PACIFIC JOURNAL OF MATHEMATICS
Vol. 152, No. 2, 1992

SUPERHARMONICITY OF CURVATURES FOR SURFACES OF CONSTANT MEAN CURVATURE

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For a hypersurface of constant mean curvature in an Euclidean space, we illustrate the superharmonicity of various forms of curvatures. Applications in parametric capillary surfaces, constant rank properties of surfaces and a Shiffman type theorem are discussed.

Given a hypersurface M^n of constant mean curvature in R^{n+1}, we study its geometry by showing the superharmonicity of various forms of curvatures, such as the scalar curvature R, the Gauss-Kronecker curvature K_n, the νth mean curvature K_ν and the level curvature L.

We first prove that, for M^n of positive sectional curvatures, R and $\log K_n$ are superharmonic [Theorems 1 and 3]. When M^n has nonnegative sectional curvatures, $\log K_n$ is not everywhere defined, but K_n still satisfies the strong minimum principle that it never attains interior nonnegative minimum unless M^n is a portion of a sphere or of a cylinder. Such findings explain various results, known or unknown, on the geometry of M^n with a simple geometric version.

As an application, we consider a liquid contained but not full within a convex closed container left in outer space, i.e. a space of zero gravity. The Euler-Lagrange equation yields that the capillary surface of the liquid has constant mean curvature with constant contact angle γ against the wall of the container. Consider a C^2-deformation of the capillary surface by pouring in an extra amount of the liquid until a small vacuum bubble remained in the middle. The superharmonicity of $\log K_n$ proves the convexity of capillary surfaces during the deformation, valid for $\gamma = 0$ and arbitrary dimension n [Theorems 2 and 4]. This can be regarded as a generalization of the results by Chen-Huang [4] and Korevaar [11] where the nonparametric case was considered. Nonconvex examples of the problem were obtained by Finn [5] for nonzero γ. The strong minimum principle for K_n also reveals the intrinsic geometric effect of $\gamma = 0$ on the convexity of the capillary surface, with the argument that K_n nonnegative on the boundary ∂M guaranteed by $\gamma = 0$ is to ensure K_n positive in the interior.

As a second application of superharmonicity of curvatures, we consider a constant rank theorem established by Korevaar-Lewis **[10]** that the Hessian of the height function $u = u(x)$, $x \in \Omega \subset R^n$ has constant rank when the mean curvature H of a convex surface $u = u(x)$ satisfies H^{-1} being strictly convex in u. Refining our previous work by considering the log-concavity of the νth mean curvature K_ν defined by the νth symmetric function of the principal curvatures, we prove [Theorem 5] that given a convex hypersurface M^n in R^{n+1} with concave mean curvature H [the case that $H \equiv$ constant is included—see §4.2], the second fundamental form (h_{ij}) has constant rank.

We also point out that the superharmonicity of varoius forms of curvatures gives rise to classical known results, such as the rigidity of spheres, the classification of complete convex surfaces by Liebman-Klotz-Osserman **[9]** and the convexity theorem of level curves on minimal surfaces by Shiffman **[14]**. For the last one, we extend to surfaces of constant mean curvature in §5, deal with the "level curvature" (i.e. the curvature of level curves) and prove its superharmonicity in a sense [Theorem 8]. The computation is relatively tedious. We thereby conclude that deforming smoothly a given annular-like soap film with two ending curves lying in two respective horizontal planes to a vertical Delaunay surface, all the level curves are convex during the deformation, if the outside pressure of the beginning soap film is no less than the inside pressure [Theorem **9**]. This generalizes Shiffman's theorem on minimal surfaces. Our method is free from using the complex analysis apparatus.

1. The superharmonicity of scalar curvature. Let M^n be an n-dimensional hypersurface of constant mean curvature immersed in R^{n+1}. We choose an orthonormal frame $\{e_A\}$ such that $\{e_1, \dots, e_n\}$ are tangent to M and e_{n+1} normal. Let the corresponding coframe be denoted by $\{w_A\}$ and the connection forms by $\{w_{AB}\}$. The pull-backs of them through the immersion are still denoted by $\{w_A\}$, $\{w_{AB}\}$ in the abuse of notation. Therefore in particular,

$$w_{n+1} = 0. \tag{1.1}$$

We use the index convention as follows,

$$1 \le i, j, k, \dots \le n; \quad 1 \le A, B, C, \dots \le n+1.$$

The second fundamental form B is defined by the matrix (h_{ij}) with

$$w_{i,n+1} = h_{ij} w_j. \tag{1.2}$$

We have the curvature tensor R_{ijkl} expressed by

$$R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk}. \tag{1.3}$$

The sectional curvature K_{ij} of the plane spanned by $\{e_i, e_j\}$ is defined by

$$K_{ij} = R_{ijij} = h_{ii}h_{jj} - h_{ij}^2 = \lambda_i\lambda_j \qquad (\text{if } i \neq j) \tag{1.4}$$

where $\lambda_1, \dots, \lambda_n$ are the principal curvatures of M^n in R^{n+1}, and the scalar curvature R is defined by

$$R = \sum_{i,j} R_{ijij} = \sum_{i\neq j} \lambda_i\lambda_j. \tag{1.5}$$

We have

THEOREM 1. *Let M^n be a hypersurface of constant mean curvature immersed in R^{n+1}. Then the Laplacian of the scalar curvature R is*

$$\Delta_M R = -\sum_{i\neq j}(\lambda_i - \lambda_j)^2 K_{ij} - 2|\nabla B|^2, \tag{1.6}$$

where $|\nabla B|^2 = \sum_{i,j,k} h_{ijk}^2$. Furthermore if M is of non-negative sectional curvatures, then R is superharmonic on M.

Proof. We have a commutation formula,

$$h_{ijkl} = h_{ijlk} + h_{mj}R_{imlk} + h_{im}R_{jmlk}, \tag{1.7}$$

while noting that h_{ijk} is symmetric in any two indices among the three. This gives

$$\begin{aligned} \Delta_M h_{ij} &= h_{ijkk} \quad (\text{using the summation convention}) \\ &= h_{ikjk} \\ &= h_{ikkj} + h_{mk}R_{imkj} + h_{im}R_{kmkj} \\ &= h_{kkij} + h_{mk}R_{imkj} + h_{im}R_{kmkj}. \end{aligned} \tag{1.8}$$

On the other hand, we have

$$\begin{aligned} \Delta_M R &= \sum_{i,j,k} (h_{ii}h_{jj} - h_{ij}^2)_{kk} \\ &= \sum_{i,j,k} (h_{iik}h_{jj} + h_{ii}h_{jjk} - 2h_{ij}h_{ijk})_k \\ &= \sum_{i,j,k} (h_{iikk}h_{jj} + h_{ii}h_{jjkk} - 2h_{ijkk}h_{ij}) \\ &\quad + 2(h_{iik}h_{jjk} - h_{ijk}^2). \end{aligned} \tag{1.9}$$

Since the mean curvature $H \equiv \sum_i h_{ii}$ is constant, it is clear that

$$\sum_i h_{iikk} h_{jj} = 0 = \sum_j h_{ii} h_{jjkk} \tag{1.10}$$

and

$$\sum_i h_{iik} = 0. \tag{1.11}$$

Hence we obtain

$$\Delta_M R = -\sum_{i,k} 2 h_{iikk} h_{ii} - \sum_{i,j,k} 2 h_{ijk}^2 \tag{1.12}$$

by suitably taking $e_1, \ldots, e_n$ to be the principal directions at a given point p. We continue the computation that at p,

$$\begin{aligned} \Delta_M R &= -2 \sum_{i,k} (h_{kkii} + h_{mk} R_{imki} + h_{im} R_{kmki}) h_{ii} - 2|\nabla B|^2 \\ &= \left(2 \sum_{i,k} h_{kk} h_{ii} R_{ikik} - 2 \sum_{i,k} h_{ii}^2 R_{kiki} \right) - 2|\nabla B|^2 \\ &= \left(2 \sum_{i \neq k} (\lambda_k \lambda_i K_{ik}) - \sum_{i \neq k} (\lambda_i^2 + \lambda_k^2) K_{ik} \right) - 2|\nabla B|^2 \\ &= -\sum_{i \neq k} (\lambda_i - \lambda_k)^2 K_{ik} - 2|\nabla B|^2. \end{aligned} \tag{1.13}$$

This proves formula (1.6).

As a corollary for $n = 2$, we obtain a stronger form.

COROLLARY 1. *Let M be a surface of constant mean curvature in R^3 and K be its Gaussian curvature. Then at non-umbilic points, we have*

$$\Delta_M K = -(\lambda_1 - \lambda_2)^2 K - \frac{4}{(\lambda_1 - \lambda_2)^2} |\nabla K|^2. \tag{1.14}$$

And therefore K attains neither interior nonnegative minimum nor interior negative maximum, unless M is a sphere or a cylinder.

Proof. We have

$$
\begin{aligned}
|\nabla K|^2 &= \sum_k ((h_{11}h_{22} - h_{12}^2)_k)^2 \\
&= \sum_k (h_{11k}h_{22} + h_{11}h_{22k} - 2h_{12}h_{12k})^2 \\
&= (h_{111}h_{22} + h_{11}h_{221})^2 + (h_{112}h_{22} + h_{11}h_{222})^2 \\
&\qquad\qquad \text{(by choosing frame so that } h_{12} = 0 \text{ at a given point)} \\
&= (h_{11} - h_{22})^2(h_{221}^2 + h_{112}^2) \\
&\qquad\qquad\qquad\qquad \text{(since } h_{111} = -h_{221}\text{, } h_{222} = -h_{112}\text{)}
\end{aligned}
$$

and

$$
\begin{aligned}
|\nabla B|^2 &= (h_{111}^2 + 2h_{121}^2 + h_{221}^2) + (h_{112}^2 + 2h_{112}^2 + h_{222}^2) \\
&= 4(h_{221}^2 + h_{112}^2).
\end{aligned}
\tag{1.15}
$$

Hence

$$
|\nabla B|^2 = \frac{4}{(\lambda_1 - \lambda_2)^2}|\nabla K|^2. \tag{1.16}
$$

By substituting $R = 2K$ into (1.6), we obtain (1.14). The second assertion is then evident.

REMARK 1.1. For $n = 2$, the formula (1.14) can also be proved in a classical way. Consider the coefficient

$$
\Phi = \frac{L - N}{2} - iM
$$

of Hopf's differential, where L, M and N are the coefficients of the second fundamental form. By virtue of the Coddazzi equations, Φ is holomorphic when the mean curvature is constant. Hence $\log|\Phi^2|$ is harmonic on the surface. By computing $\Delta(\log|\Phi^2|)$ with respect to the metric of the surface at non-umbilic points, we easily obtain formula (1.14). Since umbilic points attain maximal values for K, the superharmonicity around umbilic points is still valid.

2. Parametric capillary surfaces. Given a liquid V contained but not full in a bounded closed container C. Leave them in outer space, i.e. in the field of zero gravity. Let M denote the liquid surface inside C in equilibrium.

The Euler-Lagrange equation of energy function of the liquid implies that M has constant contact angle γ independent of the shape of C. This was originally observed by Pierre Simon Laplace **[12]**, Thomas-Young **[18]** and Carl Gauss **[6]**, when the container is a vertical tube.

A natural question is raised. How does the shape of C affect the geometry of surface M? For the case of tube with given cross section Ω where M is expressed nonparametrically in $u = u(x)$, $x \in \Omega$, related results have been obtained. In a joint work of the author with J. T. Chen **[4]**, it was proved that for $\gamma = 0$ the convexity of Ω implies the strict convexity of the surface M in a strong sense that the Gaussian curvature is positive at the interior points. (In this paper we keep on using *strictly convex* to name the convexity in this strong sense.) The result was soon generalized by Korevaar **[11]** to higher dimensions of the domain Ω, using his maximum principle of concavity function and a limiting argument that extends convexity results in the gravitational field to outer space. On the other hand, Finn **[5]** found counterexamples over trapezoids for each nonzero γ.

Why the assumption of $\gamma = 0$ is that essential has been partially revealed in the proof of Korevaar and the examples of Finn. However, based on (1.14) and the continuity method we obtain the superharmonicity of nonnegative Gaussian curvature K for general parametric surfaces of constant mean curvature. This gives a more satisfactory and geometric explanation for the essentiality of vanishing γ. In fact, that $\gamma = 0$ guarantees nonnegative Gaussian curvature on the boundary. The superharmonicity of K then shows its positiveness in the interior.

We make precise arguments of this for general parametric capillary hypersurfaces of dimension 2 and higher dimensions respectively in the subsequent theorems.

THEOREM 2. *Given a liquid contained inside a closed, strictly convex container C (C denoting the bounding wall) in outer space. We suppose that the material of the liquid and the container are made so that $\gamma = 0$ wherever they contact. Pouring more liquid gradually into the container, it is assumed that the liquid surface M deforms smoothly until the small vacuum part left in the middle of the liquid. Then the interior of M is strictly convex in the beginning as well as during the deformation. Moreover, we have the minimum principle,*

$$K_M \geq \min K_C > 0 \tag{2.1}$$

where K_M and K_C denote the Gaussian curvature of M and C respectively. And therefore the vacuum part at any stage is a strict convex set.

Proof. First we note that the small vacuum part left in the middle of the liquid is bounded by a closed surface of constant mean curvature embedded in R^3. By a theorem of Aleksandrov **[1]**, it is known that the vacuum part is simply a sphere on which the Gaussian curvature is a positive constant. We reverse the deformation process that we begin with the sphere $S^2(\varepsilon_0) \equiv M_0$ and deform M_0 smoothly into $M_1 \equiv M$ through $\{M_t\}$. Suppose that (2.1) is not satisfied for M, then there exists a smallest τ such that an interior point P_0 in M_τ exists with $K_{M_\tau}(p_0) = \min_{M_\tau} K_{M_\tau} > 0$. But this is impossible by virtue of (1.14) with the strong minimum principle. Hence

$$K_M \geq \min K_C > 0 \tag{2.2}$$

as required.

REMARK. For the non-parametric case, i.e. for the liquid contained in a tube with convex cross section Ω, the above argument is modified to deform the tube into a circular tube in which the capillary surface is a lower hemisphere. We follow the reasoning in Theorem 2 except we have to exclude the possibility that $K_{M_{t_0}}$ attains a zero minimum at a boundary point P_0 for certain $t_0 < 1$. This can be argued by a refined strong maximum principle. The remark also implies that the assumption of strict convexity of C in Theorem 2 can weaken to convexity.

3. The log-concavity of Gauss-Kronecker curvature. For a hypersurface M^n immersed in R^{n+1}, the Gauss-Kronecker curvature K_n is defined by the product of principal curvature $\{\lambda_i\}$, i.e. $K_n \equiv \lambda_1 \lambda_2 \cdots \lambda_n = \det h_{ij}$. It describes the ratios of n-dimensional infinitesimal areas under the Gauss map

$$M^n \to \text{the unit sphere } S^n(1),$$

which sends a point p to the unit normal e_{n+1} at p.

THEOREM 3. *For a hypersurface M^n immersed in R^{n+1} with constant mean curvature and nonnegative sectional curvatures, we have the strong minimum principle for K_n, i.e.*

$$\min_M K_n = \min_{\partial M} K_n \tag{3.1}$$

with K_n never attaining minimum in the interior unless K_n is constant. Furthermore we have the following results: (i) $\log K_n$ *is superharmonic on the points where* $K_n > 0$, (ii) *On general points of M^n*, $\log \widetilde{K}_n$ *is* *superharmonic where* $\widetilde{K}_n \equiv \det(h_{ij} + \varepsilon\delta_{ij})$, $\varepsilon > 0$ *and* $\tilde{h}_{ij} \equiv h_{ij} + \varepsilon\delta_{ij}$, δ_{ij} *being the Kronecker symbol.*

REMARK. In the case that K_n is a positive constant, M^n is umbilic by (3.10) and (3.11) and therefore it is a portion of n-spheres. When K_n has a zero minimum, M^n is a cylinder by (4.21).

Proof. We consider locally on M_n an orthonormal moving frame $e_1, e_2, \dots, c_n$. For a function u on M^n, the Laplacian Δu of u is defined by

$$\Delta u = u_{kk} \tag{3.2}$$

where u_{ij} is the covariant derivative with respect to the frame and the summation convention on k is used here and likewise in the sequel. We recall that the Laplacian thus defined is the so-called Beltrami-Laplace operator defined by the metric of M_n. It follows straightforwardly that

$$\Delta \log \widetilde{K}_n = \frac{\Delta \widetilde{K}_n}{\widetilde{K}_n} - \left|\frac{\nabla \widetilde{K}_n}{\widetilde{K}_n}\right|^2 \tag{3.3}$$

and

$$\begin{aligned} \Delta \widetilde{K}_n &= \sum_\sigma (-1)^\sigma (\tilde{h}_{1\sigma_1}\tilde{h}_{2\sigma_2}\cdots\tilde{h}_{n\sigma_n})_{kk} \\ &= \sum_\sigma (-1)^\sigma (\tilde{h}_{1\sigma_1 k}\tilde{h}_{2\sigma_2}\cdots\tilde{h}_{n\sigma_n} + \cdots + \tilde{h}_{1\sigma_1}\tilde{h}_{2\sigma_2}\cdots\tilde{h}_{n\sigma_n k})_k \\ &= \sum_\sigma (-1)^\sigma \{[\tilde{h}_{1\sigma_1 kk}\tilde{h}_{2\sigma_2}\cdots\tilde{h}_{n\sigma_n} + \cdots + \tilde{h}_{1\sigma_1}\tilde{h}_{2\sigma_2}\cdots\tilde{h}_{n\sigma_n kk}] \\ &\qquad + 2[\tilde{h}_{1\sigma_1 k}\tilde{h}_{2\sigma_2 k}\tilde{h}_{3\sigma_3}\cdots\tilde{h}_{n\sigma_n} + \cdots \\ &\qquad\qquad + \tilde{h}_{1\sigma_1}\tilde{h}_{2\sigma_2}\tilde{h}_{3\sigma_3}\cdots h_{n-1\sigma_{n-1}k}h_{n\sigma_n k}]\} \end{aligned} \tag{3.4}$$

where $\sigma = (\sigma_1, \dots, \sigma_n)$ is a permutation of $(1, \dots, n)$. By choosing the frame $\{e_i\}$ so that $h_{ij} = \lambda_i\delta_{ij}$ at a given point p, we have

$$\tilde{h}_{ij} = (\lambda_i + \varepsilon)\delta_{ij} \tag{3.5}$$

at p. Formula (3.4) is reduced to

$$\begin{aligned} \Delta \widetilde{K}_n = {} & [\tilde{h}_{11kk}\tilde{h}_{22}\cdots\tilde{h}_{nn} + \cdots + \tilde{h}_{11}\tilde{h}_{22}\cdots\tilde{h}_{nnkk}] \\ & + 2[(\tilde{h}_{11k}\tilde{h}_{22k} - \tilde{h}_{12k}^2)\tilde{h}_{33}\cdots\tilde{h}_{nn} + \cdots \\ & \qquad + \tilde{h}_{11}\tilde{h}_{22}\cdots\tilde{h}_{n-2,n-2}(\tilde{h}_{n-1,n-1,k}\tilde{h}_{nnk} - \tilde{h}_{n-1,n,k}^2)]. \end{aligned} \tag{3.6}$$

Hence we find

$$(3.7)\quad \begin{aligned}\Delta \log \widetilde{K}_n = \frac{1}{\widetilde{K}_n}\{&[\tilde{h}_{11kk}\tilde{h}_{22}\cdots\tilde{h}_{nn} + \cdots + \tilde{h}_{11}\tilde{h}_{22}\cdots\tilde{h}_{nnkk}] \\ &+ 2[(\tilde{h}_{11k}\tilde{h}_{22k} - \tilde{h}_{12k}^2)\tilde{h}_{33}\cdots\tilde{h}_{nn} + \cdots \\ &\qquad + \tilde{h}_{11}\tilde{h}_{22}\cdots\tilde{h}_{n-2,n-2} \\ &\qquad \times (\tilde{h}_{n-1,n-1,k}\tilde{h}_{nnk} - \tilde{h}_{n-1,n,k}^2)]\} \\ - \frac{1}{\widetilde{K}_n^2}&[\tilde{h}_{11k}\tilde{h}_{22}\cdots\tilde{h}_{nn} + \cdots + \tilde{h}_{11}\tilde{h}_{22}\cdots\tilde{h}_{nnk}]^2.\end{aligned}$$

It is clear that

$$(3.8)\qquad \tilde{h}_{ijk} = h_{ijk} \quad \text{and} \quad \tilde{h}_{ijkl} = h_{ijkl},$$

and therefore

$$(3.9)\qquad \tilde{h}_{kkij} = h_{kkij} = 0.$$

By using the commutation formula (1.7) for $\tilde{h}_{ijkl}$, the first term of (3.7) at p becomes

$$(3.10)\quad \begin{aligned}&\frac{1}{\widetilde{K}_n}\sum_{i,k}\tilde{h}_{11}\tilde{h}_{22}\cdots\tilde{h}_{iikk}\cdots\tilde{h}_{nn} \\ &\qquad\qquad (\tilde{h}_{iikk} \text{ appears at the } i\text{th position}) \\ &= \frac{1}{\widetilde{K}_n}\sum_{i,k}\tilde{h}_{11}\tilde{h}_{22}\cdots(\tilde{h}_{kkii} - \tilde{h}_{kk}R_{ikik} + \tilde{h}_{ii}R_{kiki})\cdots\tilde{h}_{nn} \\ &= \frac{1}{\widetilde{K}_n}\sum_{i,k}\tilde{h}_{11}\tilde{h}_{22}\cdots([-\tilde{h}_{kk} + \tilde{h}_{ii}]h_{ii}h_{kk})\cdots\tilde{h}_{nn} \\ &= \frac{\widetilde{K}_n}{\widetilde{K}_n}\sum_{i,k}\left[-\frac{\tilde{h}_{kk}}{\tilde{h}_{ii}} + 1\right]h_{ii}h_{kk} \\ &= \sum_{i<k}\left[-\frac{\tilde{h}_{kk}}{\tilde{h}_{ii}} + 2 - \frac{\tilde{h}_{ii}}{\tilde{h}_{kk}}\right]h_{ii}h_{kk} \\ &= -\sum_{i<k}[\tilde{h}_{kk} - \tilde{h}_{ii}]^2\left(\frac{h_{ii}}{\tilde{h}_{ii}}\right)\left(\frac{h_{kk}}{\tilde{h}_{kk}}\right) \le 0\end{aligned}$$

and the equality holds if and only if p is an umbilic or a cylindric point. The remaining terms of (3.7) equal

$$
\begin{aligned}
(3.11)\ &\frac{2}{\tilde{K}_n}[(h_{11k}h_{22k}-h_{12k}^2)\tilde{h}_{33}\cdots\tilde{h}_{nn}+\cdots] \\
&\quad-\frac{1}{\tilde{K}_n^2}[h_{11k}\tilde{h}_{22}\tilde{h}_{33}\cdots\tilde{h}_{nn}+\tilde{h}_{11}h_{22k}\tilde{h}_{33}\cdots\tilde{h}_{nn}+\cdots]^2 \\
&=-\left[\frac{2h_{12k}^2}{\tilde{h}_{11}\tilde{h}_{22}}+\cdots\right]-\left[\frac{h_{11k}^2}{\tilde{h}_{11}^2}+\cdots\right] \\
&=-\sum_{i,j,k}\frac{h_{ijk}^2}{\tilde{h}_{ii}\tilde{h}_{jj}}\le 0.
\end{aligned}
$$

Therefore at a given point p on which $K_n \geq 0$, we have shown (ii) of the theorem, i.e. we have proved the superharmonicity of $\log\tilde{K}_n$, or more precisely,

$$
\begin{aligned}
(3.12)\qquad \Delta\log\tilde{K}_n = &-\sum_{i<k}(\lambda_i-\lambda_k)^2\frac{\lambda_i\lambda_k}{(\lambda_i+\varepsilon)(\lambda_k+\varepsilon)} \\
&-\sum_{i,k}\frac{|\nabla h_{ik}|^2}{(\lambda_i+\varepsilon)(\lambda_k+\varepsilon)}\le 0
\end{aligned}
$$

where $\tilde{K}_n \equiv \det(h_{ij}+\varepsilon\delta_{ij})$, $\varepsilon > 0$, and the equality holds if and only if p is umbilic or cylindric. The same computation follows evidently for $\log K_n$ at p with $K_n > 0$. Thus we also obtain (i) of the theorem.

To show the strong minimum principle for K_n on general points of M, we note that $\tilde{K}_n$ satisfies strong minimum principle for each $\varepsilon > 0$, by virtue of (3.12). However,

$$
\begin{aligned}
(3.13)\qquad \tilde{K}_n &= \tilde{K}_n(\varepsilon) = \det(h_{ij}+\varepsilon\delta_{ij}) \\
&= K_n + K_{n-1}\varepsilon + K_{n-2}\varepsilon^2+\cdots+K_1\varepsilon^{n-1}+\varepsilon^n,
\end{aligned}
$$

where K_ν is the νth mean curvature of M^n in R^{n+1}, defined by the νth symmetric function of principal curvatures $\lambda_1, \lambda_2, \cdots, \lambda_n$.

Suppose that K_n attains a zero minimum at an interior point P_0 of M^n. A contradiction is shown to follow unless K_n is constantly zero. The function K_n is real analytic on M^n, as M^n inherits a real analytic structure from R^{n+1} with all coordinate functions real analytic in M^n. The set Z of minimal points of K_n on M^n is a finite union of real analytic submanifolds around P_0. Notice that on Z,

$$\log\tilde{K} \equiv \log\tilde{K}(\varepsilon) = \log\varepsilon + \log(K_{n-1}+\varepsilon K_{n-2}+\cdots+\varepsilon^{n-1})$$

while on $M^n - Z$,

$$\log \widetilde{K} \equiv \log \widetilde{K}(\varepsilon) = \log[K_n + \varepsilon(K_{n-1} + \varepsilon K_{n-2} + \cdots + \varepsilon^{n-2})].$$

Let γ be a geodesic segment of M^n with $\gamma(0) = P_0$ and $\gamma - \{P_0\}$ contained in $M^n - Z$. As ε tends to zero, $\gamma(-t)$ and $\gamma(t)$ have values of $\log \widetilde{K}_n$ bounded below, but at P_0, $\log \widetilde{K}_n$ tends to $-\infty$. This gives that

$$\lim_{\varepsilon \to 0} \frac{d^2}{dt^2}(\log \widetilde{K}_n \circ \gamma(t))(0) = +\infty.$$

On the other hand, the second derivatives of $\log \widetilde{K}_n$ along the directions tangent to Z as well as the gradient of $\log \widetilde{K}_n$ are convergent and therefore bounded when $\varepsilon \to 0$. It follows that as $\varepsilon \to 0$,

$$\Delta \log \widetilde{K}_n(\varepsilon) \to +\infty \quad \text{at } P_0.$$

This contradicts (3.12). The proof is completed.

REMARK. To avoid using the real analyticity of the zero set Z, a general rigorous argument of the above proof is given in [**8**].

Using Theorem 3, and following the argument in Theorem 2, we obtain

THEOREM 4. *The generalization of Theorem* 2 *to high dimensions is valid.*

4. Log-concavity of νth mean curvature and a constant rank theorem. Before considering νth mean curvatures, we mention that the superharmonicity of curvatures explained classical known results. For an example, Liebman-Klotz-Osserman [7] proved that a complete convex surface M of constant mean curvature in R^3 is a cylinder, a plane or a sphere. Since the complete surface M of nonnegative curvature is parabolic, the superharmonicity of K when $K \geq 0$ implies immediately that K is a constant. Hence M is a cylinder, a plane or a sphere. The version also explains why the standard sphere is rigid among the class of surfaces of a given constant mean curvature, while examples with K of alternating sign may possibly be deformed.

By considering superharmonicity of logarithm of the νth mean curvature K_ν of M^n in R^{n+1}, we obtain a constant rank theorem as a further application. The νth *mean curvature* K_ν *is defined by* the νth symmetric function of principal curvatures $\lambda_1, \lambda_2, \ldots, \lambda_n$. For example,

$$(4.0) \qquad \begin{aligned} K_1 &\equiv \lambda_1 + \lambda_2 + \cdots + \lambda_n = \text{mean curvtature } H, \\ K_2 &\equiv \lambda_1\lambda_2 + \lambda_1\lambda_3 + \cdots + \lambda_{n-1}\lambda_n \end{aligned}$$

and

$$K_n \equiv \lambda_1\lambda_2\cdots\lambda_n = \text{Gauss-Kronecker curvature } K_n.$$

It is then clear that

$$\det(h_{ij} + \lambda\delta_{ij}) = K_n + K_{n-1}\lambda + \cdots + K_1\lambda^{n-1} + \lambda^n. \tag{4.1}$$

THEOREM 5. *Let M^n be a connected convex hypersurface immersed in R^{n+1} with concave mean curvature H, i.e.*

$$\frac{d^2}{dt^2}H \le 0 \tag{4.2}$$

along any geodesic parametrized by t, or equivalently, the second covariant derivatives constitute a nonpositive matrix. Let the rank of the second fundamental form (h_{ij}) attain a minimum r_0 at an interior point p_0 of M_n. Then (i) *each K_μ with $r_0 < \mu \le n$ vanishes on M^n,* (ii) *the rank of (h_{ij}) is constantly equal to r_0 on M^n,* (iii) *the logarithm of the r_0th mean curvature K_{r_0} is superharmonic at the points where $K_{r_0} > 0$, and* (iv) *K_{r_0} satisfies the strong minimum principle, i.e.*

$$\min_M K_{r_0} = \min_{\partial M} K_{r_0} \tag{4.3}$$

where K_{r_0} has no interior minimum unless K_{r_0} is constant.

REMARK 4.1. The statement shows that the constant rank theorem is essentially a consequence of superharmonicity of $\log \widetilde{K}_\mu$, $r_0 < \mu \le n$ (for the meaning of $\widetilde{K}_\mu$, see below). The details are contained in the following proof.

Proof. It suffices to prove the strong minimum principle for K_ν with $r_0 \le \nu \le n$, since $K_{r_0+1}, K_{r_0+2}, \dots, K_n$ then vanish on M as they attain a zero minimum at the interior point p_0. We use the induction on ν backward until ν equals r_0. For $\nu = n$, Theorem 3 has shown that the strong minimum principle holds for K_n. Now assuming $K_{\nu+1}, K_{\nu+2}, \dots, K_n$ satisfies the strong minimum principle, we claim that K_ν does also. However this induction hypothesis yields $K_{\nu+1} = K_{\nu+2} = \cdots = K_n = 0$ at every point of M. Given an interior point p_1 of M_n, we choose around p_1 a local frame $\{e_i\}$ on M^n by the following construction:

(1) letting

$$h_{ij} = \lambda_i\delta_{ij} \quad \text{at } p_1, \tag{4.4}$$

(2) defining $e_{\nu+1}, e_{\nu+2}, \ldots, e_n$ in a neighborhood of p_1 by principal directions with zero principal curvature,

$$\lambda_{\nu+1} = \lambda_{\nu+2} = \cdots = \lambda_n = 0. \tag{4.5}$$

It is direct to verify that the tangent subspace by $\{e_{\nu+1}, \ldots, e_n\}$ on a neighborhood of p_1 is parallel in M^n. Hence we may construct a local frame $\{e_1, \ldots, e_n\}$ parallel along radial geodesics emanating from p_1 by holding (1) and (2) valid. We now define $\tilde{h}_{ij}$, the *perturbed* h_{ij}, by

$$\tilde{h}_{ij} = h_{ij} + \varepsilon_i \delta_{ij} \tag{4.6}$$

where

$$\varepsilon_i = \begin{cases} \varepsilon; & i = 1, 2, \ldots, \nu, \\ 0; & i = \nu+1, \ldots, n, \end{cases}$$

ε being a small positive number. It is not difficult to see that

$$\tilde{h}_{ijk} = \tilde{h}_{ikj} + O(\varepsilon)$$

uniformly in a compact neighborhood of p_1. We also define $\widetilde{K}_\nu$, the *perturbed* K_ν, by

$$\det(\tilde{h}_{ij} + \lambda\delta_{ij}) = \widetilde{K}_n + \widetilde{K}_{n-1}\lambda + \widetilde{K}_{n-2}\lambda^2 + \cdots + \widetilde{K}_1\lambda^{n-1} + \lambda^n. \tag{4.8}$$

It is clear that for any given μ with $\nu + 1 \le \mu \le n$,

$$\tilde{h}_{\mu\mu}(p_1) = 0, \tag{4.9}$$

$$\tilde{h}_{\mu\mu k}(p_1) = 0 \tag{4.10}$$

and

$$\begin{aligned} \tilde{h}_{\mu\mu kk}(p_1) &= (\tilde{h}_{kk\mu\mu} - \tilde{h}_{kk}R_{\mu k\mu k} + \tilde{h}_{\mu\mu}R_{k\mu k\mu})(p_1) + O(\varepsilon) \\ &= \tilde{h}_{kk\mu\mu}(p_1) + O(\varepsilon) = \frac{d^2}{dt^2}H + O(\varepsilon) \quad \text{at } p_1 \le O(\varepsilon), \end{aligned} \tag{4.11}$$

t being the parameter of the geodesic generated by e_μ through p_1.

We now compute $\Delta \log \widetilde{K}_\nu$. To avoid the complication of exhibiting long indices, we simplify notations of formulas when the meaning is obvious. We have

$$\widetilde{K}_\nu = \sum_{I,\sigma_I} (-1)^{\sigma_I} \tilde{h}_{i_1\sigma_{i_1}} \cdots \tilde{h}_{i_\nu\sigma_{i_\nu}} \tag{4.12}$$

where the summation ranges over all $I \equiv (i_1, i_2, \ldots, i_\nu)$ with

$$1 \le i_1 < i_2 < \cdots < i_\nu \le n, \tag{4.13}$$

and over all the permutations σ_I of I. It is straightforward that

$$(4.14)\quad \Delta\log\widetilde{K}_\nu = \frac{\Delta\widetilde{K}_\nu}{\widetilde{K}_\nu} - \left|\frac{\nabla\widetilde{K}_\nu}{\widetilde{K}_\nu}\right|^2$$

$$= \frac{1}{\widetilde{K}_\nu}\left\{\sum_{I,\sigma_I}(-1)^{\sigma_I}[(\tilde{h}_{i_1\sigma_{i_1}kk}\tilde{h}_{i_2\sigma_{i_2}}\cdots\tilde{h}_{i_\nu\sigma_{i_\nu}}+\cdots)\right.$$

$$\left.+2(\tilde{h}_{i_1\sigma_{i_1}k}\tilde{h}_{i_2\sigma_{i_2}k}\tilde{h}_{i_3\sigma_{i_3}}\cdots\tilde{h}_{i_\nu\sigma_{i_\nu}}+\cdots)]\right\}$$

$$-\frac{1}{(\widetilde{K}_\nu)^2}\left\{\sum_{I,\sigma_I}(-1)^{\sigma_I}[\tilde{h}_{i_1\sigma_{i_1}k}\tilde{h}_{i_2\sigma_{i_2}}\cdots\tilde{h}_{i_\nu\sigma_{i_\nu}}+\cdots]\right\}^2.$$

By virtue of (4.4), (4.5), (4.6), (4.9) and (4.10), most of the terms in (4.14) vanish at p_1 and we have

$$(4.15)\quad \Delta\log\widetilde{K}_\nu$$

$$\le \frac{1}{\tilde{h}_{11}\tilde{h}_{22}\cdots\tilde{h}_{\nu\nu}}\{[\tilde{h}_{11kk}\tilde{h}_{22}\cdots\tilde{h}_{\nu\nu}+\cdots+\tilde{h}_{11}\tilde{h}_{22}\cdots\tilde{h}_{\nu\nu kk}]$$

$$+2[(\tilde{h}_{11k}\tilde{h}_{22k}-\tilde{h}^2_{12k})\tilde{h}_{33}\cdots\tilde{h}_{\nu\nu}+\cdots$$

$$+\tilde{h}_{11}\tilde{h}_{22}\cdots(\tilde{h}_{n-1,n-1,k}\tilde{h}_{nnk}-\tilde{h}^2_{n-1,n,k})]\}$$

$$-\frac{1}{(\tilde{h}_{11}\tilde{h}_{22}\cdots\tilde{h}_{\nu\nu})^2}[\tilde{h}_{11k}\tilde{h}_{22}\cdots\tilde{h}_{\nu\nu}+\cdots+\tilde{h}_{11}\tilde{h}_{22}\cdots\tilde{h}_{\nu\nu k}]^2$$

$$+O(\varepsilon).$$

For any given i with $1\le i\le n$, we note that by (4.2),

$$(4.16)\quad \tilde{h}_{iikk} = \tilde{h}_{kkii} - \tilde{h}_{kk}R_{ikik} + \tilde{h}_{ii}R_{kiki} + O(\varepsilon)$$

$$\le 0 - \sum_{i,k=1}^{\nu}(\lambda_k+\varepsilon)(\lambda_i\lambda_k) + \sum_{i,k=1}^{\nu}(\lambda_i+\varepsilon)(\lambda_i\lambda_k) + O(\varepsilon).$$

Letting the three parts of (4.15) corresponding to the three brackets in (4.15) be denoted respectively by I, II and III, we have

$$(4.17)\quad \mathrm{I} \le -\sum_{i,k=1}^{\nu}\left(\frac{\lambda_k+\varepsilon}{\lambda_i+\varepsilon}-1\right)\lambda_i\lambda_k + O(\varepsilon)$$

$$= -\sum_{1\le i<k\le\nu}(\lambda_k-\lambda_i)^2\frac{\lambda_i\lambda_k}{(\lambda_i+\varepsilon)(\lambda_k+\varepsilon)} + O(\varepsilon) \le O(\varepsilon).$$

For the sum II + III, we cancel the terms of the sort,

$$2(\tilde{h}_{11}\tilde{h}_{22}\cdots\tilde{h}_{\nu\nu})\tilde{h}_{11k}\tilde{h}_{22k}\tilde{h}_{33}\cdots\tilde{h}_{\nu\nu} \tag{4.18}$$

in II with the cross terms of the sort

$$-2(\tilde{h}_{11k}\tilde{h}_{22}\tilde{h}_{33}\cdots\tilde{h}_{\nu\nu})(\tilde{h}_{11}\tilde{h}_{22k}\tilde{h}_{33}\cdots\tilde{h}_{\nu\nu}) \tag{4.19}$$

to obtain

$$\begin{aligned} \mathrm{II}+\mathrm{III} &= -2\left[\frac{\tilde{h}^2_{12k}}{\tilde{h}_{11}\tilde{h}_{22}}+\cdots\right]-\left[\frac{\tilde{h}^2_{11k}}{\tilde{h}^2_{11}}+\frac{\tilde{h}^2_{22k}}{\tilde{h}^2_{22}}+\cdots\right] \\ &= -\sum_{i,j=1}^{\nu}\sum_{k=1}^{\nu}\frac{\tilde{h}^2_{ijk}}{(\lambda_i+\varepsilon)(\lambda_j+\varepsilon)}. \end{aligned} \tag{4.20}$$

Combining (4.17) and (4.20), we obtain

$$\begin{aligned} \Delta\log\widetilde{K}_\nu \le &- \sum_{1\le i<j\le \nu}(\lambda_i-\lambda_j)^2\frac{\lambda_i\lambda_j}{(\lambda_i+\varepsilon)(\lambda_j+\varepsilon)} \\ &- \sum_{i,j=1}^{\nu}\sum_{k=1}^{n}\frac{\tilde{h}^2_{ijk}}{(\lambda_i+\varepsilon)(\lambda_j+\varepsilon)}+O(\varepsilon). \end{aligned} \tag{4.21}$$

We therefore conclude that $\Delta\log\widetilde{K}_\nu \le 0$ for small $\varepsilon > 0$. For otherwise $\lambda_1 = \cdots = \lambda_\nu$ and $\nabla\tilde{h}_{ij} = 0$ in a compact neighborhood U of p_1, since the term $O(\varepsilon)$ tends to zero uniformly in compact neighborhood of p_1. But this means that in U, $\lambda_1 = \cdots = \lambda_\nu \equiv$ a positive constant and $\lambda_{\nu+1} = \cdots = \lambda_n \equiv 0$, which implies $\widetilde{K}_\nu$ is a positive constant in U, contradicting the assumption of $\Delta\log\widetilde{K}_\nu > 0$ at p_1. Hence $\widetilde{K}_\nu$ satisfies the strong minimum principle for $r_0 \le \nu \le n$. A similar argument at the end of the proof of Theorem 3 now applies and passes the strong minimum principle for $\widetilde{K}_\nu = \widetilde{K}_\nu(\varepsilon)$ to that of the limit K_ν as ε tends to zero (for more precisely details, see **[8]**). This completes the proof of Theorem 5.

5. Superharmonicity for level curvatures. Brascamp-Lieb proved in **[2]** that the first eigenfunction μ of the Laplacian over a convex domain Ω in R^2 with zero boundary value on $\partial\Omega$ has its level curves all convex. In the works of Lee-Wang **[13]** and Shih **[15]**, the problem has been considered for convex domains in spheres or in hyperbolic planes, among which Shih used the expression

$$k = (-u_1^2u_{22}+2u_1u_2u_{12}-u_2^2u_{11})/|\nabla u|^3$$

for the curvature of level curves to illustrate a counterexample of the Brascamp-Lieb theorem in hyperbolic planes.

We consider in this section the level curves of surfaces of constant mean curvature. Shiffman [14] dealt with them for minimal surfaces using complex setting. In this note we tackle the problem by directly computing the Laplacian of the curvature of level curves and show its superharmonicity under C^2-deformation. This provides a more geometric version.

Let M^n be a hypersurface of constant mean curvature H immersed in R^{n+1} via

$$X: M^n \to R^{n+1},$$

and

$$u = u(X) = \langle X, \xi\rangle$$

be a height function relative to the direction ξ of R^{n+1}. For each real number a, the level set

$$N_a \equiv \{X \in M^n | u(X) = a\}$$

has *Gauss-Kronecker curvature* L in $R^n_a \equiv \{X \in R^{n+1} | u(X) = a\}$ defined by the product of principal curvatures $\lambda'_1, \lambda'_2, \dots, \lambda'_{n-1}$ of N_a in R^n_a. We call L the *level curvature of* M^n *relative to* ξ and a chosen normal field e_{n+1} of M^n in R^{n+1}. It is easy to see that

$$L = (u_i u_j H_{ij})/|\nabla u|^3 \tag{5.1}$$

where u_i, $1 \le i \le n$, are the covariant derivatives of u relative to an orthonormal local frame $\{e_i\}$ of M^n and H_{ij} the cofactor of the second fundamental form (h_{ij}) with respect to e_{n+1} of M^n in R^{n+1}. We have

$$h_{ij} H_{jk} = \delta_{ik} \det(h_{ij}) = \delta_{ik} K_n \tag{5.2}$$

where K_n is the Gauss-Kronecker curvature of M^n in R^{n+1} considered in §§3, 4. On the other hand, the Hessian u_{ij} has the forms as follows:

$$\begin{aligned} u_{ij} &= \langle X, \xi\rangle_{ij} = \langle X_{ij}, \xi\rangle = \langle h_{ij} e_{n+1}, \xi\rangle \\ &= h_{ij}\langle e_{n+1}, \xi\rangle = \pm h_{ij}\sqrt{1 - |\nabla u|^2} = h_{ij} W \end{aligned} \tag{5.3}$$

where

$$W \equiv \pm\sqrt{1 - |\nabla u|^2} = \langle e_{n+1}, \xi\rangle, \quad \nabla u = \operatorname{grad} u = \sum_{i=1}^n u_i e_i \tag{5.4}$$

and e_{n+1} is the unit normal of M^n in R^{n+1} with

$$h_{ij} = \langle X_{ij}, e_{n+1}\rangle. \tag{5.5}$$

We also remark that

$$\nabla u=\sum_{i=1}^{n} u_i e_i=\sum_{i=1}^{n}\langle X_i\,,\,\xi\rangle e_i=\sum_{i=1}^{n}\langle e_i\,,\,\xi\rangle e_i=\xi^T \tag{5.6}$$

which is the tangential component of ξ on tangent space of M^n and if we define

$$p\equiv|\nabla u| \tag{5.7}$$

then $0\leq p\leq 1$, since

$$p^2=|\nabla u|^2=|\xi^T|^2\leq|\xi|^2=1 \tag{5.8}$$

and

$$\begin{aligned}|W|&=\sqrt{1-p^2}=|\xi^N|\\&=\text{the length of normal component of } \xi.\end{aligned} \tag{5.9}$$

To abbreviate notations, we define

$$L_\alpha=(u_iu_jH_{ij})/|\nabla u|^\alpha\,, \tag{5.10}$$

by which we have

$$L_0=u_iu_jH_{ij}\,,\quad \text{and}\quad L=L_3(u_iu_jH_{ij})/|\nabla u|^3. \tag{5.11}$$

In the following, we first compute ΔL_0 and then ΔL_3. Our idea is to express

$$\Delta L_a=A+BL_\alpha+C\cdot\nabla L_\alpha$$

so that $A\leq 0$, $B\leq 0$. Then deform M into a standard surface such as a catenoid to conclude $L_\alpha\geq 0$ at every stage.

Step 1.

$$\begin{aligned}\Delta(u_iu_jH_{ij})&=(u_iu_jH_{ij})_{kk}\quad \text{(summation convention used unless otherwise declared)}\\&=(2u_{ik}u_jH_{ij}+u_iu_jH_{ijk})_k\\&=2u_{ikk}u_jH_{ij}+2u_{ik}u_{jk}H_{ij}\\&\quad+4u_{ik}u_jH_{ijk}+u_iu_jH_{ijkk}\end{aligned} \tag{5.12}$$

where the four terms are denoted respectively by

2I, 2II, 4III, and IV.

Step 2.

$$\begin{aligned}\mathrm{I}=u_{ikk}u_jH_{ij}&=(u_{kki}+u_mR_{kmki})u_jH_{ij}\\&=[(h_{kk}W)_i+u_mR_{kmki}]u_jH_{ij}\quad [\text{by } (5.3)]\\&=(h_{kki}W+h_{kk}W_i)u_jH_{ij}+u_mu_jH_{ij}R_{kmki}.\end{aligned} \tag{5.13}$$

However, we have

$$(5.14)\qquad \begin{aligned} h_{kki}W &= H_iW = 0 \quad \text{and} \\ W_i &= \langle e_{n+1}, \xi\rangle_i = \langle -h_{il}e_l, \xi\rangle = -h_{il}u_l. \end{aligned}$$

Since H is constant,

$$(5.15)\qquad \begin{aligned} h_{kk}W_iu_jH_{ij} &= -H(h_{il}u_l)u_jH_{ij} \quad \text{[by (5.3)]} \\ &= -Hu_lu_j\delta_{lj}K_n \quad \text{[by (5.2)]} \\ &= -|\nabla u|^2HK_n \end{aligned}$$

and

$$(5.16)\qquad \begin{aligned} u_mu_jH_{ij}R_{kmki} &= u_mu_jH_{ij}(h_{kk} - h_{mi} - h_{ki}h_{mk}) \\ &= u_mu_j(\delta_{jm}K_n)H - u_mu_j(\delta_{jk}K_n)h_{mk} \\ &= |\nabla u|^2HK_n - u_mu_jh_{mj}K_n \\ &= [|\nabla u|^2H - B(\nabla u, \nabla u)]K_n \end{aligned}$$

where B denotes the bilinear form defined by the second fundamental form (h_{ij}) of M^n in R^{m+1}. It follows that

$$(5.17)\qquad \mathrm{I} = -B(\nabla u, \nabla u)K_n.$$

Step 3.

$$(5.18)\qquad \begin{aligned} \mathrm{II} = u_{ik}u_{jk}H_{ij} &= W^2h_{ik}h_{jk}H_{ij} = W^2h_{ik}(\delta_{ki}K_n) \\ &= W^2HK_n = (1 - |\nabla u|^2)HK_n. \end{aligned}$$

Step 4. We will express III in terms of L and ∇L when $n = 2$. Consider a frame $\{e_1, e_2, \dots, e_n\}$ of M^n such that e_1 is perpendicular to the level sets N_a, $a \in R$. Then

$$(5.19)\qquad \nabla u = u_1e_1.$$

Now at a given point p_0 of M^n, we select $e_2, e_3, \dots, e_n$ so that $h_{ij} = \delta_{ij}$ at p_0 for $i, j \geq 2$. We call such a frame a *level principal frame* of M^n at p_0 with respect to ξ. It is evident that

$$(5.20)\ \begin{aligned} \mathrm{III} = u_{ik}u_jH_{ijk} &= u_jWh_{ik}H_{ijk} = u_jW[(\delta_{kj}K_n)_k - h_{ikk}H_{ij}] \\ &= u_jW(K_n)_j \quad [\text{since } h_{ikk} = h_{kki} = H_i = 0] \\ &= W\nabla u\cdot\nabla K_n. \end{aligned}$$

On the other hand, when $n = 2$ we consider

$$(5.21)\qquad \nabla L_0 = (L_0)_1e_1 + (L_0)_2e_2$$

where

$$(L_0)_1 = (u_iu_jH_{ij})_1 = 2u_{i1}u_jH_{ij} + u_iu_jH_{ij1} \tag{5.22}$$
$$= 2Wu_1K_n + u_1^2H_{111}$$
$$= 2|\nabla u|WK_n + |\nabla u|^2h_{221}$$

and

$$(L_0)_2 = (u_iu_jH_{ij})_2 = u_1^2H_{112} = |\nabla u|^2h_{222}. \tag{5.23}$$

Hence we have for $n = 2$,

$$\begin{aligned} \mathrm{III} &= u_{ik}u_jH_{ijk} = u_{ik}u_1H_{i1k} \\ &= u_1W(h_{11}H_{111} + h_{12}(H_{112} + H_{211}) + h_{22}H_{212}) \\ &= u_1W(h_{11}h_{221} + h_{12}(h_{222} - h_{121}) - h_{22}h_{122}) \\ &= |\nabla u|((h_{11} - h_{22})h_{221} + 2h_{12}h_{222}) \end{aligned} \tag{5.24}$$

noting that h_{ijk} is symmetric with respect to each pair of i, j, k. By substituting (5.22) and (5.23) into (5.24), we obtain

$$\mathrm{III} = \frac{W}{|\nabla u|}[(h_{11}-h_{22})e_1+2h_{12}e_2]\cdot\nabla L_0-2W^2\left(H-\frac{2L_0}{|\nabla u|^2}\right)K_n \tag{5.25}$$

where $h_{22} = L_0/|\nabla u|^2$ has been considered. Therefore we have

$$\begin{aligned} \mathrm{III} = &-2W^2HK_n + \left(4\frac{W^2}{|\nabla u|^2}K_n\right)L_0 \\ &+ \frac{W}{|\nabla u|}[(h_{11} - h_{22})e_1 + 2h_{12}e_2]\cdot\nabla L_0 \end{aligned} \tag{5.26}$$

for $n = 2$.

Step 5. Let σ_{ij} be a permutation

$$\begin{pmatrix} 1 & 2 & \cdots & i & \cdots & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_i & \cdots & \sigma_n \end{pmatrix}$$

with $\sigma_i = j$. Then

$$H_{ij} = \sum_{\sigma_{ij}}(-1)^{\sigma_{ij}}h_{1\sigma_1}\cdots\hat{h}_{i\sigma_i}\cdots h_{n\sigma_n} \tag{5.27}$$

where "$\frown$" denotes the deletion of the term, and we have

$$
\begin{aligned}
(5.28)\quad \mathrm{IV} &= u_i u_j H_{ijkk} \\
&= u_i u_j \left[\sum_{\sigma_{ij}} (-1)^{\sigma_{ij}} h_{1\sigma_1} \cdots \hat{h}_{i\sigma_i} \cdots h_{n\sigma_n} \right]_{kk} \\
&= u_i u_j \sum_{\sigma_{ij}} (-1)^{\sigma_{ij}} [h_{1\sigma_1} h_{2\sigma_2} \cdots \hat{h}_{ij} \cdots h_{n\sigma_n} \\
&\qquad\qquad + \cdots + h_{1\sigma_1} h_{2\sigma_2} \cdots \hat{h}_{ij} \cdots h_{n\sigma_n k}]_k \\
&= u_i u_j \sum_{\sigma_{ij}} (-1)^{\sigma_{ij}} \{[h_{1\sigma_1 kk} h_{2\sigma_2} \cdots \hat{h}_{ij} \cdots h_{n\sigma_n} \\
&\qquad\qquad + \cdots + h_{1\sigma_1} h_{2\sigma_2} \cdots \hat{h}_{ij} \cdots h_{n\sigma_n kk}] \\
&\qquad\qquad + 2[h_{1\sigma_1 k} h_{2\sigma_2 k} \cdots \hat{h}_{ij} \cdots h_{n\sigma_n} + \cdots \\
&\qquad\qquad + h_{1\sigma_1} h_{2\sigma_2} \cdots \hat{h}_{ij} \cdots h_{n-1,\sigma_{n-1} k} h_{n\sigma_n k}]\}.
\end{aligned}
$$

By taking a level principal frame at p_0, we have

$$
\begin{aligned}
(5.29)\quad \mathrm{IV} = u_1^2 \{ & [h_{22kk} h_{33} \cdots h_{nn} + \cdots + h_{22} h_{33} \cdots h_{nnkk}] \\
& + 2[(h_{22k} h_{33k} - h_{23k}^2) h_{44} \cdots h_{nn} \\
& + \cdots + h_{22} h_{33} \cdots (h_{n-1,n-1,k} h_{nnk} - h_{n-1,n,k}^2)]\}
\end{aligned}
$$

On the other hand, for each i with $2 \leq i \leq n$,

$$
\begin{aligned}
(5.30)\quad h_{iikk} &= h_{kkii} + h_{mi} R_{kmki} + h_{km} R_{imki} \\
&= 0 + h_{mi}(h_{kk} h_{mi} - h_{ki} h_{mk}) + h_{km}(h_{ik} h_{mi} - h_{ii} h_{mk}) \\
&= H(h_{1i}^2 + h_{ii}^2) - h_{ii} \|B\|^2
\end{aligned}
$$

where the summation convention applies only on k while i is a *fixed* index. We obtain

$$
\begin{aligned}
(5.31)\quad \mathrm{IV} = & |\nabla u|^2 H[h_{12}^2 h_{33} h_{44} \cdots h_{nn} + h_{22} h_{13}^2 h_{44} \cdots h_{nn} + \cdots] \\
& + |\nabla u|^2 H[h_{22} + h_{33} + h_{44} + \cdots + h_{nn}] h_{22} h_{33} h_{44} \cdots h_{nn} \\
& - \|B\|^2 L_0 + Q
\end{aligned}
$$

where Q denotes the second term of (5.29). However the first term of (5.31) equals to

$$
(5.32)\qquad |\nabla u|^2 H(h_{11} h_{22} \cdots h_{nn} - \det(h_{ij})).
$$

Adding this to the second term of (5.31), it simplifies to

$$
(5.33)\qquad H^2 L_0 - |\nabla u|^2 H K_n.
$$

We conclude with

$$(5.34)\qquad \begin{aligned} \mathrm{IV} &= H^2L_0 - |\nabla u|^2HK_n - \|B\|^2L_0 + Q \\ &= 2K_2L_0 - |\nabla u|^2HK_n + Q \end{aligned}$$

where K_2 is the 2nd mean curvature defined in (4.0) and the term Q has the properties:

(i) For $n = 2$, $Q = 0$,

(ii) At a zero minimal point p_0 of L_α, $\alpha \geq 0$ (for $\alpha = 0, 1$, also assume $|\nabla u| \neq 0$ at p_0), we have

$$(5.36)\qquad Q \leq 0.$$

In fact, the frame $\{e_i\}$ at p_0 can be taken such that

$$(5.37)\qquad h_{22}(p_0) = 0$$

and $h_{33}(p_0), \dots, h_{nn}(p_0)$ all nonnegative, since

$$h_{22}h_{33}\cdots h_{nn} = |\nabla u|^{\alpha-2}L_\alpha \geq 0$$

and "$=0$" holds at p_0. By parallelly transporting $\{e_i\}$ to neighboring points of p_0 along radial geodesics from p_0, we have also

$$(5.38)\qquad h_{22k}(p_0) = (dh_{22})(e_k) \quad \text{at } p_0 = 0.$$

Using (5.37) and (5.38), all the terms of Q vanish except the terms of the type

$$(5.39)\qquad -h_{23k}^2h_{44}\cdots h_{nn}$$

which are clearly not greater than zero. For $n = 2$, (5.34) reduces to

$$(5.40)\qquad \mathrm{IV} = -|\nabla u|^2HK + 2KL_0.$$

THEOREM 6. *For a surface M of constant mean curvature H immersed in R^3, let L_0 be defined by* (5.11); *then*

$$(5.41)\qquad \begin{aligned} p^2\Delta L_0 = &- 3p^2(2-p^2)HK + 4(4-3p^2)KL_0 \\ &+ 4p(1-p^2)[(h_{11}-h_{22})e_1 + 2h_{12}e_2]\cdot\nabla L_0 \end{aligned}$$

where $p = |\nabla u| \leq 1$, K is the Gaussian curvature of M, and $(h_{ij}$ is the second fundamental form relative to the level principal frame $\{e_1, e_2\})$.

Proof. Combining (5.17), (5.18), (5.26), (5.40), we obtain

$$p^2\Delta L_0 = 2p^2\mathrm{I} + 2p^2\mathrm{II} + 4p^2\mathrm{III} + p^2\mathrm{IV}$$

where

$$\begin{aligned}
2p^2\mathrm{I} &= -2p^2B(\nabla u, \nabla u)K = -2p^4h_{11}K = -2p^4(H-h_{22})K \\
&= -2p^4HK + 2p^2KL_0, \\
2p^2\mathrm{II} &= 2p^2(1-p^2)HK = 2p^2HK - 2p^4HK, \\
4p^2\mathrm{III} &= -8p^2(1-p^2)HK + 16(1-p^2)KL_0 \\
&\quad + 4p(1-p^2)[(h_{11}-h_{22})e_1 + 2h_{12}e_2]\cdot\nabla L_0, \\
p^2\mathrm{IV} &= -p^4HK + 2p^2KL_0.
\end{aligned}$$

Hence (5.41) follows.

Step 7. In order to compute ΔL_α, we note that

$$L_0 = p^\alpha L_\alpha, \tag{5.42}$$

$$\begin{aligned}
p^2\Delta L_0 &= p^2\Delta(p^\alpha L_\alpha) \\
&= p^{\alpha+2}\Delta L_\alpha + 2p^2\nabla L_\alpha\cdot\nabla p^\alpha + p^2L_\alpha(\Delta p^\alpha)
\end{aligned} \tag{5.43}$$

and

$$\Delta p^\alpha = (p^\alpha)_{kk} = (\alpha p^{\alpha-1}p_k)_k = (\alpha p^{\alpha-2}u_ih_{ik}\sqrt{1-p^2})_k. \tag{5.44}$$

A straightforward calculation implies

$$\begin{aligned}
\Delta p^\alpha &= \alpha p^{\alpha-2}((\alpha-2)-(\alpha-1)p^2)(h_{11}^2+h_{22}^2) \\
&\quad + \alpha p^{\alpha-1}(1-p^2)(h_{11}^2+2h_{12}^2+h_{22}^2)
\end{aligned} \tag{5.45}$$

and

$$(L_0)_k = (p^\alpha L_\alpha)_k = p^\alpha(L_\alpha)_k + \alpha p^{\alpha-1}(p_k)L_\alpha. \tag{5.46}$$

Substituting (5.41), (5.44), (5.45) and (5.46) into (5.43), we get for $\alpha = 3$ the following main formula:

THEOREM 7. *For a surface M^2 of constant mean curvature H immersed in R^3, let L be the level curvature of M^2 relative to a given direction ξ. Then*

$$\begin{aligned}
p^5\Delta L = &-3p^2(2-p^2)HK \\
&+ \{p^3K + 3p^3(2-p^2)H^2 - 3p^4(3-2p^2)HL\}L \\
&+ 2p^4\sqrt{1-p^2}[-(h_{11}+2h_{22})e_1 + 2h_{12}e_2]\cdot\nabla L
\end{aligned} \tag{5.47}$$

where the notations are defined as in Theorem 6.

When the mean curvature $H \equiv 0$, we have the theorem:

THEOREM 8. *Let M^2 be a compact surface with boundary ∂M and*

$$X: M^2 \to R^3$$

be a minimal immersion of M^2 into R^3. Given a unit vector ξ of R^3, consider the height function u defined by

$$u(X) = \langle X, \xi\rangle.$$

Let L denote the level curvature (i.e. the curvature of level curves) relative to the vertical direction ξ. Then

$$|\nabla u|^2 \Delta L \equiv KL \ (\mathrm{mod}\, \nabla L) \tag{5.48}$$

and therefore L attains neither nonnegative interior minimum nor nonpositive interior maximum.

Proof. This is a direct consequence of (5.47).

As an application, we illustrate a plausible but simple geometric proof of Shiffman's theorem. Given an annular minimal surface M in R^3 with its two boundary curves Γ_1, Γ_2 convex in two horizontal planes respectively. Suppose there is a C^2-deformation of M into a catenoid as the two boundary curves are deformed into two circles on the given horizontal planes. Since $K \leq 0$ on minimal surfaces, the formula (5.48) together with the continuity method shows by Shiffman's theorem that

$$L \geq 0. \tag{5.49}$$

In fact, we obtain a stronger estimate that

$$L \geq \min\{\text{curvature of } \Gamma_1, \Gamma_2\} \geq 0. \tag{5.50}$$

However with this argument, we have to assume additionally that $|\nabla u| \neq 0$ during the deformation.

Using the main formula (5.47) we prove our main theorem of this section, which can be regarded as a partial generalization of Shiffman's theorem to surfaces of constant mean curvature.

THEOREM 9. *Given an annular surface M of constant mean curvature H in R^3 such that $\partial M = \Gamma_1 \cup \Gamma_2$, each Γ_i being a convex curve in a plane $\Pi_{i, i=1,2}$, and Π_1 is parallel to Π_2. Suppose that the mean curvature vector is outward from the tubular compartment V enclosed by Π_1, Π_2 and M. This is equivalent to that $H \leq 0$ in the above setting, or equivalent to that the exterior pressure of the soap film M*

is no less than the inner pressure in V. Consider a C^2-deformation $\{M_t; 0 \le t \le 1\}$, each M_t having constant mean curvature H_t, of $M = M_1$ with H_t increasing into a minimal surface M_0 (for an example, we increase the inner pressure in V until both sides of the soap film reach equal pressures). Assume also that each M_t has no point tangent to a "horizontal" plane (i.e. a plane parallel to Π_i) so that the curvature of level curves on M_t is well defined. Then the level curves of each M_t, $0 \le t \le 1$, are all convex.

REMARK 5.1. We can also consider the deformation by taking M into a Delaunay surface corresponding to a rotating hyperbola, while deforming each Γ_i into a circle in Π_i.

Before proving the theorem, we consider a stronger form of constant rank theorem.

LEMMA 5.1. *Given a surface M of constant mean curvature H in R^3, let the level curvature L relative to a given vertical direction ξ attain zero minimum at a point x_0, interior in M or on its boundary ∂M. Additionally, if the tangent of the level curve at x_0 is a principal direction of zero normal curvature, then M is either a circular cylinder or a plane.*

REMARK 5.2. This is a stronger form of constant rank theorem for $n = 2$, that instead of assuming $K \ge 0$ around x_0, we assume the level curvature $L \ge 0$ around x_0. I do not know whether the lemma is true for higher dimensions.

REMARK 5.3. Caffarelli-Friedman [3] have proved that for

$$\begin{cases} \Delta u = \gamma(u) \text{ in } \Omega; \quad \gamma(u) \nearrow \text{ in } u,\ \gamma(0) = 0, \\ u|_{\Gamma_1} = M,\ u|_{\Gamma_2} = m; \quad M > m > 0 \text{ constants}, \end{cases}$$

Γ_1 and Γ_2 being inner and outer strictly convex $C^{2,\alpha}$-boundary curves of Ω, the solution surface $u = u(x)$ has all level curves strictly convex. Compared with the condition $H \le 0$ in our Theorem 9, the assumption $\Delta u = \gamma(u) \ge \gamma(0) = 0$ are parallel in the linear case.

Proof of Lemma 5.1. We use the second-order comparison method developed as in [2]. Let Z be a cylinder of radius $1/|H|$, contacting M at x_0 of at least second order. This is possible since

$$L(x_0) = 0 = K(x_0) \tag{5.51}$$

and therefore the horizontal direction tangent to M at x_0 is a principal direction with zero principal curvature. On every horizontal plane Π, the straight line $Z \cap \Pi$ intersects the convex level curve in at most two points unless $M \equiv Z$. This implies that the function $u - v$ alternates its sign at most four times when turning around x_0, where $u = u(x, y)$, $v = v(x, y)$ are nonparametric functions of the surfaces M and Z on (x, y) of Π, respectively. However a theorem of Hopf shows that

$$u - v = R_e(\lambda(\xi + i\eta)^k) + o((\xi^2 + \eta^2)^{k/2}) \tag{5.52}$$

where λ is a complex number, (ξ, η) is a coordinate change of (x, y) with $(0, 0)$ corresponding to x_0 and

$$k = \text{contact order } + 1. \tag{5.53}$$

By the above construction of Z, $k \geq 3$, this leads to a contradiction. Hence $M \equiv Z$.

Proof of Theorem 9. Suppose there is M_t with some level curves nonconvex; then there would exist M_τ, $\tau \in (0, t]$, with zero minimal point x_0 for the level curvature L on M_τ. It is clear that

$$K(x_0) = h_{11}h_{22} - h_{12}^2 = h_{11} \cdot 0 - h_{12}^2 \leq 0 \tag{5.54}$$

for $h_{22} = L/p = 0$ at x_0, $p = |\nabla u|$. By Lemma 5.1, unless $M \equiv Z$ the possibility of $K(x_0) = 0$ is also excluded. Hence $K(x_0) < 0$ and the main formula (5.47) implies

$$p^5 \Delta L = -3p^2(2 - p^2)HK < 0 \quad \text{at } x_0 \tag{5.55}$$

contradicting the minimality of L at x_0. This completes the proof.

6. Concluding remarks. Given a compact surface M of constant mean curvature in R^3 with M convex and the boundary ∂M, is M then convex in the interior? i.e. does

$$K|_{\partial M} \geq 0 \text{ imply } K \geq 0 \text{ in } M, \tag{6.1}$$

where K is the Gaussian curvature of M? The implication is seen not necessarily true in general by considering, for an example, one of Wente's immersed tori **[13]** and cutting off a small neighborhood of a point of positive K. Adding topological conditions such as embedding or restriction on topological type is insufficient. In fact, we consider a Delaunay surface defined by revolving around an axis the locus of one of the two foci of a given ellipse which rotates on the axis.

The Delaunay surface is a periodic surface of revolution. We take a segment of the Delaunay surface bounded by two largest meridians. It gives an example of embedded nonconvex surfaces of constant mean curvature in R^3 which are convex around the boundary.

Certain stability seems required. A connection with capillary treatment leads to the capillary stability defined as follows. Let M^n be a hypersurface of constant mean curvature H_0 embedded in R^{n+1}, and let Ω^{n+1} be a bounded convex domain with C^2-boundary $C \equiv \partial\Omega^{n+1}$, such that $M^n \subset \Omega^{n+1}$ and ∂M^n meets C in a constant angle γ_0. Let E be the energy function given by

$$E = \operatorname{Area} M - (\cos\gamma_0)\operatorname{Area} C_* + nH_0 \operatorname{Vol}(\Omega_*)$$

where Ω_* is a component of $\Omega - M$ away from which points the mean curvature vector of M. Also C_* denotes $\overline{\Omega}_0 \cap C$. The energy function E is indeed defined on the space Σ_{γ_0} of all the surfaces embedded in Ω with their boundary meeting with C in the given constant angle γ_0. It is easily seen that E is critical on Σ_{γ_0} at the given surface M of constant mean curvature H_0 with constant contact angle γ_0 with C at the boundary points, or equivalently, the first variation of E at such M is zero for any perturbation of M in Σ_{γ_0}, i.e.

$$E' = 0 \quad \text{at } M. \tag{6.2}$$

We now call M *capillarily stable* with respect to the container C and the given contact angle γ_0 if the second variation of E at M is nonnegative for any perturbation of M in Σ_{γ_0}, i.e.

$$E'' \geq 0 \quad \text{at } M. \tag{6.3}$$

Problem 1. Let $\gamma_0 = 0$ and suppose M minimizes E for a given convex container C. Does there exist a 1-parameter C^2-family $\{M_t;\ 0 \leq t \leq 1\}$ of surfaces in Σ_{γ_0}, each M_t minimizing E with a corresponding $H_0 = H_0(t)$ (but $\gamma_0 \equiv 0$, independent of t) such that $M_0 = M$ and $M_1 =$ a small n-sphere? If it does, a capillary stable surface M with respect to a convex container C is convex in the interior by Theorems 2 and 4.

Whether the Dirichlet stability is related with capillary stability and is involved in order for (6.1) to be valid seems also interesting to know.

Another equally interesting *question* first proposed by Rosenberg [personal conversation, 1984 Stanford] is the following.

Problem 2. Does an embedded disk-type surface M of constant mean curvature in R^3 having its boundary ∂M a convex closed curve on a plane Π intersect with every plane parallel to Π (such a plane is called later "horizontal") in a convex closed curve?

The problem is also a type of Shiffman's problem. Suppose there is a C^2-deformation $\{M_t\}$ of M into a spherical cap as ∂M deformed into a circle, each M_t having constant mean curvature. We reverse the process as in the proofs of Theorem 2 and Theorem 4 and notice that a point of zero Gaussian curvature is impossible to appear first in the interior. However it may show up on the boundary and whence spread into a region of concave set M_-, i.e. the set with $K < 0$. It is evident that for a while after a point of zero K sneaks in across the boundary, the set D of the directions of nonpositive normal curvature at a point of the concave set M_- does not contain a horizontal direction. Formula (5.47) yields that at an interior zero minimal point of the level curvature L, $\Delta L = -3p^{-3}(2-p^2)HK > 0$. This does not exclude the possibility that at a certain stage in the reversed deformation, the set D finally touches a horizontal direction. And then there may start to appear a region of negative L. With this observation, the existence of counter examples for Problem 2 perhaps should not be excluded.

Even for the annular type, the generalized Shiffman's theorem established in Theorem 9 is not completed. In Theorem 9 we only consider the case $H \leq 0$, i.e. the inner pressure is not greater than the outer one. How is the case $H > 0$? It nevertheless has no essential difference in this case and the case of Problem 2.

Finally, we would like to mention the following related questions of interest.

Problem 3. Is Shiffman's theorem of $H \leq 0$ valid for hypersurfaces of higher dimensions? Does there exist a dimension bound?

Problem 4. For a nonparametric surface M of constant mean curvature in R^3 given by $u = u(x)$, $x \in \Omega \subset R^2$, with $u|_{\partial\Omega} = 0$, what is the least pinching number of the curvature κ of the boundary curve $\partial\Omega$ in order for the surface M to be convex? Here the pinching number of κ is defined by the ratio, $\kappa_{\min}$ to $\kappa_{\max}$, the minimum to the maximum of κ of the curve $\partial\Omega$ in R^2.

Note. A. N. Wang **[16]** has found, after the preprint of this paper was circulated, a counterexample showing that the level curves are not all convex when $H > 0$ as mentioned in the comments of Problem 2. Therefore the assumption $H \leq 0$ in Theorem 9 is essential.

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Received May 5, 1989.

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