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MULTI-TUPLE HULLS

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We study two general families of hulls related to vector-valued functions and their interrelations with multi-tuple Shilov boundaries of uniform algebras.

1. Introduction. The classical hulls—polynomial, rational, holomorphic, A-convex etc. are tightly and naturally connected with functional approximations and interpolations. Recently several new families of hulls have come into appearance. Namely Basener [**2**] has used a generalization of the family of polynomial hulls in his study of q-holomorphic functions. Recently Slodkowski [**8**] has used a generalization of rational hulls in his investigation of analytic perturbation of Taylor spectrum, and Corach and Suárez [**3**] have introduced a general family of rational hulls in order to evaluate the topological stable rank of some algebras. In [**11**] there were investigated the properties of two families of hulls, the so called *n-tuple rational A-convex hulls* and *n-tuple A-convex hulls.*

The multi-tuple Shilov boundaries of commutative Banach algebras have proved to be essential tools in the investigation of multi-dimensional analytic structures in algebra spectra. Results concerning relationships between these boundaries and the analyticity in algebra spectra have appeared often during the last fifteen years (e.g. Basener [**1**], Kumagai [**5**], Sibony [**6**], Tonev ([**11**], [**12**]) etc.). Various properties of multi-dimensional Shilov boundaries have been investigated by Basener [**1**], [**2**], Sibony [**6**], Slodkowski ([**7**], [**8**]), Tonev ([**10**], [**11**]) and others.

In this paper we establish a unified approach to the above mentioned families of hulls, study their properties and investigate their interrelations with the vector valued functions and multi-tuple Shilov boundaries of uniform algebras.

2. Multi-tuple rational A-convex hulls. Let A be a *uniform algebra* over $\mathbf{C}$ with unit. That is A is a separating closed subalgebra of the space of all continuous complex valued functions on some compact Hausdorff space X which contains the constants and the norm of

$f \in A$ is the maximum of $|f(x)|$ on X. As usual $\operatorname{sp} A$ denotes *the maximal ideal space* of A and $\hat{f}$ denotes the *Gelfand extension* of a given function f of A. The *Shilov boundary* ∂A of A is the smallest closed subset of $\operatorname{sp} A$ on which the Gelfand extensions of all functions of A assume the maximums of their absolute values. Throughout this paper we shall assume that $\operatorname{sp} A$ is identified with the set X and that the Gelfand extensions $\hat{f}(m)$ are identified with the algebra elements $f(m)$. A^n will denote the set of all n-tuples of functions from A.

DEFINITION 1. The *n-tuple rational A-convex hull* $r_n(E)$ of a subset E of $\operatorname{sp} A$ is the biggest among all closed subsets K of $\operatorname{sp} A$ for which the equality

$$\min_{x \in K} \|F(x)\| = \min_{x \in E} \|F(x)\| \tag{1}$$

holds for every n-tuple $F = (f_1, \dots, f_n)$ of functions in A. E is called *n-tuple rationally A-convex* if $r_n(E) = E$.

Obviously $r_n(E)$ is a closed subset of $\operatorname{sp} A$. One can see that $r_n(E) = \{m \in \operatorname{sp} A \colon \|S(m)\| \geq \min_{m \in E} \|S(m)\|$ for every $S \subset A$ with $\#S \leq n\}$. Naturally, the last inequality is essential for regular subsets S of A only. As a corollary from this observation we get that $E \subset \cdots \subset r_{n+1}(E) \subset r_n(E) \subset \cdots \subset r_1(E) \subset \operatorname{sp} A$.

The next proposition gives a useful characterization of the hulls $r_n(E)$.

PROPOSITION 1. *The n-tuple rational A-convex hull of a subset E of* $\operatorname{sp} A$ *coincides with the set* $r_n(E) = \{m \in \operatorname{sp} A : F(m) \in F(E)$ *for all* $F \in A^n\}$.

Proof. Denote for a while the set $\{m \in \operatorname{sp} A : F(m) \in F(E)$ for all $F \in A^n\}$ by K. Let $m_0 \in r_n(E)$ and let $F \in A^n$ be such that $F(m_0) = 0$. By (1) F vanishes within E so that $0 = F(m_0) \in F(E)$. Hence $K \supset r_n(E)$. Assuming conversely that $K \setminus r_n(E) \neq \varnothing$, for any point $m_0 \in K \setminus r_n(E)$ we have $\|F(m_0)\| < \min_{m \in E} \|F(m)\|$ for some $F \in A^n$. Hence $H(m_0) = 0$ but $0 \notin H(E)$ for $H = F - F(m_0) \in A^n$ in contradiction with $m_0 \in K$. The proposition is proved.

The n-tuple rational A-convex hull is a multi-tuple version of the *rational A-convex hull* $r(E) = \{m \in \operatorname{sp} A : f(m) \in f(E)$ for all $f \in A\}$ of any subset E of the spectrum of a uniform algebra A—namely, as it is easy to check, the 1-tuple rational A-convex hull $r_1(E)$ coincides with $r(E)$.

From Proposition 1 it follows that the n-tuple rational A-convex hulls coincide with the sets $\bigcap_{F\in A^n} F^{-1}\circ F(E)$, i.e. with the *generalized rational hulls* introduced by Corach and Suárez in [3] and utilized by them in their recent investigations on topological stable ranks of commutative Banach algebras.

Since $F(m) \in F(E)$ if and only if the function $H = F - F(m)$ vanishes on E, Proposition 1 implies the following

PROPOSITION 2. *The n-tuple rational A-convex hull $r_n(E)$ of a closed subset E of* $\operatorname{sp} A$ *coincides with the set of these points m in* $\operatorname{sp} A$ *such that every n-tuple $F \in A^n$ with $F(m) = 0$ vanishes on E.*

In general the n-tuple rational A-convex hull of a subset of $\operatorname{sp} A$ does not coincide with the algebra spectrum. For instance if A is the disc-algebra $A(\Delta)$, then $r_1(S^1) = r_1(\partial A) = S^1 \neq \overline{\Delta} = \operatorname{sp} A$, as one can see by applying, say, (1) to the identity function in $\mathbf{C}^1$. In this respect the following corollary from Proposition 2 is of some interest.

COROLLARY 1 [3]. $r_n(E) = \operatorname{sp} A$ *if and only if every n-tuple $F(m)$ over A which does not vanish on E is regular.*

EXAMPLE 1. If $n \geq 2$, then the 1-tuple rational $A(B^n(1))$-convex hull of the unit sphere $S^n(1)$ in $\mathbf{C}^n$ is the unit ball $B^n(1)$.

Indeed, as known, if $n \geq 2$ any holomorphic function vanishes on $S^n(1)$ whenever it vanishes inside $B^n(1)$.

EXAMPLE 2. The n-tuple rational convex hulls $\rho_n(E)$.

Let Λ be an arbitrary set and let $\mathbf{C}^\Lambda$ be the Cartesian product of Λ copies of the complex plane, equipped by the natural topology. Given a compact subset E in $\mathbf{C}^\Lambda$ let $R(E)$ be the closure in $C(E)$ of all rational functions p/q in $\mathbf{C}^\Lambda$ with non-vanishing on E denominators q. Since these rational functions are dense in $R(E)$, the n-tuple rational $R(E)$-convex hull of E is the biggest among all compact subsets N of $\mathbf{C}^\Lambda$, such that $(r_1, \ldots, r_n)(N) = (r_1, \ldots, r_n)(E)$ for every n-tuple $(r_1, \ldots, r_n)$ of rational functions $r_j = p_j/q_j$ in $\mathbf{C}^\Lambda$ with $q_j \neq 0$ on N. We shall refer to this hull as *n-tuple rational convex hull* of E and shall denote it by $\rho_n(E)$.

Being the n-tuple rational $R(E)$-convex hull of E, $\rho_n(E)$ is the biggest among all sets N in $\mathbf{C}^\Lambda$ for which the inequality

$$\inf_{\mathbf{z}\in N} \|(r_1(\mathbf{z}), \ldots, r_n(\mathbf{z}))\| = \min_{\mathbf{z}\in E} \|(r_1(\mathbf{z}), \ldots, r_n(\mathbf{z}))\|$$

holds for every n-tuple $(r_1, \ldots, r_n)$ of rational functions in $\mathbf{C}^\Lambda$ with non-vanishing on E denominators.

A subset $E \subset \mathbf{C}^\Lambda$ is *n-tuple rationally convex* if $\rho_n(E) = E$. The n-tuple rational convex hulls $\rho_n(E)$ are natural generalizations of the usual rational convex hulls $r(E) = \{\mathbf{z} \in \mathbf{C}^\Lambda : |r(\mathbf{z})| \le \max_{\mathbf{z}\in E} |r(\mathbf{z})|$ for every rational function $r(\mathbf{z})$ that is bounded on $E\}$—namely by Corollary 1 one can observe that the 1-tuple rational convex hull $\rho_1(E)$ of a set $E \subset \mathbf{C}^\Lambda$ coincides with $r(E)$.

By applying the relation from Proposition 1 to the identity mapping in $\mathbf{C}^n$ we get

PROPOSITION 3. *Every compact set E in $\mathbf{C}^n$ is k-tuple rationally convex for any $k \ge n$.*

As we shall see below, $\rho_n(E)$ coincides with the n-tuple rational $P(E)$-convex hull of E, i.e. the n-tuple rational A-convex hulls of all compact subsets E in $\mathbf{C}^\Lambda$ are equal for both algebras $A = P(E)$ and $A = R(E)$.

PROPOSITION 4. *The n-tuple rational hull $\rho_n(E)$ of every compact subset E in $\mathbf{C}^\Lambda$ is equal to its n-tuple rational $P(E)$-convex hull.*

Proof. Denote for a while the n-tuple rational A-convex hull of a set by $r_n^A(E)$. Clearly $r_n^{P(E)}(E) \supset r_n^{R(E)}(E) = \rho_n(E)$, because $P(E) \subset R(E)$. If we assume that $r_n^{P(E)}(E) \setminus \rho_n(E) \neq \varnothing$, by Proposition 3 we can find a point m_0 in $r_n^{P(E)}(E)$ and an n-tuple $F \in R^n(E)$ such that $F(m_0) = 0$, but $\|F(m)\| \neq 0$ on E. If $F = (f_1, \ldots, f_n)$, $f_j = p_j/q_j$, where p_j, q_j are polynomials, $q_j \neq 0$ on E, then $(p_1(m_0), \ldots, p_n(m_0)) = 0$ but the n-tuple $(p_1, \ldots, p_n)$ does not vanish on E. Proposition 3 indicates that this contradicts the choice of the point $m_0 \in r_n^{P(E)}(E)$.

In the case $n = 1$ Proposition 4 restricts to the well known equality between the rational hull and the $P(E)$-convex hull of a set $E \in \operatorname{sp} A$, i.e. $r(E) = h^{P(E)}(E)$ (e.g. [**4**]), which by the way motivates the names "n-tuple rational hull" and "n-tuple rational A-convex hull" given to the sets $\rho_n(E)$ and $r_n^A(E)$ respectively.

Together with Proposition 2, Proposition 4 implies the following

COROLLARY 2. *The n-tuple rational convex hull $\rho_n(E)$ of a compact subset E of $\mathbf{C}^\Lambda$ coincides with the set of all points $\mathbf{z} \in \mathbf{C}^\Lambda$ such that for* *every n-tuple of polynomials* $(p_1, \ldots, p_n)$ *vanishing at* $\mathbf{z}$ *the variety* $\{\mathbf{y} \in \mathbf{C}^\Lambda : p_j(\mathbf{y}) = 0, j = 1, \ldots, n\}$ *meets* E.

The sets described in Corollary 2 are precisely the $(n-1)$*-th rational hulls* which were introduced by Slodkowski in **[8]** and used by him in his recent investigation of analytic perturbation of Taylor spectrum of n-tuples of commuting operators. Note that, as shown in **[3]**, $\bigcap_{n\geq 1} r_n(E) = E$.

Let $V(f_1, \ldots, f_n)$ be the vanishing set of a fixed n-tuple $(f_1, \ldots, f_n)$ over A, i.e.

$$V(f_1, \ldots, f_n) = \{m \in \operatorname{sp} A : f_1(m) = f_2(m) = \cdots = f_n(m) = 0\}.$$

Denote by A_E the closure in $C(E)$ of restrictions of all elements of A on a fixed closed subset E of $\operatorname{sp} A$.

THEOREM 1. *Let* k *be a fixed integer,* $1 \leq k \leq n-1$. *The n-tuple rational A-convex hull* $r_n(E)$ *of a closed subset* E *of* $\operatorname{sp} A$ *coincides with the set of these points* $m \in \operatorname{sp} A$ *which belong to the k-tuple rational* $A_{V(S)}$*-convex hulls* $r_k(E \cap V(S))$ *of the sets* $E \cap V(S)$ *for every set* S *in* A *such that* $\#S \leq n-k$ *and with* m *belonging to* $V(S)$.

Proof. Let $K = \{m \in \operatorname{sp} A : m \in r_k(E \cap V(S))$ for every $S \subset A$ with $\#S \leq n-k$ and $m \in V(S)\}$. Suppose that $K \setminus r_n(E) \neq \varnothing$ and let $m_0 \in K \setminus r_n(E)$. By Definition 1 we can find an n-tuple $F = (f_1, \ldots, f_n)$ over A such that $\|F(m_0)\| < \min_{m \in E} \|F(m)\|$. By applying, if necessary, an orthogonal transformation in $\mathbf{C}^n$, we can suppose from the beginning that $F(m_0) = (f_1(m_0), \ldots, f_k(m_0), 0, \ldots, 0)$. Hence $m_0 \in V(S)$ where $S = (f_{k+1}, \ldots, f_n)$. For $T = (f_1, \ldots, f_k)$ we have $\|T(m_0)\| = \|F(m_0)\| < \min_{m \in E} \|F(m)\| \leq \min_{E \cap V(S)} \|F(m)\| = \min_{E \cap V(S)} \|T(m)\| = \min_{r_k(E \cap V(S))} \|T(m)\|$. Consequently $m_0 \notin r_k(E \cap V(S))$ in contradiction with $m_0 \in K$. We conclude that $K \subset r_n(E)$.

Suppose conversely that $r_n(E) \setminus K \neq \varnothing$ and let $m_0 \in r_n(E) \setminus K$. Let S be an $(n-k)$-tuple over A such that $m_0 \in V(S) \setminus r_k(E \cap V(S))$. Consequently there exists a k-tuple $T \in A^k_{V(S)}$, such that $\|T(m_0)\| < r = \min_{E \cap V(S)} \|T(m)\|$. Without loss of generality we can assume that $T \in A^k$. For every positive $\varepsilon < r$ we can find a neighborhood V_ε of the set $E \cap V(S)$ in E on which $\|T(m)\| > r - \varepsilon$. Hence for every $m \in E$ we have

$$(C_\varepsilon^2 \|S(m)\|^2 + \|T(m)\|^2)^{1/2} > r - \varepsilon \tag{2}$$

for some positive constant C_ε large enough. Because $(C_\varepsilon S, T) \in A^n$, (2) holds also on $r_n(E)$. In particular at $m_0 \in V(S)$ we have $\|T(m_0)\| > r - \varepsilon$ and henceforth $\|T(m_0)\| \geq r$ because of the liberty of the choice of ε. Since this contradicts with the initial inequality $\|T(m_0)\| < r$, we conclude that $r_n(E) \subset K$. The theorem is proved.

The case $k = 1$ from Theorem 1 in particular says

COROLLARY 3. *The n-tuple rational A-convex hull $r_n(E)$ of a closed set E in* $\operatorname{sp} A$ *coincides with the set of these points $m \in \operatorname{sp} A$ which belong to the rational $A_{V(S)}$-convex hulls $r(E \cap V(S))$ of the sets $E \cap V(S)$ for any set S in A whose cardinality does not exceed $n-1$ and such that m belongs to $V(S)$, i.e.*

$$r_n(E) = \{m \in \operatorname{sp} A : m \in r(E \cap V(S)) \text{ for any } S \subset A \text{ with } \#S \leq n-1,\ m \in V(S)\}.$$

3. Multi-tuple A-convex hulls. In this section we introduce another family of hulls in algebra spectra by putting some limitations on the n-tuples F from (1). Recall that an n-tuple $F = (f_1, \dots, f_n) \in A^n$ is called *regular* if the functions $f_1, \dots, f_n$ have no common zeros on $\operatorname{sp} A$, i.e. if $V(f_1, \dots, f_n) = \varnothing$, or in other words, if the mapping F does not vanish on $\operatorname{sp} A$.

DEFINITION 2. The *n-tuple A-convex hull* $h_n(E)$ of a closed subset E of $\operatorname{sp} A$ is the union of all closed subsets N of $\operatorname{sp} A$ which contain E and such that the inequality

$$\min_{m \in N} \|F(m)\| = \min_{m \in E} \|F(m)\| \tag{3}$$

holds for every non-vanishing on N n-tuple F of functions from A. E is an *n-tuple A-convex set* if it coincides with its n-tuple A-convex hull $h_n(E)$.

The n-tuple A-convex hulls are closed subsets of $\operatorname{sp} A$, since the closure $[N]$ of a set $N \in \operatorname{sp} A$ that satisfies (3) also satisfies (3). In fact $h_n(E)$ coincides with the union of all subsets N of $\operatorname{sp} A$ which satisfy (3).

The next proposition, which proof is analogical to that of Proposition 1, gives a useful characterization of the hulls $h_n(E)$.

PROPOSITION 5. *Let E be a closed subset of* $\operatorname{sp} A$. *The n-tuple A-convex hull $h_n(E)$ of E coincides with the biggest among all subsets*

N of $\operatorname{sp} A$ *which contain E and for which*

$$bF(N) \subset F(E) \tag{4}$$

for every $F \in A^n$.

Proof (see also **[11]**). First we show that if $N = h_n(E)$ then (3) holds for every non-vanishing on $h_n(E)$ mapping $F \in A^n$. Clearly $0 \notin F(E)$ for every such $F \in A^n$. If $c = \min_{m\in E} \|F(m)\|$ then $F(E) \subset \mathbf{C}^n \setminus B(c) = \{\mathbf{z} \in \mathbf{C}^n : \|\mathbf{z}\| \geq c\}$ and by the definition of $h_n(E)$ we have that $bF(h_n(E)) \subset \mathbf{C}^n \setminus B(c)$. This implies $F(h_n(E)) \subset \mathbf{C}^n \setminus B(c)$. Consequently

$$\min_{m\in h_n(E)} \|F(m)\| \geq c = \min_{m\in E} \|F(m)\|.$$

Since the opposite inequality is obviously fulfilled, we conclude that (3) holds with $N = h_n(E)$ for every $F \in A^n$ with $\|F(m)\| \neq 0$ on $h_n(E)$.

Let now N be a closed subset of $\operatorname{sp} A$ which satisfies (3) for every $F \in A^n$ that does not vanish on N, and assume that (4) is false, i.e. that $bF(N) \setminus F(E) \neq \varnothing$. Let $\mathbf{z}_0$ be a point from $bF(N) \setminus F(E)$ and let $m_0 \in F^{-1}(\mathbf{z}_0)$. Obviously

$$\|H(m_0)\| < \min_{m\in E} \|H(m)\| \tag{5}$$

for the n-tuple $H = F - \mathbf{z}_0$. We can find also a point $\mathbf{z}_1 \in \mathbf{C}^n \setminus F(N)$ close enough to $\mathbf{z}_0$, such that (5) holds for the n-tuple $H_1 = F - \mathbf{z}_1$. But this contradicts to (3) since obviously $H_1(m)$ does not vanish on N. The proposition is proved.

As the following example shows, the n-tuple A-convex hulls $h_n(E)$ are natural multi-tuple versions of the *A-convex hulls* $h(E) = \{m \in \operatorname{sp} A : |f(m)| \leq \max_{m\in E} |F(m)| \text{for all } f \in A\}$ of closed sets E in $\operatorname{sp} A$. Recall that $h(E)$ consists of all linear multiplicative functionals of A that possess continuous extensions on A_E. A set $E \in \operatorname{sp} A$ is *A-convex* if $h(E) = E$, i.e. if $E = \{m \in \operatorname{sp} A : |f(m)| \leq \max_{m\in E} |f(m)|$ for all $f \in A\}$, or, equivalently, if $\operatorname{sp} A_E = E$ (e.g. **[4]**). The vanishing set $V(S)$ of any subset S of A is a simple example for an A-convex set.

Example 3. The 1-tuple A-convex hull $h_1(E)$ of each closed subset E of $\operatorname{sp} A$ coincides with its usual A-convex hull $h(E)$.

Indeed, since $bf(h_1(E)) \subset f(E)$, we have that $\max_{m\in h_1(E)} |f(m)| \leq \max_{m\in E} |f(m)|$ for every function $f \in A$ and consequently $h_1(E) \subset h(E)$ by the definition of the A-convex hull $h(E)$. Assume that $h(E)$ contains properly $h_1(E)$. Then there exists a function f from A such that $bf(h(E)) \not\subset f(E)$. Let m_0 be a point from $h(E)$, such that $f(m_0) \in bf(h(E)) \setminus f(E)$. By choosing a point z_0 from $\mathbf{C} \setminus f(h(E))$ close enough to $f(m_0)$, we can construct a function $g(z) = f(z) - z_0 \in A^{-1}_{h(E)}$ for which $|g(m_0)| < \min_{m\in E} |g(m)|$. Hence $1/|g(m_0)| > \max_{m\in E} 1/|g(m)|$. Since $1/g \in A_{h(E)}$ and $\operatorname{sp} A_{h(E)} = h(E)$, there exists a function g_1 from A such that $|g_1(m_0)| > \max_{m\in E} |g_1(m)|$, i.e. $m_0 \notin h(E)$ in contradiction with the choice of m_0.

In particular we obtain that $h(E)$ is the biggest among all closed subsets N of $\operatorname{sp} A$ such that $bf(N) \subset f(E)$ for every $f \in A$.

EXAMPLE 4. The n-tuple polynomial convex hulls $\pi_n(E)$.

Given a compact subset E in $\mathbf{C}^\Lambda$ denote by $P(E)$ the closure in $C(E)$ of the set of all polynomials in $\mathbf{C}^\Lambda$. Since the polynomials are dense in $P(E)$, the n-tuple $P(E)$-convex hull of E is the biggest among all closed subsets N of $\mathbf{C}^\Lambda$ such that $b(p_1, \dots, p_n)(N) \subset (p_1, \dots, p_n)(E)$ for every n-tuple $(p_1, \dots, p_n)$ of polynomials in $\mathbf{C}^\Lambda$. We shall refer to this hull as *n-tuple polynomial convex hull* of E and shall denote it by $\pi_n(E)$. Clearly $E \subset \rho_n(E) \subset \pi_n(E)$ for every compact subset E in $\mathbf{C}^\Lambda$.

Being the n-tuple $P(E)$-convex hull of E, $\pi_n(E)$ is the biggest among all sets N in $\mathbf{C}^\Lambda$ which contain E and such that the inequality

$$\inf_{\mathbf{z}\in N} \|(p_1(\mathbf{z}), \dots, p_n(\mathbf{z}))\| = \min_{\mathbf{z}\in E} \|(p_1(\mathbf{z}), \dots, p_n(\mathbf{z}))\|$$

holds for every non-vanishing on n n-tuple $(p_1, \dots, p_n)$ of polynomials in $\mathbf{C}^\Lambda$.

A subset $E \subset \mathbf{C}^\Lambda$ is *n-tuple polynomially convex* if it coincides with its n-tuple polynomial hull $\pi_n(E)$. By applying the inclusion

$$b(p_1, \dots, p_n)(\pi_n(E)) \subset (p_1, \dots, p_n)(E)$$

to the identity mapping in $\mathbf{C}^n$ we get that $b(\pi_n(E)) \subset E$ for every compact subset E in $\mathbf{C}^n$. This implies that the n-tuple polynomial convex hull $\pi_n(E)$ of E is contained in the union of E and the union of all bounded components of its complement in $\mathbf{C}^n$. Because of its maximality property, $\pi_n(E)$ actually coincides with this union. Therefore E *is n-tuple polynomially convex if and only if its complement $\mathbf{C}^n \setminus E$ does not possess bounded components.*

The n-tuple polynomial convex hulls $\pi_n(E)$ of subsets E of $\mathbf{C}^\Lambda$ are natural generalizations of their usual polynomial hulls $\widehat{E} = \{\mathbf{z} \in \mathbf{C}^\Lambda : |p(\mathbf{z})| \leq \max_{\mathbf{z} \in E} |p(\mathbf{z})|$ for every polynomial p in $\mathbf{C}^\Lambda\}$—as Example 3 shows, $\pi_1(E) = \widehat{E}$. In general hulls $\pi_n(E)$ are different from the usual polynomial hulls $\widehat{E}$. For instance the 2-tuple polynomial hull $\pi_2(E)$ of the set $E = \{(z_1, z_2) \in \mathbf{C}^2 : 1 \leq |z_1| \leq 2, \ |z_2| = 0\} \subset \mathbf{C}^2$ is the set E itself, which does not coincide with the usual polynomial hull $\widehat{E} = \{(z_1, z_2) \in \mathbf{C}^2 : |z_1| \leq 2, \ |z_2| = 0\}$. Indeed, for any $z_0 \notin E$ we can choose an $\varepsilon > 0$ small enough, so that the regular pair of polynomials $(z_1 - z_0, z_2 + \varepsilon)$ to attain the minimum of its norm near z_0 and outside E at the same time. This means that $z_0 \notin \pi_2(E)$ for any $z_0 \notin E$, i.e. that $\pi_2(E) \subset E$ and hence $\pi_2(E) = E$ since $\pi_2 \supset E$ by Definition 2.

As we know, $r_n(E) \subset h_n(E)$ for every closed subset E of $\operatorname{sp} A$. The following corollary establishes a somewhat opposite inclusion.

Corollary 4. *Let E be a closed subset of the spectrum of a uniform algebra A. Then $h_n(E) \subset r_{n-1}(E)$.*

Proof (see also [**11**]). If $m_0 \notin r_{n-1}(E)$, then by Definition 1 we can find an $(n-1)$-tuple $F = (f_1, \dots, f_{n-1})$ of functions from A with $\|F(m_0)\| < \min_{m \in E} \|F(m)\|$. The regular n-tuple $(f_1, \dots, f_n, 1) = (F, 1)$ therefore satisfies the inequality

$$\|F(m_0)\|^2 + 1 < \min_{m \in E} \|F(m)\|^2 + 1.$$

Definition 2 indicates that $m_0 \notin h_n(E)$. We conclude that $h_n(E) \subset r_{n-1}(E)$, as claimed.

Observe that $F(h_n(E)) \subset \pi_n(F(E))$ for any $F \in A^n$ because $F(E) \supset bF(h_n(E))$ and hence $h_n(E) \subset \bigcap_{F \in A^n} F^{-1}(\pi_n(F(E)))$. In fact both sets are equal. Indeed, denote the latter set by K and take an $F \in A^n$ with $\|F(m)\| \neq 0$ on K. Clearly $\|\mathbf{z}\| \geq \min_{m \in E} \|F(m)\|$ for every $\mathbf{z} \in \pi_n(F(E))$ because $F(m)$ does not vanish on the set $E \subset K$. Thus $\|F(m_0)\| \geq \min_{m \in E} \|F(m)\|$ for every point m_0 in $F^{-1}(\pi_n(F(E)))$ and therefore for every point m_0 in K as well. Hence $h^n(E) \supset K$, i.e.

Corollary 5. $h_n(E) = \bigcap_{F \in A^n} F^{-1}(\pi_n(F(E)))$.

The next theorem is an n-tuple A-convex analogue of Corollary 3. Its proof follows the same lines as the proof of the case $k = 1$ of Theorem 1.

THEOREM 2. *The n-tuple A-convex hull $h_n(E)$ of a closed set E in $\operatorname{sp} A$ coincides with the set of these points $m \in \operatorname{sp} A$ which belong to the $A_{V(S)}$-convex hulls $h(E \cap V(S))$ of the sets $E \cap V(S)$ for any set S in A whose cardinality does not exceed $n-1$ and such that m belongs to $V(S)$, i.e.*

$$h_n(E) = \{m \in \operatorname{sp} A : m \in h(E \cap V(S)) \textit{ for any } S \subset A \textit{ with } \#S \leq n-1,\, m \in V(S)\}.$$

Proof. Denote for a while the set $\{m \in \operatorname{sp} A : m \in h(E \cap V(S))$ for any $S \subset A$ with $\#S \leq n-1$, $m \in V(S)\}$ by K and suppose that $K \setminus h_n(E) \neq \varnothing$. By Definition 2 we can find an n-tuple $F = (f_1, \dots, f_n) \subset A^n$ which does not vanish on K and such that $\|F(m_0)\| < \min_{m \in E} \|F(m)\|$ for some $m_0 \in K \setminus h_n(E)$.

Without loss of generality (applying, if necessary, an orthogonal transformation in $\mathbf{C}^n$) we can assume from the beginning that $F(m_0) = (f_1(m_0), 0, \dots, 0)$. Hence $m_0 \in V(S)$ for $S = (f_2, \dots, f_n)$ and f_1 does not vanish on $K \cap V(S)$. Then $|f_1(m_0)| = \|F(m_0)\| < \min_{m \in E} \|F(m)\| \leq \min_{E \cap V(S)} \|F(m)\| = \min_{E \cap V(S)} |f_1(m)|$, i.e. $\max_{m \in E \cap V(S)} |g(m)| < |g(m_0)|$ for $g = 1/f_1 \in A_{V(S)}$. Consequently $m_0 \notin h(E \cap V(S))$ in contradiction with $m_0 \in K$. We conclude that $K \subset h_n(K)$.

Suppose conversely that $h_n(E) \setminus K \neq \varnothing$ and let $m_0 \in h_n(E) \setminus K$. Let S be an $(n-1)$-tuple over A such that $m_0 \in V(S) \setminus h(E \cap V(S))$. Consequently there exists a function $f \in A_{V(S)}$, such that $|f(m_0)| > \max_{E \cap V(S)} |f(m)|$. Without loss of generality we can assume that f does not vanish on $\operatorname{sp} A$. For $g = 1/f \in A^{-1}$ we have $|g(m_0)| < r = \min_{E \cap V(S)} |g(m)|$. For any positive $\varepsilon < r$ we can find a neighborhood V_ε of the set $E \cap V(S)$ in E on which $|g(m)| > r - \varepsilon$. Hence for every $m \in E$ we have

$$\left(C_\varepsilon^2 \sum_{j=1}^{n-1} |f_j(m)|^2 + |g(m)|^2 \right)^{1/2} > r - \varepsilon \tag{5}$$

for some positive constant C_ε large enough. (5) holds also on $h_n(E)$ because the n-tuple $(C_\varepsilon f_1, \dots, C_\varepsilon f_{n-1}, g)$ does not vanish on $h_n(E)$. In particular at m_0 we have $|g(m_0)| > r - \varepsilon$ and henceforth $\|g(m_0)\| \geq r$ because of the liberty of the choice of ε. Since this contradicts with the initial inequality $|g(m_0)| < r$, we conclude that $h_n(E) \subset K$. The theorem is proved.

The sets described in Theorem 2 are precisely the hulls which were considered by Basener in [2] and used by him in his study of q-holomorphic functions.

THEOREM 3. *Let E be a closed subset of* $\operatorname{sp} A$. *Then* $E \subset \cdots \subset h_{n+1}(E) \subset h_n(E) \subset \cdots \subset h_1(E) = h(E) \subset \operatorname{sp} A$ *and* $\bigcap_{n\geq 1} h_n(E) = E$.

Proof. The first part of the statement is obvious because if (3) is fulfilled for every $(n+1)$-tuple F of functions from A which does not vanish on K, then it is fulfilled also for every n-tuple which does not vanish on K.

The inclusion $\bigcap_n h_n(E) \supset E$ is clear. If $m_0 \notin E$ then for every m in E we can find a function $f_m \in A^{-1}$ such that $|f_m(m)| > 1$ and $|f_m(m_0)| < 1$. Let U_m be an open neighborhood of m such that $|f_m(x)| > 1$ for any $x \in U_m$. By a compactness argument there exist finitely many points $m_1, \dots, m_k$ in E such that $E \subset U_{m_1} \cup \cdots \cup U_{m_k}$. By replacing each f_{m_j} by some of its power we can assume from the beginning that $|f_{m_j}(m_0)| < 1/k$ and $|f_{m_j}(x)| > 1$ on U_{m_j}. Hence for $F = (f_{m_1}, \dots, f_{m_k}) \in A^k$ we have that $\|F(m)\| \neq 0$ on $\operatorname{sp} A$, $\|F(m)\| > 1$ on E and $\|F(m_0)\| < 1$. Hence $m_0 \notin h_k(E)$ and moreover $m_0 \notin \bigcap_n h_n(E)$. We conclude that $\bigcap_n h_n(E) \subset E$, as required.

4. Multi-tuple Shilov boundaries and multi-tuple hulls.

DEFINITION 3 (Basener, Sibony). The *n-tuple Shilov boundary* $\partial^{(n)}A$ of a commutative Banach algebra A is the following subset of $\operatorname{sp} A$:

$$\partial^{(n)}A = \left[\bigcup\{\partial A_{V(f_1,\dots,f_{n-1})} : (f_1, \dots, f_{n-1}) \in A^{n-1}\}\right],$$

where $A^0 = \{0\}$ (see [1], [6], also [13]).

It is easy to check that $\partial A = \partial^{(1)}A \subset \partial^{(2)}A \subset \partial^{(3)}A \subset \cdots \subset \partial^{(n)}A \subset \cdots \subset \operatorname{sp} A$. We shall recall some of the basic properties of multi-tuple Shilov boundaries.

THEOREM 4 ([10, *Theorem* 1]). *$\partial^{(n)}A$ is the smallest closed subset of* $\operatorname{sp} A$ *on which every regular n-tuple $F = (f_1, \dots, f_n) \in A$ assumes the minimum of its norm*

$$\|F(m)\| = \left(\sum_{j=1}^{n} \|\hat{f}_j(m)\|^2\right)^{\frac{1}{2}}, \qquad m \in \operatorname{sp} A. \tag{6}$$

In other words Theorem 4 says that $\partial^{(n)}A$ is the smallest closed subset of $\operatorname{sp} A$ on which every regular n-tuple $F = (f_1, \dots, f_n) \in A$ assumes the minimum of its norm (6).

THEOREM 5 ([**10**, *Theorem* 3]). *$\partial^{(n)}A$ is the smallest closed subset of $\operatorname{sp} A$ such that the inclusion*

$$F(\partial^{(n)}A) \supset bF(\operatorname{sp} A) \tag{7}$$

holds for every n-tuple $F = (f_1, \dots, f_n) \in A^n$.

The next proposition in particular says that, in the case of algebra $A = A(B^n(1))$, $r_k(S^n(1)) = B^n(1)$ for any $k < n$ unlike the case $k = n$ when, according to Proposition 3, $S^n(1)$ is n-tuple rationally A-convex and hence $r_n(S^n(1)) = S^n(1)$.

PROPOSITION 6. *$r_k(\partial^{(n)}A) = \operatorname{sp} A$ for each $k < n$; if $r_n(\partial^{(n)}A) = \operatorname{sp} A$ then $r_n(E) \neq \operatorname{sp} A$ for every proper closed subset E of $\partial^{(n)}A$.*

Indeed, as shown in [**10**], $V(G) \cap \partial^{(n)}A \neq \varnothing$ for every irregular k-tuple $G \in A^k$, $1 \le k \le n-1$. From Proposition 2 we conclude that $r_k(\partial^{(n)}A) = \operatorname{sp} A$ for every $k < n$. If $E \neq \partial^{(n)}A$ then there is an n-tuple $F \in A^n$ with $\min_{m \in E} \|F(m)\| > \min_{m \in \operatorname{sp} A} \|F(m)\| = 0$ and consequently, by Definition 1, $r_n(E) \neq r_n(\partial^{(n)}A) = \operatorname{sp} A$.

PROPOSITION 7. *$h_n(E) = \operatorname{sp} A$ if and only if E contains the n-tuple Shilov boundary $\partial^{(n)}A$ of A.*

Indeed, from Definition 2 and Theorem 4 it follows that $h_n(\partial^{(n)}A) = \operatorname{sp} A$. Theorem 4 shows that $h_n(E) \neq \operatorname{sp} A$ if $E \setminus \partial^{(n)}A \neq \varnothing$.

Given an n-tuple $(f_1, \dots, f_n) \in A^n$ let $\sigma(f_1, \dots, f_n)$ be the *joint spectrum* of the n-tuple $(f_1, \dots, f_n)$, i.e.

$$\sigma(f_1, \dots, f_n) = \{(f_1(m), \dots, f_n(m)) : m \in \operatorname{sp} A\}.$$

PROPOSITION 8. *The joint spectrum of every n-tuple F over A is contained in the n-tuple polynomial hull of the set $F(\partial^{(n)}A)$, i.e.*

$$\sigma(F) = F(\operatorname{sp} A) \subset \pi_n(F(\partial^{(n)}A)), \qquad F \in A^n.$$

Proof (see also [**11**]). Since by the Theorem 4

$$\min_{m \in \partial^{(n)}A} \|G(m)\| = \min_{m \in \operatorname{sp} A} \|G(m)\|$$

for every regular n-tuple $G \in A^n$, the equality

$$\min_{m \in \partial^{(n)} A} \|(p_1 \circ F(m), \ldots, p_n \circ F(m))\| = \min_{m \in \operatorname{sp} A} \|(p_1 \circ F(m), \ldots, p_n \circ F(m))\|$$

and, equivalently,

$$\min_{m \in F(\partial^{(n)} A)} \|(p_1, \ldots, p_n)(\mathbf{z})\| = \min_{m \in F(\operatorname{sp} A)} \|(p_1, \ldots, p_n)(\mathbf{z})\|$$

are fulfilled for every n-tuple of polynomials $p_1, \ldots, p_n$ in $\mathbf{C}^n$ without joint zeros on $\sigma(F) = F(\operatorname{sp} A)$. Definition 2 indicates that the sets $F(\partial^{(n)} A)$ and $\sigma(F)$ have equal n-tuple polynomial hulls and consequently $\sigma(F) \subset \pi_n(F(\partial^{(n)} A))$, as claimed.

A well known theorem from the uniform algebra theory says that if an algebra is generated (linearly) by its subset Λ, then the range of its Shilov boundary ∂A via the *spectral mapping* $\widehat{\Lambda} : \operatorname{sp} A \to \mathbf{C}^{\Lambda} : m \longmapsto \{f(m) : f \in \Lambda\}$ of Λ is the smallest closed subset of $\mathbf{C}^{\Lambda}$ whose polynomial hull is equal to the polynomial hull of the set $\widehat{\Lambda}(\operatorname{sp} A)$. In the next theorem we make use of the n-tuple A-convex boundaries in order to obtain an extension of this result for the multi-tuple Shilov boundaries $\partial^{(n)} A$. Namely

THEOREM 6. *Let $S = \{b_\lambda\}_{\lambda \in \Lambda}$ be a set which generates linearly a uniform algebra A. Then the range $\widehat{\Lambda}(\partial^{(n)} A)$ of the n-tuple Shilov boundary via $\widehat{\Lambda}$ is the smallest among all compact subsets E in $\mathbf{C}^{\Lambda}$ whose n-tuple polynomial hulls $\pi_n(E)$ are equal to the n-tuple polynomial hull $\pi_n(\widehat{\Lambda}(\operatorname{sp} A))$ of the range of $\widehat{\Lambda}$.*

Proof (see also **[11]**). Without loss of generality we can assume that E is a subset of $\widehat{\Lambda}(\operatorname{sp} A)$ and consequently that $E = \widehat{\Lambda}(K)$ for some compact set $K \in \operatorname{sp} A$. The n-tuple polynomial hulls $\pi_n(E) = \pi_n(\widehat{\Lambda}(K))$ and $\pi_n(\widehat{\Lambda}(\operatorname{sp} A))$ are equal if and only if $\min_{\mathbf{z} \in \widehat{\Lambda}(K)} \|P(\mathbf{z})\| = \min_{\mathbf{z} \in \widehat{\Lambda}(\operatorname{sp} A)} \|P(\mathbf{z})\|$ for every n-tuple $P = (p_1, \ldots, p_n)$ of polynomials in $\mathbf{C}^{\Lambda}$ with $\|P(\mathbf{z})\| \neq 0$ on $\pi_n(\widehat{\Lambda}(\operatorname{sp} A))$. Equivalently, $\pi_n(\widehat{\Lambda}(K)) = \pi_n(\widehat{\Lambda}(\operatorname{sp} A))$ if and only if $\min_{m \in K} \|P \circ \widehat{\Lambda}(m)\| = \min_{m \in \operatorname{sp} A} \|P \circ \widehat{\Lambda}(m)\|$ for any n-tuple of type $P \circ \widehat{\Lambda} \in A^n$ which does not vanish on the set $\widehat{\Lambda}^{-1}(h_n(\widehat{\Lambda}(\operatorname{sp} A))) = \operatorname{sp} A$. Since the set of functions $p_j \circ \widehat{\Lambda}(m)$ is dense in A, $\pi_n(\widehat{\Lambda}(K)) = \pi_n(\widehat{\Lambda}(\operatorname{sp} A))$ if and only if $\min_{m \in K} \|F(m)\| = \min_{m \in \operatorname{sp} A} \|F(m)\|$ for every regular n-tuple $F \in A^n$. By Theorem

4 the n-tuple Shilov boundary $\partial^{(n)}A$ is the smallest closed subset of $\operatorname{sp} A$ with the last property; and therefore $\widehat{\Lambda}(\partial^{(n)}A)$ is the smallest closed subset of $\mathbf{C}^{\Lambda}$ whose n-tuple polynomial hull is the same as the n-tuple polynomial hull of the set $\widehat{\Lambda}(\operatorname{sp} A))$, as claimed.

5. Remarks. Recall that *function space* is called any linear subspace of the space $C(X)$, where X is a compact Hausdorff space, which is closed under the *uniform norm* $\|f(x)\| = \max_{x\in X} |f(x)|$, contains the constants and separates the points of X. It can be shown that for every function space B over X there exists a smallest set $\operatorname{Sh}_B(X)$ among all closed subsets E of X such that

$$\min_{x\in X} |f(x)| = \min_{x\in E} |f(x)|$$

for every *nonvanishing* on X function f in B. In general $\operatorname{Sh}_B(X)$ does not coincide with the usual Shilov boundary ∂B of B which, by definition is the smallest among all closed subsets E of X such that

$$\max_{x\in X} |f(x)| = \max_{x\in E} |f(x)|$$

for *every* function f in B. The *n-tuple Shilov boundary* of a function space B is the set

$$\operatorname{Sh}_{B^n}(X) = \left[\bigcup \operatorname{Sh}_B(V(F)) : F = (f_1, \dots, f_{n-1}) \in B^{n-1}\right],$$

where $V(F) = (f_1, \dots, f_{n-1})^{-1}(\mathbf{0})$.

It is a matter of a simple verification to check that all the above results hold not only for uniform algebras but for function spaces as well with the Shilov boundary ∂A replaced with the boundary $\operatorname{Sh}_B(X)$ and the n-tuple Shilov boundary $\partial^{(n)}A$ replaced with the n-tuple Shilov boundary $\operatorname{Sh}_{B^n}(X)$.

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