lower bounds for $\chi(1)$ in **[16]** yield the desired upper bound for $|\chi(x)/\chi(1)|$.

We may then assume that x is contained in a Levi complement L_J of a suitable standard maximal parabolic P_J. We write $\chi_{P_J} = \chi_1 + \chi_2 + \chi_3 + \chi_4$, where the irreducible constituents of χ_1 are linear characters of P_J, the irreducible constituents of χ_2 are nonlinear but have U_J in their kernels, the irreducible constituents of χ_3 lie over nonprincipal L'_J-invariant irreducible characters of U_J, and the irreducible constituents of χ_4 lie over irreducible characters of U_J which are not L'_J-invariant.

We show that $|\chi_1(x)|$ is absolutely bounded by finding absolute upper bounds both for the multiplicities of the linear constituents of χ_{P_J} and for the number of distinct linear constituents of χ_{P_J}. We essentially get the best possible absolute upper bound for the multiplicities. The theory developed in **[13]** and **[17]** is used to bound these multiplicities in terms of corresponding multiplicities in the Weyl group W of G. We obtain only a crude absolute upper bound for the number of distinct linear constituents of χ_{P_J}. Our bound involves the indices of certain large reflection subgroups of W in their normalizers.

Since L_J is a group of Lie type, an inductive hypothesis yields the desired bound for $|\chi_2(x)/\chi_2(1)|$.

To estimate χ_3, the results of [2] and Glauberman's character correspondence [**15**, 13.1] are used to show that if θ is an irreducible constituent of $(\chi_3)_{U_J}$, then $U_J/\mathrm{Ker}\,\theta$ is extraspecial. It is here that we must exclude the groups in characteristic 2 and 3 mentioned above. Standard Hall-Higman type results then yield that $|\chi_3(x)/\chi_3(1)| \leq 1/q$.

Finally, χ_4 is handled by restricting to L_J and using an inductive hypothesis.

While our result is obviously not the last word on the problem of finding upper bounds for character values in groups of Lie type, we hope the reader will agree that our method offers a conceptual and effective approach to this problem.

I would like to thank P. Fong, P. Johnson, G. Seitz, and D. White for helpful conversations and correspondence related to this paper.

1. Preliminaries. This section contains preliminaries to our estimation of χ_1, χ_2, χ_3, and χ_4. Since we will apply an inductive hypothesis to Levi complements of parabolic subgroups, we must work with groups which are not necessarily quasisimple. Appropriate definitions and inductive machinery are introduced in this section. We also define the excluded "special" groups. The reason for their exclusion will not be apparent before §3. Finally, we estimate $|\chi(x)/\chi(1)|$ when x is contained in no proper parabolic of G.

We begin with some important conventions. All fields $\mathrm{GF}(q)$ considered in this paper will have at least 4 elements. All algebraic groups $\overline{G}$ will be over the algebraic closure $\overline{\mathrm{GF}(p)}$ of the prime field $\mathrm{GF}(p)$. Algebraic groups and objects associated with them will be labeled with bars. We will denote by σ an endomorphism of $\overline{G}$ such that $\overline{G}_\sigma$ is finite. Since $q \geq 4$, we then have $O^{p'}(\overline{G}_\sigma) = \overline{G}'_\sigma$. Following the usual convention, a simple algebraic group is semisimple with a simple root system and a possibly nontrivial center. A component of a finite group is a subnormal quasisimple subgroup.

DEFINITION 1.1. Let G be a finite group and p a fixed prime number. Let x be a p'-element of G. We say that x is admissible if x has prime order in $G/Z(G)$.

DEFINITION 1.2. Let $\overline{G}$ be a simple algebraic group over $\overline{\mathrm{GF}(p)}$. Let σ be an endomorphism of $\overline{G}$ such that $\overline{G}_\sigma$ is finite. We call $\overline{G}$

and $\overline{G}_\sigma$ special (following [2]) if $p = 2$ and the root system of $\overline{G}$ is B_n, C_n, F_4 or G_2, or if $p = 3$ and the root system of $\overline{G}$ is G_2.

DEFINITION 1.3. Let G be a finite group. We say G is admissible if $G = \langle y, \overline{G}'_\sigma\rangle$, where $\overline{G}$ is a connected reductive algebraic group of characteristic p whose commutator subgroup is a product of simply connected and non-special simple components, and y is an admissible element of $\overline{G}_\sigma$.

REMARK. In the definition above, we don't exclude the possibility that $y \in \overline{G}'_\sigma$. Thus G is either a central product of quasisimple groups of Lie type, or an extension of such a group by a cyclic group. The element y then induces inner times diagonal automorphisms of the quasisimple factors of $\overline{G}'_\sigma$. We note that an admissible group has a split BN-pair obtained by intersecting the B and N subgroups of $\overline{G}_\sigma$ with G. The parabolic subgroups of G are also obtained from those of $\overline{G}_\sigma$ by intersection with G; see [3, p. 103].

DEFINITION 1.4. Let G be a finite group. We say that G is simple admissible if G is admissible and G' is quasisimple.

DEFINITION 1.5. We say that (G, x) is an admissible pair if G is an admissible group and x is an admissible element of G. If G is also simple admissible, we say that (G, x) is a simple admissible pair.

LEMMA 1.6. *Let G be an admissible group and let L be a Levi complement of a parabolic subgroup of G. Let K be a product of components of L. Let y be an admissible element of L such that $[y, K] \neq 1$. Then $\langle y, K\rangle$ is an admissible group.*

Proof. Let $\overline{G}$ and σ be as in Definition 1.3. Then $L = \overline{L}_\sigma \cap G$, where $\overline{L}$ is a σ-stable Levi complement in $\overline{G}$. Also $K = \overline{K}_\sigma \leq G$, where $\overline{K}$ is a product of simple components of $\overline{L}$. Let $\overline{T}$ be a σ-stable torus of $\overline{L}$ containing y. Then $\overline{K}\,\overline{T}$ is a connected reductive group with $((\overline{K}\,\overline{T})_\sigma)' = K$. To complete the proof, it suffices to show that the simple components of $\overline{K}$ are simply connected. To prove this, we may assume that $\overline{G}'$ is simple.

Let $\overline{L}_1$ be a simple component of $\overline{L}$. We may assume that $\overline{L}$ is a standard Levi subgroup $\overline{L}_J$, where J is (by abuse) a subset of a fundamental set Π for the root system $\overline{G}$. Then $\overline{L}_1$ corresponds to a connected subset J_1 of J.

Since $\overline{G}'$ is simply connected, its diagonal subgroup $\overline{H}$ is the direct product of subgroups $\overline{H}_\alpha$, for $\alpha \in \Pi$, each isomorphic to the multiplicative group of $\overline{\mathrm{GF}(p)}$; see [7, pp. 197–198]. Let $\overline{H}_1$ be the direct product of the $\overline{H}_\alpha$ for $\alpha \in J_1$. Then $\overline{H}_1$ is the diagonal subgroup of $\overline{L}_1$. Let $\overline{M}_1$ be the simply connected covering group of $\overline{L}_1$. Let $\theta\colon \overline{M}_1 \to \overline{L}_1$ be the natural epimorphism, as in [7, p. 190]. Clearly θ maps the diagonal subgroup of $\overline{M}_1$ isomorphically onto $\overline{H}_1$. It follows that $\operatorname{Ker}\theta = 1$ and so $\overline{L}_1$ is simply connected.

LEMMA 1.7. *Let (G, x) be a simple admissible pair with rank $(G) > 1$. Suppose x lies in a proper parabolic of G. Then for some standard maximal parabolic P_J of G, a G-conjugate of x lies in L_J and centralizes no component of L_J.*

Proof. View G as a group with a split BN-pair. Up to conjugacy, $x \in L_J$, the standard Levi complement of a standard maximal parabolic of G.

Suppose that x centralizes every component of L_J. Then $x \in H$, the diagonal subgroup of G, and so $x \in L_{J'}$, where $J' \neq J$ is another maximal subset of the index set I of the fundamental roots. If also $[x, L'_{J'}] = 1$, then x centralizes the standard Borel subgroup of G and its "opposite", so $[x, G] = 1$, a contradiction. Hence $x \in L_{J'}$ and x doesn't centralize every component of $L_{J'}$.

Thus we may choose J so that $x \in L_J$ and x doesn't centralize every component of L_J. If x centralizes no component of L_J, then we are done. Otherwise write $J = J_1 \cup J_2$, where J_1 corresponds to the union of the components of L_J centralized by x and J_2 corresponds to the union of the components of L_J not centralized by x. Write $x = hx_1x_2$, with $h \in H$, $x_1 \in L'_{J_1}$ and $x_2 \in L'_{J_2}$. Then hx_1 centralizes L'_{J_1}. Since h normalizes every root subgroup of L_{J_1}, so does $x_1 = h^{-1}(hx_1)$. It follows that $x_1 \in H$.

Thus $x = kx_2$, with $k \in H$ and $x_2 \in L'_{J_2}$. Thus $x \in L_{J_2}$. If J_2 is connected, we may choose a maximal and connected subset J_3 of I with $J_2 \leq J_3$. If J_2 is not connected, then J has 3 components, the root system of G is D_n, E_6, E_7, or E_8, and one checks that it is still possible to choose a maximal and connected subset J_3 of I with $J_2 \leq J_3$.

Hence $x \in L_{J_3}$, and since x doesn't centralize $L'_{J_2} \leq L'_{J_3}$, x doesn't centralize the unique component of L_{J_3}.

DEFINITION 1.8. Let (G, x) be a simple admissible pair. Let L_J be the standard Levi complement of a standard maximal parabolic of G. Suppose $x \in L_J$ and x centralizes no component of L_J. Then (G, x, L_J) is called an admissible triple.

We now turn to the problem of estimating $|\chi(x)/\chi(1)|$ when x lies in no proper parabolic of G.

LEMMA 1.9. *Let (G, x) be a simple admissible pair. Suppose x is contained in no proper parabolic of G. Then $C_{G'}(x)$ is a maximal torus of G'.*

Proof. By the Borel-Tits theorem [**3**, p. 103], $C_G(x)$ contains no unipotent elements. Let $\overline{G}$ and σ be as in Definition 1.3. Since $C_{\overline{G}}(x)$ admits σ, it follows that $C_{\overline{G}}(x)$ contains no unipotent elements. By a theorem of Steinberg [**8**, 3.5.6], $C_{\overline{G}}(x)$ is connected, and so is a (necessarily maximal) torus $\overline{T}$ of $\overline{G}$. By [**8**, p. 88], $\overline{T} \cap \overline{G}'$ is a maximal torus of $\overline{G}'$, and so taking σ-fixed points yields that $C_{G'}(x)$ is a maximal torus of G'.

LEMMA 1.10. *Let (G, x) be an admissible pair with $G' = \mathrm{SL}(2, q)$. Let $\chi \in \mathrm{Irr}(G)$ be nonlinear. Then $|\chi(x)/\chi(1)| = |\psi(x_1)/\psi(1)|$, where ψ is an irreducible character of* $\mathrm{SL}(2, q)$ *or* $\mathrm{GL}(2, q)$ *and x_1 is a noncentral semisimple element of* $\mathrm{SL}(2, q)$ *(resp.* $\mathrm{GL}(2, q)$*), and $\psi(1) = \chi(1)$.*

Proof. Let $G = \langle G', y\rangle$, as in Definition 1.3. Suppose y induces an inner automorphism of G'. Let $g \in G'$ induce the same automorphism of G' as y. Let $z = yg^{-1}$ and let r be the prime order of $y \bmod Z(G)$. Then $z^r \equiv y^r \pmod{G'}$, and so $z^r \in G'O_{p'}(Z(G))$. Since $z \in Z(G)$, $z^r \in (G'O_{p'}(Z(G))) \cap Z(G) \leq O_{p'}(Z(G))$. Thus G is a central product $G'\langle z\rangle$, where z is a p'-element. The conclusion of the lemma follows with $\psi \in \mathrm{Irr}(\mathrm{SL}(2, q))$.

Next suppose y induces an outer automorphism α of G'. Then q is odd. Since y has prime order $\bmod Z(G)$ and the diagonal automorphism group of G' has order 2, we have $\alpha^2 = 1$. Let H be the semidirect product $\langle\alpha\rangle G' = \langle\alpha\rangle\mathrm{SL}(2, q)$. If we can find abelian p'-groups Z, Z_1, W, and W_1 such that $G * Z \cong H * W$ and $\mathrm{GL}(2, q) * Z_1 \cong H * W_1$, where $*$ denotes a central product, then the conclusion of the lemma follows with $\psi \in \mathrm{Irr}(\mathrm{GL}(2, q))$.

To do this, let $Z = \langle z\rangle$ be a cyclic group of order $|y|$. Form the central product $G * Z$, where $z^2 = y^2 \in Z(G)$. Then yz^{-1} is an involution which induces α on G', and $G * Z = (\langle yz^{-1}\rangle G') * Z \cong H * Z$. Next let $t \in \mathrm{GL}(2, q)$ induce α on $\mathrm{SL}(2, q)$. Then t is a semisimple element. Let $Z_1 = \langle z_1\rangle$ be a cyclic group of order $|t|$ and form the central product $\mathrm{GL}(2, q) * Z_1$, where $z_1^2 = t^2 \in Z(\mathrm{GL}(2, q))$.

Then z_1t^{-1} is an involution which induces α on $\mathrm{SL}(2, q)$, and so $\mathrm{GL}(2, q) * Z_1 \cong (\langle z_1t^{-1}\rangle \mathrm{SL}(2, q)) * (Z(\mathrm{GL}(2, q))Z_1) = H * W_1$.

THEOREM 1.11. *Let (G, x) be a simple admissible pair. Suppose that x is contained in no proper parabolic of G. Let $\chi \in \mathrm{Irr}(G)$ with $\chi(1) > 1$. Then $|\chi(x)/\chi(1)| \leq 6/q$.*

Proof. If $G' = \mathrm{SL}(2, q)$, then Lemma 1.10 implies that $|\chi(x)/\chi(1)| = |\psi(x_1)/\psi(1)|$, as in the conclusion of Lemma 1.10. Checking character tables (**[11]**, **[18]**) shows that the last ratio is at most $2/(q-1) \leq 6/q$. Hence we assume $G' \neq \mathrm{SL}(2, q)$.

We claim that there exists a group H and an element $h \in H$ such that $H' = G'$, $C_{G'}(h) = C_{G'}(x)$, $\{|\chi(x)/\chi(1)|: \chi \in \mathrm{Irr}(G) \text{ and } \chi(1) > 1\} = \{|\psi(h)/\psi(1)|: \psi \in \mathrm{Irr}(H) \text{ and } \psi(1) > 1\}$, and $|H : G'| \leq d$, where d is the order of the diagonal automorphism group of G'.

Write $G = \langle y, \overline{G}'_\sigma\rangle$, as in Definition 1.3. Clearly $\overline{G}'_\sigma = G'$, so $G = \langle y\rangle G'$. To prove the claim we may assume that $y \notin G'$.

Let $Z = \langle z\rangle$ be a cyclic group of order $|y|$. Let r be the prime order of $y \bmod Z(G)$. Form the central product $G * Z$, where $z^r = y^r \in Z(G)$. Then $|yz^{-1}| = r$ and $G * Z = (\langle yz^{-1}\rangle G') * Z$. Let $G^* = \langle yz^{-1}\rangle G'$. Write $x = x_1w$ where $x_1 \in G^*$ and $w \in Z$. Then $(G^*)' = G'$, $C_{G'}(x) = C_{G'}(x_1)$, and $\{\chi(x)/\chi(1)|: \chi \in \mathrm{Irr}(G) \text{ and } \chi(1) > 1\} = \{\zeta(x_1)/\zeta(1)|: \zeta \in \mathrm{Irr}(G^*) \text{ and } \zeta(1) > 1\}$.

If yz^{-1} induces an outer automorphism of G', then since yz^{-1} has prime order r, we have $r|d$. Hence our claim holds for $H = G^*$, $h = x_1$.

We therefore assume that yz^{-1} induces an inner automorphism of G'. If r divides $|Z(G')| = d$, the claim holds as above. Hence we assume $(r, |Z(G')|) = 1$. For some element v in the coset $yz^{-1}G'$, G^* is a central product $G'\langle v\rangle$. We have $v^r \in G' \cap Z(G) \leq Z(G')$. Since r doesn't divide $|Z(G')|$, we have $|O_r(\langle v\rangle)| = r$ and $G^* = G'(O_r(\langle v\rangle) \times O_{r'}(\langle v\rangle)) = G' \times O_r(\langle v\rangle)$. Write $x_1 = gu$, with $g \in G'$ and $u \in O_r(\langle v\rangle)$. Then $C_{G'}(x_1) = C_{G'}(g)$ and $\{|\zeta(x_1)/\zeta(1)|: \zeta \in \mathrm{Irr}(G^*) \text{ and } \zeta(1) > 1\} = \{|\psi(g)/\psi(1)|: \psi \in \mathrm{Irr}(G') \text{ and } \psi(1) > 1\}$. Hence our claim holds with $H = G'$, $h = g$. This proves the claim in all cases.

To prove the theorem, it suffices to show that $|\psi(h)/\psi(1)| \leq 6/q$ for any nonlinear irreducible character ψ of H. Using Lemma 1.9, $|\psi(h)|^2 \leq |C_H(h)| \leq |C_{G'}(h)|d = |C_{G'}(x)|d = |T|d$, where T is a torus of G'. By the order formula for tori (see **[8**, p. 98**]**), $|T| \leq (q+1)^l$, where l is the rank of $\overline{G}$, which is greater than the rank of G when G is twisted. Since $G' \neq \mathrm{SL}(2, q)$, $l > 1$.

We have $d \leq l+1$. By [**16**, p. 419], $\psi(1) \geq (q^l - 1)/2$. Then

$$\frac{|\psi(h)|}{\psi(1)} \leq \frac{2\sqrt{l+1}(q+1)^{l/2}}{q^l - 1} \leq \frac{32\sqrt{l+1}(q+1)^{l/2}}{15(q^2-1)^{l/2}} = \frac{32\sqrt{l+1}}{15(q-1)^{l/2}} \leq \frac{6}{q}$$

for $l \geq 2$ and $q \geq 4$. The second inequality uses $(q^l-1)/(q^2-1)^{l/2} \geq (q^l-1)/(q^2)^{l/2} = 1-q^{-l} \geq 15/16$. The third inequality holds because $\sqrt{l+1}/(q-1)^{l/2}$ is decreasing in q and l for $l \geq 2$ and $q \geq 4$.

2. Estimating χ_1. Let G be a simple admissible group and let P_J be a standard maximal parabolic of G. Let $\chi \in \mathrm{Irr}(G)$ and let λ be a linear character of P_J. In the first part of this section we establish an absolute upper bound for the multiplicity (χ_{P_J}, λ).

Our work will be based on the following "comparison theorem" of McGovern [**17**]. By abuse, we will use the same symbol λ to denote the restriction of λ to the diagonal subgroup of G. Since $q \geq 4$, this diagonal subgroup covers P_J/P_J' and L_J/L_J'. Let $W(\lambda)$ be the stabilizer of λ in W, the Weyl group of G. In [**17**], a certain set $D(\lambda, J)$ of $(W(\lambda), W_J)$-double coset representatives is defined. We may assume that $1 \in D(\lambda, J)$.

THEOREM 2.1. *Let G be a finite group with a split BN-pair of charcteristic p. There is a one-to-one correspondence between the constituents of λ_B^G and the irreducible characters of $W(\lambda)$. Suppose χ is the constituent of λ_B^G corresponding to $\phi \in \mathrm{Irr}(W(\lambda))$. Let κ be an irreducible character of L_J with inflation $\tilde{\kappa}$ to P_J, satisfying $(\chi, \tilde{\kappa}^G) \neq 0$. Then κ is a constituent of $(\lambda^v)_{B_J}^{L_J}$ and $\tilde{\kappa}$ is a constituent of $(\lambda^v)_B^{P_J}$ for a unique $v \in D(\lambda, J)$. Both κ and $\tilde{\kappa}$ correspond to the same unique irreducible character ψ of $W_J \cap W(\lambda^v)$, and $(\chi, \tilde{\kappa}^G) = (\phi^v, \psi^{W(\lambda^v)})$.*

Proof. This is [**17**, Theorem A] with some minor changes in wording. In the statement of this theorem, λ denotes an arbitrary linear character of the diagonal subgroup of G. Note that $1 \in W$ is "λ-special" ([**17**, p. 426]), so we may assume $1 \in D(\lambda, J)$.

We will apply Theorem 2.1 with $\kappa = \lambda_{L_J}$, $\tilde{\kappa} = \lambda$. Then the multiplicity we wish to bound, (χ, λ^G), equals $(\phi^v, \psi^{W(\lambda^v)})$, where $\psi \in \mathrm{Irr}(W_J \cap W(\lambda^v))$ corresponds to λ and λ_{L_J}. We will show that $v = 1$ and $\psi = 1$. Since $W_J \cap W(\lambda) = W_J$ by Lemma 2.2 below, we will need only to establish an absolute upper bound for $(\phi, 1_{W_J}^{W(\lambda)})$ as ϕ ranges over $\mathrm{Irr}(W(\lambda))$. But this reduces to bounding the multiplicities in $1_{W_J}^{W}$, which have been investigated in **[9]** and **[1]**.

LEMMA 2.2. *With notation as above, we have* $W_J \leq W(\lambda)$ *and* $v = 1$.

Proof. Since $N_J \leq P_J$ stabilizes $\lambda \in \mathrm{Irr}(P_J)$, we have $W_J \leq W(\lambda)$. Since $\kappa = \lambda_{L_J}$ is a constituent of $\lambda_{B_J}^{L_J}$ and $\tilde{\kappa} = \lambda$ is a constituent of $\lambda_B^{P_J}$, uniqueness in Theorem 2.1 implies that $v = 1$.

The next remark, which is copied from **[17**, p. 421**]**, summarizes some of the main results of **[13]**. Here Φ is the root system of G, with fundamental system Π.

REMARK 2.3. Let G be a group with a split BN-pair of characteristic p. Let λ be a linear character of the diagonal subgroup of G. Define the λ-parameters $q_a(\lambda)$ as in **[13]**. There is a prime power q of p such that $q_a(\lambda) = q^{c_a(\lambda)}$ for all roots a in the root system Φ of W. In case $G = G(q)$ is a finite group of Lie type, this prime power may be taken as the characteristic power q of G. Howlett and Kilmoyer proved that there is a semidirect product decomposition $W(\lambda) = AC$, where C is the reflection group with root system $\Gamma = \{a \in \Phi \colon q_a(\lambda) \neq 1\}$ and fundamental system $\Sigma \leq \Phi^+$, and A is an abelian p'-group which normalizes C. Then a generic algebra $A(\lambda)$ was constructed, which is an associative $\overline{Q}[t]$-algebra with basis $\{X_w \colon w \in W(\lambda)\}$, satisfying the following multiplication (for $a \in A$, $b \in \Sigma$, and $w \in W(\lambda)$):

$$X_a X_w = X_{aw}, \qquad X_w X_a = X_{wa},$$
$$X_w X_{w_b} = \begin{cases} X_{ww_b} & \text{if } w(b) \in \Gamma^+, \\ t_b(\lambda) X_{ww_b} + (t_b(\lambda) - 1) X_w & \text{if } w(b) \in \Gamma^-, \end{cases}$$
$$X_{w_b} X_w = \begin{cases} X_{w_b w} & \text{if } w^{-1}(b) \in \Gamma^+, \\ t_b(\lambda) X_{w_b w} + (t_b(\lambda) - 1) X_w & \text{if } w^{-1}(b) \in \Gamma^-, \end{cases}$$

where $t_b(\lambda) = t^{c_b(\lambda)}$.

DEFINITION. In the situation of Remark 2.3, let $W_J(\lambda) = W_J \cap W(\lambda)$. Let $A_J(\lambda)$ be the generic algebra associated with $W_J(\lambda)$. (Note that $W_J(\lambda)$ is the stabilizer of λ in the Weyl group of L_J, so it has an associated generic algebra as in Remark 2.3.)

LEMMA 2.4. *Let* G, P_J *and* λ *be as in Lemma* 2.2. *Let* $q_a(\lambda)$ *be as in Remark* 2.3. *Then* $q_a(\lambda) = q_a(1)$ *for* $a \in \Phi_J$.

Proof. For any $a \in \Phi$, [**13**, 2.6(b)] says that $q_a(\lambda) + q_a(\lambda)^{-1} = [\sum \lambda^w(h_i(x))]^2 \lambda^w((r_i)^2)$, whenever $w \in W$ is chosen so that $a^w \in \Pi$. By [**13**, p. 577], $h_i(x)$ and $(r_i)^2$, which are independent of λ, may be chosen to lie in the diagonal subgroup of $O^{p'}(L_i)$, where L_i is the standard Levi complement of the standard minimal parabolic of G corresponding to $\{a^w\} \leq \Pi$.

Since $a \in \Phi_J$ by hypothesis, we may take $w \in W_J$, so that $a^w \in \Pi_J$. Then $L_i \leq L_J$. Since $O^{p'}(L_i) \leq O^{p'}(P_J) = P'_J \leq \operatorname{Ker} \lambda$, we have $q_a(\lambda) + q_q(\lambda)^{-1} = q_a(1) + q_a(1)^{-1}$. By [**13**, §4], for any $a \in \Phi$, $q_a(\lambda) = q^{c_a(\lambda)}$, where $c_a(\lambda)$ is a non-negative integer. Thus $q_a(\lambda) = q_a(1)$.

REMARK 2.5. Let $a \in \Phi$ and choose $w \in W$ so that $a^w \in \Pi$. Let s be the fundamental reflection corresponding to a^w. Then $q_a(1) = \operatorname{ind} s = |B : B \cap B^s|$, the usual index parameter defined in [**10**, p. 610]. See [**13**, p. 552]. If $a \in \Phi_J$, then $q_a(\lambda)$ is the same, whether computed in G or L_J.

LEMMA 2.6. *Let G, P_J, and λ be as in Lemma 2.2. Write $W_J(\lambda) = A_J C_J$, as in Remark 2.3. Then $C_J = W_J = W_J(\lambda)$ and $A_J = 1$. The generic algebra $A_J(\lambda)$ can be identified with the generic algebra of the Coxeter system (W_J, Π_J) over $\overline{Q}[t]$.*

Proof. By [**17**, Theorem 1.5], the reflection factor C_J of $W_J(\lambda)$ equals $W_J \cap C$, where C is the reflection factor of $W(\lambda)$. Since the root system of C contains Φ_J by Lemma 2.4, we have $W_J \leq C$, proving the first assertion. The second assertion is clear from the definitions of the respective generic algebras in Remark 2.3 and [**10**, p. 637], and the fact that $c_a(\lambda) = c_a(1)$ for $a \in \Phi_J$ by Lemma 2.4.

PROPOSITION 2.7. *Let G, P_J, and λ be as in Lemma 2.2. Let $\psi \in \operatorname{Irr}(W_J)$ correspond to $\kappa = \lambda_{L_J}$ in Theorem 2.1. Then $\psi = 1_{W_J}$.*

Proof. By [**17**, p. 431], the bijection between the irreducible characters of the Hecke algebra $\mathscr{H}(L_J, B_J, \lambda)$ and those of $\overline{Q}[W_J]$ is obtained via [**17**, Theorem 2.1]. An irreducible character of one of these algebras corresponds to an irreducible character of the other when both are (extended) specializations of the same irreducible character of $A_J(\lambda)$.

Following [**10**, p. 637], we define a $\overline{Q}[t]$-algebra homomorphism IND: $A_J(\lambda) \to \overline{Q}[t]$ by $\operatorname{IND} X_{s_i} = t^{c_{a_i}(\lambda)} = t_{a_i}(\lambda)$, where a_i is a fundamental root in Π_J and s_i is the corresponding fundamental reflection. Clearly IND specializes to the principal character of character of $\overline{Q}[W_J]$ under the specialization $t \to 1$. To compute the specialization of IND to $\mathscr{H}(L_J, B_J, \lambda)$, we must check that the specialization $t \to q$ sends the generator X_{s_i} of $A_J(\lambda)$ to the element $\beta_{s_i} = (\operatorname{ind} s_i)e_\lambda(s_i)e_\lambda$, where $s_i \in W_J$ is a fundamental reflection, $(s_i) \in N_J \cap L'_J$ maps onto s_i, and e_λ is the primitive central idempotent of $\overline{Q}[B_J]$ corresponding to λ. By Theorem 2.17 and p. 567 of **[13]**, X_{s_i} is sent to $\gamma_{s_i} \in \mathscr{H}(L_J, B_J, \lambda)$, where $\gamma_{s_i} = \lambda((s_i))q_{s_i}^{-1/2}q_{s_i}(\lambda)^{1/2}\beta_{s_i}$ by **[13**, Eq. 2.19]. Also $q_{s_i} = q_{a_i}(1)$ and $q_{s_i}(\lambda) = q_{a_i}(\lambda)$ by the definitions in **[13]**. By Lemma 2.14 and the fact that $(s_i) \in L'_J$, we have $\gamma_{s_i} = \beta_{s_i}$. (Note that the definition of β_w in **[13**, Def. 2.2] is incorrect. The given formula for β_w must be multiplied by ind w, since otherwise **[13**, Theorem 2.4] would be incorrect and equation 2.19 would be inconsistent with Theorem 2.17.)

Let ind: $\mathscr{H}(L_J, B_J, \lambda) \to \overline{Q}$ be the specialization of IND under $t \to q$. Then $\operatorname{ind}(\beta_{s_i})$ is the specialization of $t^{c_{a_i}(\lambda)}$ under $t \to q$. Hence $\operatorname{ind}(\beta_{s_i}) = q_{a_i}(\lambda) = q_{a_i}(1) = \operatorname{ind} s_i$, as in Remark 2.5. On the other hand, $\lambda(\beta_{s_i}) = (\operatorname{ind} s_i)\lambda(e_\lambda)\lambda((s_i))\lambda(e_\lambda) = \operatorname{ind} s_i$. Since the β_{s_i} generate $\mathscr{H}(L_J, B_J, \lambda)$ as a $\overline{Q}$-algebra, ind equals the restriction of λ to $\mathscr{H}(L_J, B_J, \lambda)$, so λ_{L_J} corresponds to the principal character of W_J, as desired.

COROLLARY 2.8. *Let G, P_J, and λ be as in Lemma 2.2. Let $M(\lambda) = \max\{(\lambda^G, \chi): \chi \in \operatorname{Irr}(G)\}$. Let $m(W, J) = \max\{(1_{W_J}^W, \phi): \phi \in \operatorname{Irr}(W)\}$, where W is the Weyl group of G. Then $M(\lambda) \le m(W, J)$.*

Proof. Choose $\chi \in \operatorname{Irr}(G)$ so that $(\lambda^G, \chi) = M(\lambda)$. Let $\phi \in \operatorname{Irr}(W(\lambda))$ correspond to χ in Theorem 2.1. By Theorem 2.1 and Proposition 2.7, $1_{W_J}^{W(\lambda)} = M(\lambda)\phi + \alpha$, where α is a sum of other irreducible characters of $W(\lambda)$. Thus some irreducible constituent of $1_{W_J}^W$ has multiplicity at least $M(\lambda)$, as desired.

It now suffices to show that there is an absolute upper bound for $m(W, J)$ as W ranges over all irreducible Weyl groups and W_J ranges over all maximal parabolic subgroups of W. For W of type A_n, B_n, or C_n, $m(W, J) = 1$ for all J by **[9**, p. 90]. For the exceptional Weyl groups, Alvis' tables **[1]** yield $m(W, J) \le 13$ for all W and J, with 13 occurring for $W(E_8)$.

It remains to determine $m(W, J)$ for W of type D_n. By [**9**, p. 90], $m(W(D_5), J) > 1$ for at least one J.

PROPOSITION 2.9. *$m(W(D_n), J) \leq 2$ for all n and J.*

Proof. Let $W = W(D_n)$. Let $Y = W(B_n)$. Then $W \leq Y \leq S_n E_n$, where S_n is the symmetric group of degree n, and E_n is an elementary abelian group of order 2^n on which S_n acts by permuting coordinates. Write elements of E_n as row vectors with entries in GF(2). For $1 \leq i \leq n-1$, let $w_i = (i, i+1) \in S_n$. Let $w_n = w_{n-1}[00\cdots 011]$ and let $y_n = [00\cdots 01]$. Let $y_i = w_i$ for $1 \leq i \leq n-1$. Then $\{w_1, \ldots, w_n\}$ and $\{y_1, \ldots, y_n\}$ are standard sets of fundamental reflections for W and Y, respectively.

Let $J = \{1, \ldots, n\} - \{j\}$. If $j \leq n-2$, then $|Y_J : W_J| = 2$, $WY_J = Y$, and $Y_J \cap W = W_J$. By Mackey's theorem, $(1_{Y_J}^Y)_W = 1_{W_J}^W$. By [**9**, p. 90], $1_{Y_J}^Y$ is a sum $\phi_1 + \cdots + \phi_k$ of distinct irreducible characters of Y. If $1 \leq i < r \leq k$, then $(\phi_i)_W$ and $(\phi_r)_W$ have a common irreducible constituent if and only if $\phi_i = \mu\phi_r$, where μ is the nonprincipal linear character of Y/W. Since each $(\phi_i)_W$ is either irreducible or is the sum of two distinct Y-conjugate irreducible characters, it follows that all multiplicities in $1_{W_J}^W$ are at most 2.

If $j = n-1$, then $Y_J \cap W$ is the non-maximal standard parabolic subgroup of W corresponding to $\{1, \ldots, n-2\}$ and $Y_J W = Y$. The argument in the preceding paragraph shows that the multiplicities in $1_{Y_J \cap W}^W$ are at most 2. Since $Y_J \cap W \leq W_J$, it follows that the multiplicities in $1_{W_J}^W$ are at most 2.

If $j = n$, then $W_J = Y_J$. Since $1_{Y_J}^Y$ is multiplicity-free, so is $1_{Y_J}^W = 1_{W_J}^W$.

We summarize our work on multiplicities in the following theorem.

THEOREM 2.10. *Let G be a simple admissible group. Let P_J be a maximal parabolic subgroup of G. Let $\chi \in \mathrm{Irr}(G)$ and let λ be a linear character of P_J. Then $(\chi_{P_J}, \lambda) \leq 13$.*

Proof. This follows from Theorem 2.1, Proposition 2.7, Corollary 2.8, Proposition 2.9, and the remarks preceding Proposition 2.9.

Let G, χ, and P_J be as in Lemma 2.2. Having bounded the multiplicities of the linear constituents of χ_{P_J}, we must now bound the number of distinct linear constituents of χ_{P_J}. Linear characters of P_J are cuspidal characters of the diagonal subgroup of G. Hence if λ and μ are two linear constituents of χ_{P_J}, Harish-Chandra's theorem [**10**, 70.15(A)] implies that λ and μ, viewed as characters of the diagonal group of G, are conjugate under W. Thus we must bound the number $n(\lambda)$ of linear characters of P_J which are W-conjugate to a fixed linear character λ of P_J.

Let $P_0 = P_J \cap G'$. Then P_0 is a maximal parabolic of G'. Let H denote the diagonal subgroup of G'. Let λ' be the restriction of λ to P_0. We may also view λ' as a linear character of H. The semidirect product decomposition in Remark 2.3 applies to both $W(\lambda)$ and $W(\lambda')$. Write $W(\lambda) = A(\lambda)C(\lambda)$ and $W(\lambda') = A(\lambda')C(\lambda')$. Let $\widehat{C}(\lambda)$ be the group generated by all reflections in $W(\lambda)$, so that $C(\lambda) \leq \widehat{C}(\lambda) \leq W(\lambda)$. Define $\widehat{C}(\lambda')$ similarly.

We will use the fact, proved below, that P_0 has a cyclic commutator factor group to reduce the problem of bounding $n(\lambda)$ to known results on the indices of reflection subgroups of W in their normalizers.

LEMMA 2.11. *Let P_0 and H be as above. Then P_0/P_0' is cyclic.*

Proof. Since $[H, X] = X$ for every root subgroup X of G', it follows that $P_0' = O^{p'}(P_0)$ and $P_0 = P_0'H$. Hence $P_0/P_0' \cong H/(H \cap P_0')$.

First suppose G' is untwisted. Then, since G' is simply connected by hypothesis, H is the direct product of groups H_α, each isomorphic to the multiplicative group of $\mathrm{GF}(q)$. See [7, p. 197–8]. Moreover, the H_α correspond to the fundamental roots $\alpha \in \Pi$. By [7, p. 92], $H_\alpha \leq \langle X_\alpha, X_{-\alpha}\rangle$. Hence $\prod_{\alpha\in J} H_\alpha \leq O^{p'}(P_0) = P_0'$. Thus $H/(H \cap P_0')$ is cyclic.

Next suppose G' is a twisted group over $\mathrm{GF}(q)$. Write $G' = \overline{G}_\sigma$, where $\overline{G}$ is a simply connected algebraic group over $\overline{\mathrm{GF}(p)}$, and $\sigma = q\tau$, where q is the qth power Frobenius endomorphism of $\overline{G}$, and τ is a nontrivial graph automorphism. Let $P_0 = \overline{P}_\sigma$, where $\overline{P}$ is a standard σ-stable parabolic of $\overline{G}$. Since $\overline{G}$ is a simply connected Chevalley group over $\overline{\mathrm{GF}(p)}$, its diagonal subgroup $\overline{H}$ is a direct product of subgroups $\overline{H}_\alpha$, where α ranges over a fundamental set of roots for $\overline{G}$ and each $\overline{H}_\alpha$ is isomorphic to the multiplicative group of $\overline{\mathrm{GF}(p)}$. As in the preceding paragraph, we may write $\overline{H} = \overline{H}_1 \times \overline{H}_2$, where $\overline{H}_2 \leq \overline{P}'$ and $\overline{H}_1$ is the direct product of the $\overline{H}_\alpha$ as α ranges over a single τ-orbit of fundamental roots of $\overline{G}$. Then $H = \overline{H}_\sigma =$

$(\overline{H}_1)_\sigma \times (\overline{H}_2)_\sigma$ and $(\overline{H}_2)_\sigma \leq H \cap (\overline{P}')_\sigma = H \cap P_0'$. Hence it suffices to show that $(\overline{H}_1)_\sigma$ is cyclic. Let $\pi\colon \overline{H}_1 \to (\overline{\mathrm{GF}(p)})^*$ be the projection onto one fixed direct factor of $\overline{H}_1$. Since τ cyclically permutes the direct factors of $\overline{H}_1$ and $\sigma = q\tau$, it follows that the restriction of π to $(\overline{H}_1)_\sigma$ is injective, and so $(\overline{H}_1)_\sigma$ is cyclic.

LEMMA 2.12. *Let $n(\lambda)$ and λ' be as in the remarks preceding Lemma 2.11. Then $n(\lambda) \leq |N_W(\operatorname{Ker} \lambda')\colon W(\lambda)|$.*

Proof. Let $w \in W$. Suppose that λ^w, considered as a linear character of the diagonal subgroup of G, is the restriction of a linear character of P_J. Then $(\lambda')^w$ is the restriction to H of a linear character of P_0, and so $H \cap P_0' \leq \operatorname{Ker}((\lambda')^w)$, and also $H \cap P_0' \leq \operatorname{Ker} \lambda'$. Since $H/(H \cap P_0')$ is cyclic, and since $\operatorname{Ker} \lambda'$ and $\operatorname{Ker}(\lambda')^w = (\operatorname{Ker} \lambda')^w$ have the same index in H, we have $\operatorname{Ker} \lambda' = (\operatorname{Ker} \lambda')^w$. Hence w normalizes $\operatorname{Ker} \lambda'$. The desired inequality follows.

LEMMA 2.13. *$N_W(\operatorname{Ker} \lambda') \leq N_W(\widehat{C}(\lambda'))$.*

Proof. Since $H \cap P_0' \leq \operatorname{Ker} \lambda'$ and $H/H \cap P_0'$ is cyclic, $H/\operatorname{Ker} \lambda'$ has an abelian automorphism group. Hence $(N_W(\operatorname{Ker} \lambda'))' \leq W(\lambda')$. Clearly $W(\lambda') \leq N_W(\operatorname{Ker} \lambda')$. Thus $W(\lambda')$ is normal in $N_W(\operatorname{Ker} \lambda')$. Then $N_W(\operatorname{Ker} \lambda')$ permutes the reflections in $W(\lambda)'$ and so normalizes $\widehat{C}(\lambda')$.

LEMMA 2.14. *$C(\lambda) = C(\lambda')$.*

Proof. By Remark 2.3, $C(\lambda)$ is the reflection group with root system $\Gamma = \{a \in \Phi\colon q_a(\lambda) \neq 1\}$, and $C(\lambda')$ is the reflection group with root system $\Gamma' = \{a \in \Phi\colon q_a(\lambda') \neq 1\}$. By the proof of Lemma 2.4, $q_a(\lambda)$ is determined by the value of λ on elements of H. Hence $q_a(\lambda) = q_a(\lambda')$ for all $a \in \Phi$. Thus $\Gamma = \Gamma'$ and $C(\lambda) = C(\lambda')$.

LEMMA 2.15. *$n(\lambda) \leq |N_W(\widehat{C}(\lambda'))\colon \widehat{C}(\lambda')|\,|\widehat{C}(\lambda')\colon C(\lambda')|$.*

Proof. By Lemmas 2.12 and 2.13, $n(\lambda) \leq |N_W(\widehat{C}(\lambda'))\colon W(\lambda)|$. Hence $n(\lambda) \leq |N_W(\widehat{C}(\lambda'))\colon \widehat{C}(\lambda')|\,|\widehat{C}(\lambda')\colon C(\lambda)|$. Now Lemma 2.14 yields the desired result.

LEMMA 2.16. *Let W_J be a maximal parabolic subgroup of an irreducible Weyl group W. Let W_1 be a reflection subgroup of W which contains W_J. Then $|N_W(W_1)\colon W_1| \leq 72$.*

Proof. We follow Carter [**6**]. Let $\Phi_1 = \{a \in \Phi\colon w_a \in W_1\}$. Then Φ_1 is a root system with Weyl group W_1. Note that $\Phi_J \leq \Phi_1$. Let V_1 be the vector space spanned by Φ_1. The roots in Φ orthogonal to Φ_1 form a subsystem Φ_2 with Weyl group W_2. Clearly W_2 has order 1 or 2 in our situation. By [**6**, Proposition 28], $N_W(W_1)/(W_1 \times W_2)$ is isomorphic to a group of symmetries of Δ_1, the Dynkin diagram of Φ_1.

If $W_1 > W_J$, then Φ_1 is obtained by deleting a node from the extended Dynkin diagram of Φ, or is the dual of the diagram obtained by deleting a node from the extended Dynkin diagram of the root system dual to Φ. See [**6**, p. 8]. An easy case-by-case check shows that the largest value of $|\mathrm{Aut}(\Delta_1)|\,|W_2|$ is 72, which occurs when the middle node is deleted from the extended Dynkin diagram of D_8.

THEOREM 2.17. $n(\lambda) \leq 576$.

Proof. By Lemma 2.16, $|N_W(\widehat{C}(\lambda'))\colon \widehat{C}(\lambda')| \leq 72$. Since $\widehat{C}(\lambda')$ normalizes $C(\lambda')$ and $\widehat{C}(\lambda')/C(\lambda')$ is an elementary abelian 2-group, the proof of Lemma 2.16 yields $|\widehat{C}(\lambda')\colon C(\lambda')| \leq 8$. Lemma 2.15 then gives the desired conclusion.

3. Estimating χ_3. Let (G, x, L_J) be an admissible triple (see Def. 1.8) with rank $(G) \geq 2$. Let $\theta \in \mathrm{Irr}(U_J)$ be invariant under L'_J. Let $\chi \in \mathrm{Irr}(P_J)$ lie over θ. (In the situation of the introduction to this paper, χ plays the role of an irreducible constituent of χ_3.) In this section, we establish the bound $|\chi(x)/\chi(1)| \leq 1/q$.

Relying heavily on results from [**2**], we show that $[U'_J, U_J] \leq \mathrm{Ker}\,\theta$, that $U_J/\mathrm{Ker}\,\theta$ is extraspecial, and that x preserves a $\mathrm{GF}(q)$-bilinear symplectic form on U_J/U'_J. We then restrict to $\langle x\rangle U_J$ and use standard Hall-Higman type results to obtain the desired bound.

For an admissible triple (G, x, L_J), let $\widehat{H}$ denote the diagonal subgroup of G. Let $G' = \overline{G}_\sigma$ where $\overline{G}$ is a simple algebraic group. Here $\sigma = q\tau$, where q is the qth power map on $\overline{\mathrm{GF}(p)}$ extended to a Frobenius morphism of $\overline{G}$ and τ is a possibly trivial graph automorphism of $\overline{G}$.

Let $P = P_J \cap G'$, $L = L_J \cap G'$, and $H = \widehat{H} \cap G'$. Let $L_0 = L' = L'_J$ and let $H_0 = H \cap L_0 = \widehat{H} \cap L_0$. This notation differs from that of §2.

Let Σ be the root system of $\overline{G}$, and let Π be a fundamental system for Σ. Choose $K \leq \Pi$, abusing notation, so that $P = (\overline{P}_K)_\sigma$, where

$\overline{P}_K$ is the standard parabolic of $\overline{G}$ corresponding to K. Thus $K' = \Pi - K$ will consist of a single τ-orbit of fundamental roots. The unipotent radical $\overline{U}_K$ of $\overline{P}_K$ is the product of the root groups $\overline{U}_\beta$ for $\beta \in \Sigma^+ - \Sigma_K$.

We differ slightly from [2], where instead of $\overline{P}_K$, the authors work with the "opposite" parabolic generated by $\overline{L}_K$ and the negative root groups in $\Sigma - \Sigma_K$. By [2, p. 561], passing from $\overline{P}_K$ and $(\overline{P}_K)_\sigma$ to their opposites merely replaces the modules W_S defined below by their duals, which is harmless for our purposes. We also remark that the results of [2] apply to all parabolic subgroups, not just the maximal parabolics considered here.

For $\beta \in \Sigma^+ - \Sigma_K$, write $\beta = \beta_K + \beta_{K'}$, where β_K is a linear combination of fundamental roots in K, and similarly for $\beta_{K'}$. Write $\beta_{K'} = d_1\alpha_1 + \cdots + d_i\alpha_i + \cdots$, where α_i ranges over K'. Following [2, p. 3], define the shape of β to be $\beta_{K'}$ and the level of β to be the sum of the d_i above.

Let $\overline{U}(i) = \prod \overline{U}_\beta$, the product over all $\beta \in \Sigma^+ - \Sigma_K$ with level $(\beta) \geq i$. Let $U(i) = \overline{U}(i)_\sigma$. Then $U(i)$ is the product of the corresponding root groups of $G' = \overline{G}_\sigma$. By [2, Lemma 4 and Lemma 6], $U(1) > U(2) > \cdots$ is the descending central series of $U_J = (\overline{U}_K)_\sigma$.

Let $M(i) = U(i)/U(i{+}1)$. By [2, Lemma 5], $M(i)$ is L-isomorphic to $(\overline{U}(i)/\overline{U}(i+1))_\sigma$. By [2, Theorem 2a and Theorem 3], $M(i)$ has a direct decomposition as a product of P-chief factors W_S, which we will describe below. Each W_S is an irreducible $\mathrm{GF}(q^c)[L]$-module, where c is the number of shapes in the τ-orbit of the shape S.

The results above need not hold without our assumption that $\overline{G}$ is not "special".

To describe W_S, we need more notation. For G untwisted and S a shape on level i, define V_S to be the image in $M(i)$ of the product of all root subgroups U_β, for β of shape S. Define $\overline{V}_S$ similarly for $\overline{G}$.

If G is untwisted put $W_S = V_S$. If G is twisted and $S^\tau = S$, put $W_S = (\overline{V}_S)_\sigma$. If G is twisted and $S^\tau \neq S$ let W_S equal $(\overline{V}_S \oplus \overline{V}_S^\sigma)_\sigma = (\overline{V}_S \oplus \overline{V}_{S^\tau})_\sigma$ when $|\tau| = 2$, or the obvious analog when $|\tau| = 3$. By [2, Theorem 2b and Lemma 7], W_S is an irreducible $\mathrm{GF}(q^c)[L]$-module. There is one module W_S for each τ-orbit of shapes in $\Sigma^+ - \Sigma_K$.

By [2, Theorem 2d and Lemma 7], W_S remains irreducible as a $\mathrm{GF}(q^c)[L_0]$-module. Finally, the W_S are also P_J-chief factors, since $P_J = P\widehat{H}$ and the W_S are invariant under diagonal automorphisms; see [2, p. 552].

In Lemma 3.1 below, we give a criterion for W_S to be centralized by L_0. In Lemmas 3.2 and 3.3, we locate the trivial L_0-composition factors in U_J.

LEMMA 3.1. *L_0 centralizes W_S iff there is only one root in $\Sigma^+ - \Sigma_K$ of shape S.*

Proof. First suppose $\tau = 1$. Then, since $W_S = V_S$ is an irreducible GF$(q)[L_0]$-module, L_0 centralizes V_S iff V_S has GF(q)-dimension 1, iff there is only one root in S.

Next suppose $|\tau| = 2$. Then τ induces a linear transformation on $\overline{M(i)} = \overline{U}(i)/\overline{U}(i+1)$ which permutes the images of the root groups in $\overline{G}$ on level i. Suppose first that $(\overline{V}_S)^\sigma = \overline{V}_S$. Then $S = S^\tau$, the roots of shape S comprise a union of τ-orbits, and $(\overline{V}_S)_\sigma = W_S$ is the direct sum of the σ-fixed point spaces for each of these τ-orbits. For each such τ-orbit, the σ-fixed point space is the image in $M(i)$ of a root group of the twisted group $\overline{G}_\sigma = G'$. This image has cardinality q if the τ-orbit consists of one root, and cardinality q^2 if the τ-orbit consists of two roots. If L_0 centralizes $(\overline{V}_S)_\sigma = W_S$, then $(\overline{V}_S)_\sigma$ has dimension 1 over GF(q). Hence S consists of only one τ-orbit, which in turn consists of only one root. Conversely, if S consists of only one root, then $(\overline{V}_S)_\sigma$ has cardinality q, and so is centralized by L_0.

Next suppose $|\tau| = 2$ and $(\overline{V}_S)^\sigma \neq \overline{V}_S$. Again $W_S = (\overline{V}_S \oplus \overline{V}_S^\sigma)_\sigma$ is the direct sum of the σ-fixed point spaces for the τ-orbits of roots in $S \cup S^\tau$. Since S and S^τ are disjoint, each such τ-orbit consists of two roots, and so its σ-fixed point space in $M(i)$ has cardinality q^2, as above. If L_0 acts trivially on W_S, then GF$(q^2)[L_0]$-irreduciblity forces W_S to have dimension 1 over GF(q^2). Hence $S \cup S^\tau$ contains only one τ-orbit of roots, and so S contains only one root. The converse is clear.

Finally, the case $|\tau| = 3$ is entirely similar to $|\tau| = 2$.

LEMMA 3.2. *Let S be a shape in $\Sigma^+ - \Sigma_K$ with level $(S) = 1$. Then S contains more than one root.*

Proof. First suppose G is untwisted. Let $K' = \{\alpha\}$. All roots on level 1 have the same shape. To show that this shape contains a root distinct from α, we need only show that for some fundamental root $\beta \neq \alpha$, that $\alpha + k\beta$ is a root for some positive integer k. Root chain considerations [7, p. 37] show that this is the case whenever β is not orthogonal to α.

Now suppose G is twisted. If $|K'| = 1$, then the unique shape on level 1 contains more than one root, as above. We then assume $|K'| > 1$. The shape S consists of all roots of the form $\alpha + \sum_{i \in K} c_i \alpha_i$, where α is a fixed fundamental root in K'. An easy case-by-case check shows that there is always a fundamental root $\alpha_i \in K$ which is not orthogonal to α. (Recall that G has rank at least 2 as a twisted group.) Hence we get two roots of shape S as above.

LEMMA 3.3. *Let S be a shape in $\Sigma^+ - \Sigma_K$ which contains only one root. Then level $(S) = 2$ and L_0 centralizes $M(2)$. The commutator form from $M(1) \times M(1)$ to $M(2)$ is nondegenerate,* GF(q)*-bilinear, and L_0-invariant.*

Proof. First we claim that level $(S) = 2$. Suppose G is untwisted. If G is a classical group, there are at most two levels of roots in $\Sigma^+ - \Sigma_K$, and so our claim follows from Lemma 3.2. For the exceptional groups, we merely check the lists of positive roots in **[4]** or **[19]**. We find that for F_4, E_6, E_7, and E_8, there is exactly one maximal subset K of Π for which $\Sigma^+ - \Sigma_K$ has a shape (i.e. level) consisting of just one root, and this always occurs on level 2. For such K, there are only two levels of roots in $\Sigma^+ - \Sigma_K$. For type G_2, both possibilities for K have a unique root (only) on level 2, and there are two or three levels of roots in $\Sigma^+ - \Sigma_K$.

Next suppose G is twisted. If $|K'| = 1$, then a shape in $\Sigma^+ - \Sigma_K$ is the same thing as a level, and the claim follows as in the preceding paragraph. Hence we assume $|K'| > 1$. For types ${}^2A_n$ and ${}^2D_n$, the highest level in $\Sigma^+ - \Sigma_K$ is 2. For type ${}^3D_4$, with $K = \{\alpha_2\}$, the fundamental root fixed by τ, there are three levels. Level 2 consists of single τ-orbit of shapes, each containing only one root. Level 3 consists of one τ-invariant shape, which contains two roots. For type ${}^2E_6$, there are only two levels when $K' = \{\alpha_1, \alpha_6\}$, the outer pair of nodes exchanged by τ. When $K' = \{\alpha_3, \alpha_5\}$, the inner pair of nodes exchanged by τ, there are four levels. The shapes of level at least 2 for these sets K' are $\alpha_1 + \alpha_6$, $\alpha_3 + \alpha_5$, $2\alpha_3 + \alpha_5$, $\alpha_3 + 2\alpha_5$, and $2\alpha_3 + 2\alpha_5$. All of these shapes contain at least two roots. This proves our first claim.

We next claim that L_0 centralizes $M(2)$. By Lemma 3.1, it suffices to show that there is only one τ-orbit of shapes on level 2, since $M(2)$ would then be a trivial GF$(q^c)[L_0]$-module of the form W_S. If G is untwisted, or if G is twisted with $|K'| = 1$, this is clear since shape and level are then the same thing. If G is twisted with $|K'| > 1$, then our assumption that $\Sigma^+ - \Sigma_K$ has a shape containing only one root implies that G is not of type ${}^2E_6$. If G is of type ${}^3D_4$, then there is only one τ-orbit of shapes on level 2 as mentioned above. If G is of type ${}^2D_n$, then $|K'| > 1$ implies that $K' = \{\alpha_{n-1}, \alpha_n\}$, the pair of fundamental roots exchanged by τ. The only shape on level 2 is then $\alpha_{n-1} + \alpha_n$, which contains at least two roots. Our claim is therefore vacuously satisfied for type ${}^2D_n$. Note, by the way, that the description of the positive roots of D_n on [**4**, p. 256] is incorrect. If G is of type ${}^2A_n$, then we get a one-root shape only when $K' = \{\alpha_1, \alpha_n\}$, the outermost pair of nodes exchanged by τ. Here there is only one shape on level 2, namely $\alpha_1 + \alpha_n$. This proves our second claim.

We now claim that there is only one τ-orbit of shapes on level 1. As in the preceding paragraph, this is clear when G is untwisted or when $|K'| = 1$. The only other cases we have to consider are ${}^3D_4$ with $K = \{\alpha_2\}$ and ${}^2A_n$ with $K' = \{\alpha_1, \alpha_n\}$. In both cases we see that there is a unique τ-orbit of shapes on level 1. This proves our claim. It follows that $M(1)$ is a single P-chief factor.

Since $U(1)' = U(2)$ and $[U(1), U(2)] = U(3)$, there is a well-defined commutator form from $M(1) \times M(1)$ to $M(2)$. Since L_0 centralizes $M(2)$, this form is L_0-invariant. Since $M(1)$ is a P-chief factor, $Z(U(1)/U(3)) = U(2)/U(3) = M(2)$. Hence the commutator form is nondegenerate.

It remains to show that the commutator form is $\mathrm{GF}(q)$-bilinear. If G is untwisted, the $\mathrm{GF}(q)$-structure on each $M(i)$ is determined by $sx_\beta(t) = x_\beta(st)$, where s and t are scalars in $\mathrm{GF}(q)$ and $x_\beta(t)$ is a root element for $\beta \in \Sigma^+ - \Sigma_K$; see [**2**, p. 554]. For G twisted, we have $s(x_\beta(t)x_{\beta^\tau}(t^q)) = x_\beta(st)x_{\beta^\tau}(st^q)$ when $|\tau| = 2$ and $s \in \mathrm{GF}(q)$, $t \in \mathrm{GF}(q^2)$, etc. The Chevalley commutator formula now implies that the commutator form is $\mathrm{GF}(q)$-bilinear. This completes the proof of Lemma 3.3.

The key to our next result is the Glauberman character correspondence [**15**, 13.1], which says that when a solvable group S acts coprimely on a group G, there is a one-to-one correspondence between the set of S-invariant irreducible characters of G and the irreducible characters of $C_G(S)$.

PROPOSITION 3.4. *Let $1 \neq \theta \in \mathrm{Irr}(U_J)$ be L_0-invariant. Then $U(3) \leq \mathrm{Ker}\,\theta$ and $U_J/\mathrm{Ker}\,\theta$ is extraspecial. Also L_0 centralizes $M(2)$.*

Proof. Since H_0 fixes θ, [**15**, 13.24] implies that H_0 fixes a nonidentity conjugacy class of U_J. By [**15**, 13.10], H_0 fixes a nonidentity element $u \in U_J$. Writing u as a unique product of elements from root groups, we see that H_0 centralizes a nonidentity element v of some root group in U_J.

We claim that $C_{U_J}(H_0)$ is a root group of order q^3 when G is of type ${}^3D_4$, and is a root group of order q in all other cases, and also $C_{U_J}(H_0) = C_{U_J}(L_0)$.

First suppose G is untwisted. Then $v = x_\gamma(t)$, for some $\gamma \in \Sigma^+ - \Sigma_K$ and $t \in \mathrm{GF}(q)^*$. For $\beta \in \Pi_J$ and $s \in \mathrm{GF}(q)^*$, there is an element $h_\beta(s) \in H_0$ which conjugates v into $x_\gamma(s^{A_{\beta\gamma}}t)$; see [7, p. 194]. Here $A_{\beta\gamma}$ is the Cartan integer associated with β and γ ([7, p. 38]).

Since H_0 centralizes v, it follows that $s^{A_{\beta\gamma}} = 1$ for all $s \in \mathrm{GF}(q^*)$. Since $|A_{\beta\gamma}| \leq 3$, with equality only when Σ is G_2, this implies that $A_{\beta\gamma} = 0$ when $q \geq 5$. The same is true when $q = 4$, since we are excluding the "special" group $G_2(4)$.

Hence γ is orthogonal to Σ_J. It follows that $[U_\beta, U_\gamma] = 1$ for all $\beta \in \Sigma_J$, and so L_0 centralizes U_γ. The maximality of J implies that γ must be the unique positive root orthogonal to Σ_J. It follows that $C_{U_J}(H_0) = C_{U_J}(L_0) = U_\gamma$, which has order q.

Now suppose that G is twisted. Viewing G' as a subgroup of an untwisted Chevalley group $\widehat{G}$ over $\mathrm{GF}(q^{|\tau|})$, write v as a product of root elements of $\widehat{G}$ as in [7, 13.6.3]. (Since we are only interested in computing $C_{U_J}(H_0)$, it doesn't matter that G' is simply connected, while adjoint groups are considered in [7].) Then $v = x_\gamma(t)y$, where $\gamma \in \Sigma^+ - \Sigma_K$, $t \neq 0$, and y is a product of elements in other root subgroups of $\widehat{G}$ which belong to the τ-equivalence class of γ. Since H_0 normalizes each root subgroup of $\widehat{G}$, we see that H_0 centralizes $x_\gamma(t)$ in $\widehat{G}$. By [7, p. 239–41], the generators of H_0 have the form $h_\beta(s)$ $(s \in \mathrm{GF}(q)^*)$, or $h_\beta(s)h_{\beta^\tau}(s^q)$ $(s \in \mathrm{GF}(q^2)^*)$, or the analog for $s \in \mathrm{GF}(q^3)^*$, where in all cases $\beta \in \Pi_K$. It follows that, respectively, $s^{A_{\beta\gamma}} = 1$ for all $s \in \mathrm{GF}(q)^*$, $s^{A_{\beta\gamma}+qA_{\beta^\tau\gamma}} = 1$ for all $s \in \mathrm{GF}(q^2)^*$, or the analogous equation holds for all $s \in \mathrm{GF}(q^3)^*$. This implies that γ is orthogonal to Π_K, and hence to Σ_K when $q \geq 4$.

It follows as above that every root subgroup U_β of $\widehat{G}$, for $\beta \in \Sigma_K$, centralizes U_γ and the root groups of $\widehat{G}$ in the τ-equivalence class of U_γ. Hence L_0 centralizes the root subgroup of G which corresponds to the τ-equivalence class of γ. The preceding argument shows that any root group in U_J centralized by H_0 is also centralized by L_0.

We claim that, as in the untwisted case, there is only one root group in U_J centralized by H_0. When $|K'| = 1$, this is clear because γ, in the preceding paragraph, must be the unique positive root orthogonal to the maximal parabolic subsystem Σ_K. Since γ is then τ-invariant, $|C_{U_J}(H_0)| = q$ when $|K'| = 1$.

Now suppose $|K'| > 1$. Since L_0 centralizes a root group in U_J, L_0 centralizes the image in some $M(i)$ of this root group. This root group image is then invariant under $L = L_0H$. Hence L_0 centralizes some P-chief factor in U_J, so L_0 centralizes W_S for some shape S. By Lemma 3.1, there is only one root in $\Sigma^+ - \Sigma_K$ of shape S. Thus S satisfies the hypotheses of Lemma 3.3. The proof of Lemma 3.3 shows that, since $|K'| > 1$, G must be of type ${}^2A_n$ or ${}^3D_4$ with $K' = \{\alpha_1, \alpha_n\}$ or $\{\alpha_1, \alpha_3, \alpha_4\}$ respectively. In both cases, L_0 acts trivially on $M(2)$, which is isomorphic to a root subgroup of G, namely $U_{\alpha_1+\cdots+\alpha_n}$ or $(\overline{U}_{\alpha_1+\alpha_2+\alpha_3}\overline{U}_{\alpha_1+\alpha_2+\alpha_4}\overline{U}_{\alpha_2+\alpha_3+\alpha_4})_\sigma$. Moreover, L_0 has no other trivial composition factors in U_J. Since any root subgroup centralized by H_0 is also centralized by L_0, as remarked above, it follows that $C_{U_J}(H_0)$ is a single root subgroup of G. This root subgroup has order q when G is of type ${}^2A_n$ and order q^3 when G is of type ${}^3D_4$.

Thus we have proved the claim in the second paragraph of this proof.

Having just observed that $C_{U_J}(H_0)$ maps isomorphically onto $M(2)$ when G is twisted and $|K'| > 1$, we claim the same is true when G is untwisted or when $|K'| = 1$. Under these hypotheses, shape and level are the same thing, and each shape is τ-invariant. It follows that each $M(i)$ is an irreducible $\mathrm{GF}(q)[L_0]$-module. If L_0 centralizes a vector in $M(i)$, then L_0 centralizes $M(i)$ and, by Lemmas 3.1 and 3.3, we have $i = 2$ and the unique shape on level 2 contains only one root. Hence if L_0 centralizes a root subgroup $(\overline{U}_\gamma)_\sigma$ in U_J, then γ is the unique root in $\Sigma^+ - \Sigma_K$ on level 2. Hence $C_{U_J}(H_0) = (\overline{U}_\gamma)_\sigma$ maps isomorphically onto $M(2)$.

Thus in all cases $C_{U_J}(H_0) = C_{U_J}(L_0)$ maps isomorphically onto $M(2)$. In particular, L_0 centralizes $M(2)$.

Let $1 \neq \lambda$ be a linear character of $M(2)$. View λ as an irreducible character of $U(2)$.

Suppose first that G is not of type ${}^3D_4$. Then the nondegeneracy and the $\mathrm{GF}(q)$-bilinearity of the commutator form from $M(1)\times M(1)$ to $M(2)$ and the fact that $|M(2)| = q$ implies that $[v, M(1)] = M(2)$ for any nonzero vector $v \in M(1)$. It follows that $Z(U_J/\operatorname{Ker}\lambda) = U(2)/\operatorname{Ker}\lambda$. Hence $U_J/\operatorname{Ker}\lambda$ is extraspecial. There is a unique character $\theta_\lambda \in \mathrm{Irr}(U_J/\mathrm{Ker}\,\lambda)$ which lies over λ. Since L_0 centralizes $M(2)$, θ_λ is an L_0-invariant character of U_J. Together with the principal character of U_J, this yields $|M(2)| = q$ distinct L_0-invariant irreducible characters of U_J. Since $|M(2)| = |C_{U_J}(H_0)|$ and since $C_{U_J}(H_0)$ is abelian, Glauberman's correspondence [**15**, 13.1] implies that there are no other H_0-invariant irreducible characters of U_J, and therefore no other L_0-invariant irreducible characters of U_J. Since $\mathrm{Ker}\,\theta_\lambda = \mathrm{Ker}\,\lambda$, $U_J/\mathrm{Ker}\,\theta_\lambda$ is extraspecial, as desired.

Next suppose G has type ${}^3D_4$. Let $Z = Z(U_J/\mathrm{Ker}\,\lambda)$. Since L_0 fixes λ, Z and $\overline{Z} = Z/(U(2)/\mathrm{Ker}\,\lambda)$ are L_0-invariant. If $\overline{z} \in \overline{Z} \leq M(1)$, and $c \in \mathrm{GF}(q)^*$, then $[c\overline{z}, M(1)] = [\overline{z}, cM(1)] = [\overline{z}, M(1)] \leq \mathrm{Ker}\,\lambda$, and so $c\overline{z} \in \overline{Z}$. It follows that $\overline{Z}$ is a proper $\mathrm{GF}(q)[L_0]$-submodule of $M(1)$.

Since $C_{U_J}(H_0) \leq U(2)$, H_0 has no fixed points on $M(1)$. Hence $C_Z(H_0) = U(2)/\mathrm{Ker}\,\lambda$, and so $Z = [H_0, Z] \times (U(2)/\mathrm{Ker}\,\lambda)$.

Let $\theta \in \mathrm{Irr}(U_J|\lambda)$ be H_0-invariant. View θ as a character in $\mathrm{Irr}(U_J/\mathrm{Ker}\,\lambda)$. Let $\lambda \times \mu \in \mathrm{Irr}(Z) = \mathrm{Irr}((U(2)/\mathrm{Ker}\,\lambda) \times [H_0, Z])$ be the unique and linear irreducible consituent of θ_Z. Since $\lambda \times \mu$ is H_0-invariant and H_0 has no fixed points on $[H_0, Z]$, we have $\mu = 1$.

Let $E = (U_J/\mathrm{Ker}\,\lambda)/[H_0, Z]$. We claim that E is extraspecial. Let $W = Z/[H_0, Z] \cong U(2)/\mathrm{Ker}\,\lambda$. Since E/W is elementary abelian and $|W| = p$ it suffices to show that $Z(E) = W$. Let $y \in U_J/\mathrm{Ker}\,\lambda$ and suppose that $y[H_0, Z]$ belongs to $Z(E)$. Then $[y, U_J/\mathrm{Ker}\,\lambda] \leq [H_0, Z]$. Since $(U_J/\mathrm{Ker}\,\lambda)' = U(2)/\mathrm{Ker}\,\lambda$, this implies that $y \in Z$, and so $y[H_0, Z] \in W$, as desired. Thus E is extraspecial.

Let θ_λ be the unique character in $\mathrm{Irr}(E|\lambda)$. Since $[H_0, Z] \leq \mathrm{Ker}\,\theta$, we have $\theta = \theta_\lambda$. In other words, θ_λ is the unique H_0-invariant character in $\mathrm{Irr}(U_J|\lambda)$. For each linear character λ of $M(2)$, there is exactly one H_0-invariant character in $\mathrm{Irr}(U_J|\lambda)$. As above, the Glauberman correspondence shows that there are no other H_0-invariant irreducible characters of U_J. For $\lambda \neq 1$, $U_J/\mathrm{Ker}\,\theta_\lambda$ is extraspecial, as desired.

THEOREM 3.5. *Let (G, x, L_J) be an admissible triple with rank $(G) \geq 2$. Let $1 \neq \theta \in \mathrm{Irr}(U_J)$ be L_0-invariant. Let $\chi \in \mathrm{Irr}(P_J|\theta)$. Then $|\chi(x)/\chi(1)| \leq 1/q$.*

Proof. Let I be the inertia group of θ in P_J. Since $P_J' = L_0U_J \leq I$, I is normal in P_J. Hence if $x \notin I$, then $\chi(x) = 0$. Therefore we assume that $x \in I$.

Let $\chi = \psi^{P_J}$ with $\psi \in \mathrm{Irr}(I|\theta)$. Since $\chi(x) = \sum \psi(x^t)$, where t ranges over a transversal to I in P_J, it suffices to show that $|\psi(x^t)/\psi(1)| \leq 1/q$ for each t. Fix t and set $y = x^t$. Note that $y \in I$.

By Proposition 3.4, $\theta|_{U(2)}$ has a unique and linear irreducible constituent λ. Since $y \in I$, y fixes λ. The proof of Proposition 3.4 shows that $M(2)$, which has order q or q^3, is a single P-chief factor W_S. Hence $M(2)$ is an irreducible $\mathrm{GF}(q)[L_J]$-module. Since L_0 centralizes $M(2)$ and L_J/L_0 is abelian, $C_{M(2)}(y)$ is a $\mathrm{GF}(q)[L_J]$-submodule of $M(2)$. Since y fixes a linear character of $M(2)$, y fixes a nonzero vector in $M(2)$. Irreduciblity then forces $C_{M(2)}(y) = M(2)$.

Let $E = U_J/\operatorname{Ker}\theta$. By Proposition 3.4 and its proof, E is extraspecial and

$$\begin{aligned} E/Z(E) &= (U_J/\operatorname{Ker}\theta)/(U(2)\operatorname{Ker}\theta/\operatorname{Ker}\theta) \\ &\cong U_J/U(2)\operatorname{Ker}\theta. \end{aligned}$$

By the proof of Proposition 3.4, $U(2)\operatorname{Ker}\theta = U(2)$ when G is not of type ${}^3D_4$. Whether G has type ${}^3D_4$ or not, $U(2)/\operatorname{Ker}\theta/U(2)$ is the image in $M(1)$ of $Z(U_J/\operatorname{Ker}\lambda)$, and is a proper (possibly zero) $\mathrm{GF}(q)[\langle L_0, y\rangle]$-submodule of $M(1)$. Hence $E/Z(E)$ can be identified with a $\mathrm{GF}(q)[\langle L_0, y\rangle]$-complement M to $U(2)\operatorname{Ker}\theta/U(2)$ in $M(1)$.

By Lemmas 3.1 and 3.3, L_0 centralizes no P-chief factor $W_S \leq M(1)$. Since W_S is an irreducible $\mathrm{GF}(q)[L]$-module, Clifford's Theorem implies that no $\mathrm{GF}(q)[L_0]$-submodule of W_S is centralized by L_0. Hence $M(1)$ contains no trivial $\mathrm{GF}(q)[L_0]$-submodule. In particular, L_0 does not centralize M. Since y centralizes no component of L_0, it follows that y doesn't centralize M.

Since y centralizes $M(2)$, y preserves the commutator form from $M(1) \times M(1)$ to $M(2)$, and so y preserves the restriction of this form to $M \times M$. Since E is extraspecial, this restricted form is nondegenerate.

Now M is the orthogonal direct sum of $[M, y]$ and $C_M(y)$. Since M is nondegenerate, so is $[M, y]$. Hence $[M, y]$ admits a nondegenerate $\mathrm{GF}(q)$-bilinear alternating form. Thus $|[M, y]| \geq q^2$ and so $|[E/Z(E), y]| \geq q^2$.

To show that $|\psi(y)/\psi(1)| \leq 1/q$, it suffices to show that $|\nu(y)/\nu(1)| \leq 1/q$, where ν is an arbitrary irreducible constituent of $\psi|_{\langle y\rangle U_J}$.

Since ψ_{U_J} is a multiple of θ, we may view ν as a character in $\mathrm{Irr}(\langle y\rangle E|\theta)$.

Let $Z(E) \leq D \leq E$, with $D/Z(E) = [E/Z(E), y]$. Then D is extraspecial, since $E/Z(E)$ is the orthogonal direct sum of $[E/Z(E), y]$ and $C_{E/Z(E)}(y)$.

Let ω be an arbitrary irreducible constituent of $\nu|_{\langle y\rangle D}$. After multiplying ω by a linear character of $\langle y\rangle D/D$, we may assume that $C_{\langle y\rangle}(D) \leq \mathrm{Ker}\,\omega$. Then **[14**, V, 17.13] yields that $\omega(1) = |[E/Z(E), y]|^{1/2}$ and $|\omega(y)| = 1$. Applying this to each irreducible constituent of $\nu|_{\langle y\rangle D}$, we obtain $|\nu(y)/\nu(1)| \leq |[E/Z(E), y]|^{-1/2}$. As shown above, the last quantity is at most $1/q$. This completes the proof.

4. Main theorem. Let G be a simple admissible group, P_J a standard maximal parabolic of G, and $\chi \in \mathrm{Irr}(G)$. There exists an absolute constant N such that χ_{P_J} has at most $N-1$ linear constituents, counting multiplicities. By Theorems 2.10 and 2.17, we may take $N = 1 + 13 \cdot 576$. We now state our main theorem, which will be proved at the end of this section.

MAIN THEOREM. *Let (G, x) be a simple admissible pair, with $q \geq 3N$, for N as above. Let χ be a nonlinear irreducible character of G. Then $|\chi(x)| \leq (3N/q)\chi(1) - N$.*

LEMMA 4.1. *Let (G, x) be a simple admissible pair with rank $(G) = 1$. Then $|\chi(x)/\chi(1)| \leq 2/(q-1)$, for any nonlinear $\chi \in \mathrm{Irr}(G)$.*

Proof. We have $G' = \mathrm{SL}(2, q)$ or $\mathrm{SU}(3, q)$. By Lemma 1.10, or an entirely similar argument when $G' = \mathrm{SU}(3, q)$, we see that it suffices to establish the inequality in the statement of this lemma for all noncentral semisimple elements of $\mathrm{SL}(2, q)$, $\mathrm{GL}(2, q)$, $\mathrm{SU}(3, q)$, and $\mathrm{U}(3, q)$.

For $\mathrm{SL}(2, q)$, $\mathrm{GL}(2, q)$, and $\mathrm{U}(3, q)$, we can do this by checking character tables (**[11]**, **[18]**, **[12]**). It remains to consider $\mathrm{SU}(3, q)$, whose character table seems not to be available in the literature.

If $x \in G = \mathrm{SU}(3, q)$ is contained in no proper parabolic of G, then $C_G(x)$ is a torus of G, so $|\chi(x)| \leq |C_G(x)|^{1/2} \leq q+1$. By **[16**, p. 419], $\chi(1) \geq q^2 - q$. It follows that $|\chi(x)/\chi(1)| \leq 2/(q-1)$.

We may then assume that $x \in B = HU$, the standard Borel subgroup. Furthermore, we may assume $x \in H$. Let λ be a linear constituent of χ_B. Since G is doubly transitive on the cosets of B,

$(\lambda^G, \lambda^G) \leq (1_B^G, 1_B^G) = 2$, and so $(\chi_B, \lambda) = 1$. By Harish-Chandra's theorem [**10**, 70.15(A)], all the linear constituents of χ_B are conjugate under the Weyl group of G, which has order 2. Hence χ_B has at most two linear constituents, counting multiplicities.

Let θ be a nonlinear constituent of χ_B and let ω be an irreducible constituent of θ_U. If $U' = Z(U) \leq \operatorname{Ker} \omega$, then, since $H/Z(G)$ is fixed point free on U/U', we have $x \notin I_B(\omega) \triangleleft B$, and so $\theta(x) = 0$.

If $Z(U) \nleq \operatorname{Ker} \omega$, then $\operatorname{Ker} \omega \leq Z(U)$ and $U/\operatorname{Ker} \omega$ is extraspecial. Since $I_B(\omega) \triangleleft B$, we may assume that $x \in I_B(\omega)$, since otherwise $\theta(x) = 0$. Then $\theta|_{\langle U, x\rangle}$ is a sum of extensions of ω. Let ζ be one such extension. After multiplying ζ by a linear character, we may assume that $C_{\langle x\rangle}(U) \leq \operatorname{Ker} \zeta$. Now $\langle U, x\rangle/\operatorname{Ker} \zeta$ satisfies the hypotheses of [**14**, V, 17.13]. We conclude that $\zeta(1) = q$ and $|\zeta(x)| = 1$. It follows that $|\theta(x)/\theta(1)| \leq 1/q$.

Hence $|\chi(x)| \leq 2 + (\chi(1) - 2)/q$, assuming that x lies in a proper parabolic of G. Since $\chi(1) \geq q^2 - q$, we have $|\chi(x)| \leq 2\chi(1)/q$, which completes the proof.

LEMMA 4.2. *Let (G, x, L_J) be an admissible triple with rank $(G) \geq 2$ and $q \geq 3N$. Let χ be a nonlinear irreducible character of $\langle x, L'_J\rangle$ and assume that the Main Theorem holds for groups of smaller rank than G. Then $|\chi(x)| \leq (3N/q)\chi(1) - N$.*

Proof. Let K be a component of L_J. Lemma 1.6 implies that $\langle x, K\rangle$ is a simple admissible group. In particular, if L_J has only one component, then $(\langle x, L'_J\rangle, x)$ is a simple admissible pair, and the inductive hypothesis yields the desired bound for $|\chi(x)|$.

We now assume that $L'_J = K_1K_2$, the case of three components being entirely similar.

If $x \in L'_J$, write $x = x_1x_2$, with $x_1 \in K_1$ and $x_2 \in K_2$. By the definition of admissible triple, $x_i \notin Z(K_i)$ for $i = 1, 2$, and so each x_i has prime order modulo $Z(K_i)$. Then (K_i, x_i) is a simple admissible pair for $i = 1, 2$. We may write $\chi(x) = \chi_1(x_1)\chi_2(x_2)$ with $\chi_i \in \operatorname{Irr}(K_i)$, for $i = 1, 2$. We assume both χ_1 and χ_2 are nonlinear, the other case being easier. The inductive hypothesis yields $|\chi_i(x_i)| \leq (3N/q)\chi_i(1)N$ for $i = 1, 2$. Hence

$$|\chi(x)| \leq ((3N/q)\chi_1(1) - N)((3N/q)\chi_2(1) - N).$$

If a and b are real numbers, both greater than N, one checks that $(a - N)(b - N) \leq ab - N$. Hence $|\chi(x)| \leq (3N/q)^2\chi(1) - N$. Since $q \geq 3N$, the desired bound follows.

We now assume that $x \notin L'_J$. Let y be the automorphism of K_1K_2 induced by x and let y_i be the restriction of y to $\mathrm{Aut}(K_i)$ for $i = 1, 2$. Since x centralizes no component of L_J, the orders of y, y_1 and y_2 are the same prime number r. We form the semidirect products $L = \langle y\rangle K_1K_2$, $L_1 = \langle y_1\rangle K_1$, and $L_2 = \langle y_2\rangle K_2$.

Our inductive hypothesis implies that $|\alpha(x)| \leq (3N/q)\alpha(1) - N$ for any nonlinear irreducible character α of $\langle K_i, x\rangle$, since $(\langle K_i, x\rangle, x)$ is a simple admissible pair. Let $W_i = \langle w_i\rangle$ be a cyclic group of order $|x|$ and form the central product $\langle K_i, x\rangle * W_i$, where $w_i^r = x^r \in Z(\langle K_i, x\rangle)$. Then xw_i^{-1} has order r and $\langle K_i, x\rangle * W_i = \langle K_i, xw_i^{-1}\rangle W_i$. If $\alpha \in \mathrm{Irr}(\langle K_i, x\rangle)$, then $\alpha(x) = \alpha^*(xw_i^{-1})\lambda(w_i)$, where $\alpha^* \in \mathrm{Irr}(\langle K_i, xw_i^{-1}\rangle)$ has the same degree as α and λ is a linear character of W_i. Similarly if $\beta \in \mathrm{Irr}(\langle K_i, xw_i^{-1}\rangle)$, then $\beta(xw_i^{-1}) = \beta^*(x)\mu(w_i^{-1})$, where $\beta^* \in \mathrm{Irr}(\langle K_i, x\rangle)$ has the same degree as β and μ is a linear character of W_i. There is an obvious isomorphism from L_i to $\langle xw_i^{-1}, K_i\rangle$ which takes y_i to xw_i^{-1}. It follows that $|\gamma(y_i)| \leq (3N/q)\gamma(1) - N$ for every nonlinear irreducible character γ of L_i.

Let $Z = \langle z\rangle$ be a cyclic group of order $|x|$. Form the central product $(\langle x\rangle K_1K_2) * Z$, where $z^r = x^r$. Then $(\langle x\rangle K_1K_2) * Z = (\langle xz^{-1}\rangle K_1K_2) * Z \cong L * Z$. The argument of the preceding paragraph shows that if $|\zeta(y)| \leq (3N/q)\zeta(1) - N$ for every nonlinear irreducible character ζ of L, then the conclusion of the lemma holds.

We are therefore reduced to working with the split extensions L, L_1 and L_2.

Let $\pi\colon K_1 \times K_2 \to K_1K_2$ be the natural map. Then y acts on $K_1 \times K_2$, stabilizing (in fact centralizing) $\mathrm{Ker}\,\pi$. Let

$$\eta\colon \langle y\rangle(K_1 \times K_2) \to L_1 \times L_2$$

be the map sending $y^k(x_1, x_2)$ to $(y_1^k x_1, y_2^k x_2)$, for $k \in \mathbf{Z}$, $x_i \in K_i$, $i = 1, 2$. Then η is an injective homomorphism. Since $Z(K_i) \leq Z(\overline{K}_i)$ for an appropriate semisimple algebraic group $\overline{K}_i$ (see [**8**, 3.6.8]), it follows that x, which belongs to a reductive overgroup of $\overline{K}_i$, centralizes $Z(K_i)$. This implies that $\eta(\mathrm{Ker}\,\pi)$ is normal (in fact central) in $L_1 \times L_2$, and so η induces an embedding of $L \cong \langle y\rangle(K_1 \times K_2)/\mathrm{Ker}\,\pi$ into $(L_1 \times L_2)/\eta(\mathrm{Ker}\,\pi)$.

Let $\zeta \in \mathrm{Irr}(L)$ be nonlinear. Let $\chi_1\chi_2$ be an irreducible constituent of $\zeta|_{K_1K_2}$ with $\chi_i \in \mathrm{Irr}(K_i)$, $i = 1, 2$. If $\chi_1\chi_2$ is not $\langle y\rangle$-invariant, then $\zeta = (\chi_1\chi_2)^L$ and $\zeta(y) = 0$. We may therefore assume that $\chi_1\chi_2$ is $\langle y\rangle$-invariant. It follows that $\zeta|_{K_1K_2} = \chi_1\chi_2$ and χ_i is $\langle y_i\rangle$-invariant for $i = 1, 2$. Since L_i/K_i is cyclic, χ_i extends to $\zeta_i \in \mathrm{Irr}(L_i)$ for $i = 1, 2$, and so $\chi_1 \times \chi_2 \in \mathrm{Irr}(K_1 \times K_2)$ extends to $\zeta_1 \times \zeta_2 \in \mathrm{Irr}(L_1 \times L_2)$. Hence $\chi_1\chi_2 \in \mathrm{Irr}(K_1K_2) = \mathrm{Irr}((K_1 \times K_2)/\eta(\mathrm{Ker}\,\pi))$ extends to $\zeta_1\zeta_2 \in \mathrm{Irr}((L_1 \times L_2)/\eta(\mathrm{Ker}\,\pi))$.

Viewing L as a subgroup of $(L_1 \times L_2)/\eta(\mathrm{Ker}\,\pi)$, we see that $\chi_1\chi_2 \in \mathrm{Irr}(K_1K_2)$ is extendible to $(\zeta_1\zeta_2)_L \in \mathrm{Irr}(L)$. Hence $\zeta = \lambda(\zeta_1\zeta_2)_L$ where λ is a linear character of L/K_1K_2. Hence $|\zeta(y)| = |\zeta_1\zeta_2(\eta(y))| = |\zeta_1(y_1)|\,|\zeta_2(y_2)|$. As proved above, our inductive hypothesis implies that $|\zeta_i(y_i)| \le (3N/q)\zeta_i(1) - N$ if ζ_i is nonlinear. The desired bound for $|\zeta(y)|$ now follows as in the third paragraph of this proof. As remarked above, this completes the proof.

LEMMA 4.3. *Let (G, x) be an admissible pair. Suppose x centralizes no component of G. Let $\theta \in \mathrm{Irr}(G)$ with $1 + \theta$ a (not necessarily faithful) doubly transitive permutation character of G. Assume $q > 11$. Then $|\theta(x)| \le (3N/q)\theta(1) - 2N$.*

Proof. By [5, p. 8], $S \le G/\mathrm{Ker}\,\theta \le \mathrm{Aut}(S)$ for a simple group S. Either S has Lie rank 1 or S is some $\mathrm{PSL}(n, q)$. Since x centralizes no component of G, $x \notin \mathrm{Ker}\,\theta$. Set $\overline{G} = G/\mathrm{Ker}\,\theta$.

In the rank 1 case, since $q > 11$, θ is the Steinberg character of $\overline{G}$. Hence $|\theta(x)| \le \theta(1)/q \le (3N/q)\theta(1) - 2N$, since $\theta(1) \ge q$.

We now assume $S = \mathrm{PSL}(n, q)$ with $n \ge 3$. Then $\overline{G}$ is $\mathrm{PSL}(n, q)$ or $\langle y\rangle\mathrm{PSL}(n, q)$, where y is a p'-element of prime order. Since y induces an inner times diagonal automorphism of $\mathrm{PSL}(n, q)$, $\overline{G}$ is isomorphic to a subgroup of $\mathrm{PGL}(n, q)$. By [5, p. 8] $\overline{G}$ has two doubly transitive permutation representations, which are conjugate under the inverse transpose automorphism of $\mathrm{PGL}(n, q)$. Hence we may assume that $1 + \theta$ is the permutation character for the action of $\overline{G}$ on the points of $\mathrm{PG}(n-1, q)$. Let $d_1, \dots, d_r$ be the dimensions of the eigenspaces, if any, of a preimage of $\overline{x}$ in $\mathrm{GL}(n, q)$. Then $1 + \theta(x) = 0$ or $1 + \theta(x) = ((q^{d_1} - 1) + \cdots + (q^{d_r} - 1))/(q-1) \le ((q^{n-1} - 1) + (q - 1))/(q - 1)$. If $\theta(x) \ne -1$, then $0 \le \theta(x)/\theta(1) \le (1 + \theta(x))/(1 + \theta(1)) \le (q^{n-1} + q - 2)/(q^n - 1) \le 2/q$. In all cases $|\theta(x)| \le 2\theta(1)/q \le (3N/q)\theta(1) - 2N$.

LEMMA 4.4. *Let (G, x, L_J) be an admissible triple with rank $(G) \ge 2$. Let $\chi \in \mathrm{Irr}(P_J)$ and suppose that the irreducible constituents of χ_{U_J} are not L'_J-invariant. Assume that $q \ge 3N$ and that the Main Theorem holds for groups of smaller rank than G. Then $|\chi(x)| \le (3N/q)\chi(1) - 2N + 1$.*

Proof. By Clifford's Theorem, χ is induced from an inertia group MU_J, where $M \leq L_J$. By hypothesis, $L'_J \not\leq M$. Mackey's Theorem yields $\chi_{L_J} = \rho^J$, for some character ρ of M. If x is not conjugate in L_J to an element of M, then $\chi(x) = 0$. Hence we may assume $x \in M$. Let $G_0 = L'_J M$. Then $\chi(x) = \chi_{L_J}(x) = (\rho^{G_0})^{L_J}(x) = \sum \rho^{G_0}(x^h)$, where h ranges over a transversal to G_0 in L_J.

It suffices to show that $|\rho^{G_0}(x^h)| \leq (3N/q)\rho^{G_0}(1) - 2N + 1$ for each h. A computation using the formula for induced characters shows that $|\rho^{G_0}(x^h)/\rho^{G_0}(1)| \leq 1_M^{G_0}(x^h)/1_M^{G_0}(1)$. Thus if $1_M^{G_0}(x^h) \leq (3N/q)1_M^{G_0}(1) - 2N + 1$, then

$$\begin{aligned}|\rho^{G_0}(x^h)/\rho^{G_0}(1)| &\leq (3N/q) + (1-2N)/1_M^{G_0}(1)\\ &\leq (3N/q) + (1-2N)/\rho^{G_0}(1),\end{aligned}$$

and so $|\rho^{G_0}(x^h)| \leq (3N/q)\rho^{G_0}(1) - 2N + 1$, the desired inequality. Hence we need only show that $1_M^{G_0}(x^h) \leq (3N/q)1_M^{G_0}(1) - 2N + 1$. Since $L'_J \not\leq M$, we have $M < G_0$.

Clearly 1_{G_0} has multiplicity 1 in $1_M^{G_0}$. If λ is a linear constituent of $1_M^{G_0}$, then $\lambda_M = 1_M$. Also $L'_J = G'_0 \leq \operatorname{Ker}\lambda$. Hence $ML'_J = G_0 \leq \operatorname{Ker}\lambda$, and so $\lambda = 1_{G_0}$. Thus $1_M^{G_0} - 1_{G_0}$ has no linear constituents.

Let $G_1 = \langle x^h, L'_J\rangle$. Then (G_1, x^h) is an admissible pair by Lemma 1.6, and x^h centralizes no component of G_1. We have $G_1 \leq G_0$ and $MG_1 = G_0$. If $\alpha \neq 1_{G_0}$ is an irreducible constituent of $1_M^{G_0}$, then α is nonlinear by the preceding paragraph. Then $L'_J \not\leq \operatorname{Ker}\alpha$, and so no irreducible constituent of α_{G_1} has L'_J in its kernel. Since $G'_1 = L'_J$, all irreducible constituents of α_{G_1} are nonlinear. By Mackey's theorem, $1_{M\cap G_1}^{G_1} = (1_M^{G_0})_{G_1}$. Hence $1_{M\cap G_1}^{G_1} - 1_{G_1} = (1_M^{G_0} - 1_{G_0})_{G_1}$ is a sum of nonlinear irreducible constituents. Lemma 4.2 implies that $|\beta(x^h)| \leq (3N/q)\beta(1) - N$ for every nonprincipal irreducible constituent β of $1_{M\cap G_1}^{G_1}$. If $1_{M\cap G_1}^{G_1} - 1_{G_1}$ is not irreducible, then $1_M^{G_0}(x^h) = 1_{M\cap G_1}^{G_1}(x^h) < 1 + (3N/q)1_{M\cap G_1}^{G_1}(1) - 2N = (3N/q)1_M^{G_0}(1) - 2N + 1$, as desired.

Thus we may assume that G_1 is doubly transitive on the cosets of $M \cap G_1$. Let $1_{M\cap G_1}^{G_1} = 1 + \theta$, with $\theta \in \operatorname{Irr}(G_1)$. Then Lemma 4.3, applied to the admissible pair (G_1, x^h), yields $1_M^{G_0}(x^h) = 1_{M\cap G_1}^{G_1}(x^h) \leq 1 + |\theta(x^h)| \leq 1 + (3N/q)\theta(1) - 2N < (3N/q)1_M^{G_0}(1) - 2N + 1$, as desired.

Proof of Main Theorem. We proceed by induction on the rank of G. If G has rank 1, then Lemma 4.1 yields $|\chi(x)| \leq 2\chi(1)/(q-1)$. Hence $|\chi(x)| \leq 3\chi(1)/q$. Thus it suffices to show that $3\chi(1)/q \leq (3N/q)\chi(1) - N$. This is equivalent to $\chi(1) \geq qN/(3N-3)$, which holds since $\chi(1) \geq (q-1)/2$ and $q \geq 3N \geq 15$.

Hence we assume that rank $(G) > 1$. If x lies in no proper parabolic of G, then Theorem 1.11 says that $|\chi(x)| \leq 6\chi(1)/q$, so we need to show that $6\chi(1)/q \leq (3N/q)\chi(1) - N$. This simplifies to $\chi(1) \geq qN/(3N-6)$, which holds since $\chi(1) \geq (q^2-1)/2$ by [**16**, p. 419]. We therefore assume that x lies in a proper parabolic of G. Let (G, x, L_J) be an admissible triple, where x may be replaced by a G-conjugate. Write $\chi_{P_J} = \chi_1 + \chi_2 + \chi_3 + \chi_4$, as in the introduction to this paper. By Theorems 2.10 and 2.17, Lemma 4.2, Theorem 3.5, Lemma 4.4 and our inductive hypothesis, $|\chi_1(x)| \leq N-1$, $|\chi_2(x)| \leq (3N/q)\chi_2(1) - N$, $|\chi_3(x)| \leq \chi_3(1)/q$, and $|\chi_4(x)| \leq (3N/q)\chi_4(1) - 2N + 1$.

Let θ be an irreducible constituent of $(\chi_3)_{U_J}$. Then $U_J/\operatorname{Ker}\theta$ is extraspecial by Proposition 3.4 and its commutator factor group has order at least q^2 by the proof of Theorem 3.5. Hence $\theta(1) \geq q$, and so $\chi_3(1) \geq q$. It follows that $\chi_3(1)/q \leq (3N/q)\chi_3(1) - N$, and so $|\chi_3(x)| \leq (3N/q)\chi_3(1) - N$.

If $\chi_4 \neq 0$, the triangle inequality shows that $|\chi(x)| \leq (3N/q)\chi(1) - N$. We may therefore assume $\chi_4 = 0$. Then $\chi_3 \neq 0$, since otherwise $U_J \leq \operatorname{Ker}\chi$, which is impossible. If both χ_2 and χ_3 are nonzero, the triangle inequality and the conclusion of the preceding paragraph yield the desired bound for $|\chi(x)|$, so we assume that $\chi_2 = \chi_4 = 0$.

Then $|\chi(x)| \leq N - 1 + (\chi_3(1)/q)$. It suffices to show that $N - 1 + (\chi_3(1)/q) \leq (3N/q)\chi_3(1) - N$. This is equivalent to $\chi_3(1) \geq ((2N-1)/(3N-1))q$, which holds because $\chi_3(1) \geq q$.

COROLLARY 4.5. *Let $\overline{G}$ be a connected reductive algebraic group over $\overline{\mathrm{GF}(p)}$ whose commutator subgroup is simple, simply connected, and not special. Let q and N be as in the Main Theorem. Let x be an admissible element of $\overline{G}_\sigma$. Let G be a group satisfying $\langle \overline{G}'_\sigma, x\rangle \leq G \leq \overline{G}_\sigma$. Let $\chi \in \operatorname{Irr}(G)$ be nonlinear. Then $|\chi(x)| \leq (3N/q)\chi(1) - N$.*

Proof. Apply the Main Theorem to the constituents of the restriction of χ to $\langle \overline{G}'_\sigma, x\rangle$.

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Received January 22, 1990 and in revised form July 17, 1990. Research partially supported by a grant from the National Security Agency and by a Wayne State University Career Development Chair.

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